Feedback Linearization of Multi-Input Nonlinear Differential Algebraic Control Systems

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Abstract

The problem of feedback linearization of index one multi-input nonlinear differential algebraic control systems via feedback transformations is addressed. Although necessary and sufficient geometric conditions for this problem have been provided in the early 2000. A complete solution to the feedback linearization problem is provided by defining an algorithm allowing to compute explicitly the linearizing feedback coordinate for index one multi-input nonlinear differential algebraic control systems without solving the partial differential equations. The algorithm consists of steps (the dimension of the system).

Keywords: Feedback linearization, Differential algebraic control, Control systems.

1. Introduction

1. INTRODUTION

The problem of transforming a nonlinear differential algebraic control system (NDACS)

$$\Sigma : \begin{cases} \dot{x} = f(x,z) + g_1(x,z)u_1 + \cdots + g_m(x,z)u_m \\ 0 = \sigma(x,z) \end{cases} \quad (1)$$

into the linear system

$$\Lambda : \begin{cases} \dot{\varpi} = A\varpi + B_1\upsilon_1 + \cdots + B_m\upsilon_m \\ 0 = \sigma(\varpi,z) \end{cases} \quad (2)$$

by a feedback transformation of the form

$$\Gamma : \begin{cases} \varpi = \varphi(x,z), & (x,z) \in M \\ u = \alpha(x,z) + \beta(x,z)\upsilon \end{cases} \quad (3)$$

where

$$M = \left\{ (x,z) \,\middle|\, \begin{array}{l} (x,z) \subset \Re^n \times \Re^m,\ \sigma(x,z) = 0, \\ \text{rank}\left(\dfrac{\partial \sigma(x,z)}{\partial z}\right) = m \end{array} \right\}$$

, is called feedback linearization problem to the system (1). The linearization problem of nonlinear differential algebraic control system is an important one and has been studied sparsely. Some investigation have been carried out McClamroch et al. with constrained mechanical systems[(1),(2)] and also by Kaprielian et al. with an AC/DC power system model[(3),(4)]. Their approaches consist of using transformations to obtain a state realization (state space representation) of the nonlinear descriptor system and then apply differential geometry for linearization. For single-input nonlinear differential algebraic control systems[(5)]. have defined

$$F(x,z) = \begin{pmatrix} I_n \\ -\left(\dfrac{\partial\sigma}{\partial z}\right)^{-1} \dfrac{\partial\sigma}{\partial x} \end{pmatrix}$$

where (I_n is an $n \times n$ identity matrix) and deal with the index one NDAE locally as the following nonlinear control,

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = F(x,z)f(x,z) + F(x,z)g(x,z)u$$

to study the exact feedback linearization for this class of NDAS. On the other hand, C. Chen et al. used the ideas of differential geometric control theory to define M derivative and M bracket in order to investigate the necessary and sufficient geometric conditions for exact feedback linearization of index one single-input nonlinear differential algebraic control systems[6]. The problem of feedback linearization is solved if and only if

$(F'1) \equiv \operatorname{rank}\left(g, Mad_f g, \ldots, Mad_f^{n-1} g\right) = n$

$(F'2)$ vector field sets $\Delta = \left\{g, Mad_f g, \ldots, Mad_f^{n-2} g\right\}$ are involutive in $(x,z) \in M$. Although, the conditions $(F'1)$ and $(F'2)$ provide a way of testing whether a given system is feedback linearizable but they offer little on how to find the linearizing change of coordinates $\varphi(x,z)$ except by solving a systems of partial differential equations (PDEs) which is, in general, not straightforward. For the problem of feedback linearization of single-input nonlinear differential algebraic control systems, Ayad and Nada[7],[8] provide a complete solution by defining an algorithm that allows to compute explicitly the linearizing state coordinates and feedback for index one nonlinear differential algebraic control systems. Each algorithm is performed using a maximum of $n-1$ steps (n being the dimension of the system). The objective of this paper is to provide an algorithm giving linearizing feedback coordinates for index one multi-input nonlinear differential algebraic control systems without solving the partial differential equations. The algorithm based on Frobenius Theorem.

2. Notations and Preliminaries

Consider the index one multi-input nonlinear differential algebraic control systems NDACS (1)

$$\Sigma : \begin{cases} \dot{x} = f(x,z) + g_1(x,z)u_1 \\ \quad + \cdots + g_m(x,z)u_m \\ 0 = \sigma(x,z) \end{cases}$$

where
$x = (x_1, \ldots, x_n)^T \in \Re^n, z = (z_1, \ldots, z_p)^T \in \Re^P$ and $u = (u_1, \ldots, u_m)^T \in \Re^m$. Also
$f(x,z) : \Re^n \times \Re^p \to \Re^n$,
$g(x,z) : \Re^n \times \Re^p \to \Re^n$ and
$\sigma(x,z) : \Re^n \times \Re^p \to \Re$ are smooth vector fields. and assume that its linear system

$$\Lambda : \begin{cases} \dot{\varpi} = A\varpi + Bu \\ \quad = A\varpi + B_1 u_1 + \cdots + B_m u_m \\ 0 = \sigma(\varpi, z) \end{cases}$$

is controllable, that is, there exist positive integers $r_1 \geq 1, \ldots, r_m \geq 1$ with $r_1 + \cdots + r_m = n$ such that
$\dim \operatorname{span}\left\{A^k B_i, 0 \leq k \leq r_i - 1, 1 \leq i \leq m\right\} = n.$

Define the coordinates
$\mathrm{x}_k = \left((\mathrm{x}_k^1)^T, \ldots, (\mathrm{x}_k^m)^T\right)^T$ on
$\Re^n = \Re^{r_1} \times \cdots \times \Re^{r_m}$, where for any $1 \leq i \leq r$ we set $\mathrm{x}_\mathrm{k}^i = (\mathrm{x}_{\mathrm{k}1}^i, \ldots, \mathrm{x}_{\mathrm{k}r}^i)^T$ and we put

$$\hat{\mathrm{x}}_{\mathrm{k}i} = \left(\mathrm{x}_{\mathrm{k}1}^1, \ldots, \mathrm{x}_{\mathrm{k}i}^1, \mathrm{x}_{\mathrm{k}1}^2, \ldots, \mathrm{x}_{\mathrm{k}i}^2, \ldots, \mathrm{x}_{\mathrm{k}1}^m, \ldots, \mathrm{x}_{\mathrm{k}i}^m\right)^T$$

Let the system Σ be denoted in the coordinates $\mathrm{x_k}$ by Σ_k

$$\Sigma_\mathrm{k} : \begin{cases} \dot{\mathrm{x}}_\mathrm{k} = \mathrm{f_k}(\mathrm{x_k}, z) + \mathrm{g_{k1}}(\mathrm{x_k}, z)u_1 \\ \quad + \cdots + \mathrm{g}_{\mathrm{k}m}(\mathrm{x_k}, z)u_m \\ 0 = \sigma(\mathrm{x_k}, z) \end{cases}$$

and for any $1 \leq i \leq m$ the ith subsystem Σ_k^i by

$$\Sigma_\mathrm{k}^i : \begin{cases} \dot{\mathrm{x}}_\mathrm{k}^i = \mathrm{f}_\mathrm{k}^i(\mathrm{x_k}, z) + g_{\mathrm{k}1}^i(\mathrm{x_k}, z)u_1 \\ \quad + \cdots + g_{\mathrm{k}m}^i(\mathrm{x_k}, z)u_m \\ 0 = \sigma(\mathrm{x_k}, z) \end{cases}$$

For any $1 \le i \le m$ and any $1 \le \mathrm{k} \le r$ we define A_i^{k} in the following way: for any $x = (x_1, \ldots, x_n)^T$ we have

$$A_i^{\mathrm{k}} x = (0, \ldots, 0, x_{k+2}, \ldots, x_r, 0)^T$$

that is, A_i^{k} is the matrix A_i with the entries in the first k rows being zeros.

Definition 2.1: [9]

The minimum number of times that all or part of the constraint equation must be differentiated with respect to time in order to solve for $\dot{z}$ as a continuous function of x and z is the index of the nonlinear differential algebraic system (1).

Definition 2.2: [6]

Let $f : \Re^n \times \Re^m \to \Re^n$ be a smooth vector field and $w : \Re^n \times \Re^m \to \Re$ a smooth function. The M derivative of w along f is a function $\Re^n \times \Re^m \to \Re$, written $M_f w$ and defined as $M_f w = E(w) f$, where $E(w) = \dfrac{\partial w}{\partial x} - \dfrac{\partial w}{\partial z}\left(\dfrac{\partial \sigma}{\partial z}\right)^{-1} \dfrac{\partial \sigma}{\partial x}$. If w is differential k times along f, the function $M_f^k w$ can be defined as $M_f^k w = M_f \left(M_f^{k-1} w\right)$ with $M_f^0 w = w$.

Definition 2.3: [6]

Given two smooth vector fields $f(x,z)$ and $g(x,z)$, both are defined on $\Re^n$ then the M bracket is defined as follows:

$$Mad_{f(x,z)}\, g(x,z) = \left[f(x,z), g(x,z)\right]_M = E(g)f - E(f)g$$

Repeated M brackets are denoted as

$$Mad_{f^k(x,z)}\, g(x,z) = Mad_f\,(Mad_{f^{k-1}} g),$$

$$Mad_{f^1(x,z)}\, g(x,z) = Mad_f\, g$$

and

$$Mad_{f^0(x,z)}\, g(x,z) = g.$$

Also,

$$[f(x,z), g(x,z)]_M = -[g(x,z), f(x,z)]_M$$

and

$$[f, g]_M\, w(x,z) = M_f M_g w - M_g M_f w.$$

Theorem 2.4: [6]

Consider the partial differential equation of function $w(x,z)$ with constraint condition $0 = \sigma(x,z)$

$$E(w)\left[v_1(x,z)\, v_2(x,z)\ \cdots\ v_3(x,z)\right] = 0$$

in which

$$E(w) = \frac{\partial w}{\partial x} - \frac{\partial w}{\partial z}\left(\frac{\partial \sigma}{\partial z}\right)^{-1} \frac{\partial \sigma}{\partial x}$$

where $(x,z) \in \Re^n \times \Re^m$, $v_i(x,z)(i = 1,2,\ldots,k < n)$ are linearly independent vector fields. If vector field set

$$D = \left\{v_1(x,z)\ v_2(x,z)\ \ldots\ v_3(x,z)\right\}$$

isinvolutive at $(x,z) = (x^0, z^0)$, then there exist certainly $(n-k)$ functions $w^1(x,z), w^2(x,z), \ldots, w^{n-k}(x,z)$ which satisfy given partial differential equation groups and the vectors

$$\left[E_1(w^j)\ E_2(w^j)\ \cdots\ E_n(w^j)\right] (j = 1,2,\ldots,(n-k),$$

$$E_i = \partial / \partial x^i - \Sigma_{k=1}^m r_{k_i}\, \partial / \partial z^i,\ i = 1,2,\ldots,n)$$

are linearly independent at (x^0, z^0).

Theorem 2.5: [7]

Let υ be a smooth vector field on $\Re^n$, for any integer $1 \le k \le n$ such that $\upsilon_k(0,0) \ne 0$ and $\omega_k(x,z) = 1/\upsilon_k(x,z)$. The diffeomorphism $\xi = \varphi(x,z)$, where $\varphi : M \to \Re^n$ defined by

$$\varphi_j(x,z) = x_j + \sum_{s=1}^{\infty} \frac{(-1)^s x_k^s}{s!} M_{\omega_k \upsilon}^{s-1}(\omega_k \upsilon_j)(x,z), \quad j \ne k$$

$$\varphi_k(x,z) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1} x_k^s}{s!} M_{\omega_k \upsilon}^{s-1}(\omega_k)(x,z) \tag{4}$$

Satisfies $\varphi^*(\upsilon) = \partial_{\xi_k}$. Moreover, the diffeomorphism $\psi(\xi)$, where $\psi : \Re^n \to M$ defined by

$$\psi_j(\xi) = \xi_j + \sum_{s=1}^{\infty} \frac{\xi_k^s}{s!}\left(\sum_{i=0}^{s-1}(-1)^i C_i^s \partial_{\xi_k}^i M_\upsilon^{s-i-1}(\upsilon_j)\right)$$

$$\psi_k(\xi) = \sum_{s=1}^{\infty} \frac{\xi_k^s}{s!}\left(\sum_{i=0}^{s-1}(-1)^i C_i^s \partial_{\xi_k}^i M_\upsilon^{s-i-1}(\upsilon_k)\right)$$

(5)

is the inverse of $\xi = \varphi(x, z)$.

where

$$\partial_{\xi_k} = \frac{\partial}{\partial \xi_k},\ \partial_{\xi_k}.h = \frac{\partial h}{\partial \xi_k}, \ldots,\ \partial_{\xi_k}^i.h = \frac{\partial^i h}{\partial \xi_k^i}$$

and $C_i^s = \dfrac{s!}{i!(s-i)!}$, $\quad i \geq 2$.

3. MAIN RESULTS

Definition 3.1

The index one multi-input nonlinear differential algebraic control systems Σ_k is called $(FB)_k$ −linear form if for $1 \leq i \leq m$ the *i*th subsystem Σ_k^i decomposes

$$\Sigma_k^i : \begin{cases} \dot{x}_{kj}^i = F_{kj}^i(\hat{x}_{kk+1}, z) & if\ 1 \leq j \leq k-1 \\ \dot{x}_{kj}^i = x_{kj+1}^i & if\ k \leq j \leq r-1 \\ \dot{x}_{kj}^i = u_{ki} & \\ 0 = \sigma(\hat{x}_{kk+1}, z) & \end{cases}$$

(6)

where $F_{kj}^i(0,0) = 0$ and $\dfrac{\partial F_{kj}^i}{\partial x_{kj+1}^i}(0,0) = 0$. It follows easily that if Σ_k is $(FB)_k$ −linear, then

$$Mad_{f_k}^{j-1}(g_{ki}) = A^{j-1}B_i,$$
$$1 \leq j \leq r-k+1, 1 \leq i \leq m$$

(7)

A more compact representation of Σ_k^i is obtained as

$$\Sigma_k^i : \begin{cases} \dot{x}_k^i = A_i^k x_k^i + F_k^i(\hat{x}_{kk+1}, z) + b_i u_{ki},\ x_k^i \in \Re^r \\ 0 = \sigma(\hat{x}_{kk+1}, z) \end{cases}$$

with the last $r-k$ components of F_k^i being identically zero. By extension, a compact notation for Σ_k would be

$$\Sigma_k : \begin{cases} \dot{x}_k = A_k x_k + F_k(\hat{x}_{kk+1}, z) + B_1 u_1 + \cdots + B_m u_m \\ 0 = \sigma(\hat{x}_{kk+1}, z) \end{cases}$$

where

$$F_k(\hat{x}_{kk+1}, z) = \left((F_k^1(\hat{x}_{kk+1}, z))^T, \ldots, (F_k^m(\hat{x}_{kk+1}, z))^T\right)^T.$$

Theorem 3.2

Consider the index one multi-input NDACS

$$\Sigma_r : \begin{cases} \dot{x}_r = f_r(x_r, z) + g_{r1}(x_r, z)u_{k1} \\ \quad + \cdots + g_{rm}(x_r, z)u_{km} \\ 0 = \sigma(x_r, z) \end{cases}$$

Assume it is feedback linearizable, that is, satisfies $(F'1)$ and $(F'2)$.

There exists a sequence of explicit feedback transformations $\Gamma_r = (\varphi_r, \alpha_r, \beta_r), \ldots, \Gamma_1 = (\varphi_1, \alpha_1, \beta_1)$ giving rise to a sequence of $(FB)_k$ −linear systems $\Sigma_{r-1}, \ldots, \Sigma_0$ such that

$$\Sigma_{k-1} = \Gamma_k{}^*(\Sigma_k) = (\varphi_k, \alpha_k, \beta_k)^* \Sigma_k, \quad 1 \leq k \leq r$$

The $(FB)_k$ −linear system Σ_k can be transformed into a $(FB)_{k-1}$ −linear system Σ_{k-1} if and only if for all $1 \leq i, j \leq m$

$$\begin{cases} \text{(a)}\ \dfrac{\partial^2 f_k(\hat{x}_{kk+1}, z)}{\partial x_{kk+1}^i \partial x_{kk+1}^j} = 0 \\ \text{(b)}\ \left[\dfrac{\partial f_k}{\partial x_{kk+1}^i}, \dfrac{\partial f_k}{\partial x_{kk+1}^j}\right] \\ \quad = \dfrac{\partial^2 f_k(\hat{x}_{kk+1}, z)}{\partial x_{kk+1}^i \partial x_{kk}^j} - \dfrac{\partial^2 f_k(\hat{x}_{kk+1}, z)}{\partial x_{kk}^i \partial x_{kk+1}^j} \end{cases}$$

(8)

Thus the composition $\Gamma_1 \circ \cdots \circ \Gamma_r$ linearizes the system Σ_r.

Algorithm 3.3

Step r. Consider a feedback linearizable system Σ denoted in the coordinates $x = \mathrm{x}_{\mathrm{r}}$ by Σ_{r}

$$\Sigma_{\mathrm{r}} : \begin{cases} \dot{\mathrm{x}}_{\mathrm{r}} = \mathrm{f}_{\mathrm{r}}(\mathrm{x}_{\mathrm{r}}, z) + \mathrm{g}_{\mathrm{r}1}(\mathrm{x}_{\mathrm{r}}, z) u_{\mathrm{r}1} \\ \quad + \cdots + \mathrm{g}_{\mathrm{r}m}(\mathrm{x}_{\mathrm{r}}, z) u_{\mathrm{r}m} \\ 0 = \sigma(\mathrm{x}_{\mathrm{r}}, z) \end{cases}$$

since Σ_{r} is feedback linearizable, hence the distribution $D_{\mathrm{r}} = \left\{ g_{\mathrm{r}1}, \ldots, g_{\mathrm{r}m} \right\}$ involutive, we apply Theorem (2.5) to construct $\xi_{\mathrm{r}} = \varphi(\mathrm{x}_{\mathrm{r}}, z)$ such that $(\varphi_r)^* \left\{ g_{\mathrm{r}1}, \ldots, g_{\mathrm{r}m} \right\} = \breve{\beta}(\xi_{\mathrm{r}}, z) \left\{ \partial_{\xi_{\mathrm{r}1}}, \ldots, \partial_{\xi_{\mathrm{r}m}} \right\}$. Then apply the feedback $\breve{u}_{\mathrm{r}} = \breve{\alpha}_{\mathrm{r}}(\xi_{\mathrm{r}}, z) + \breve{\beta}(\xi_{\mathrm{r}}, z)\, u_{\mathrm{r}}$, where $\breve{\alpha}_{\mathrm{r}}(\xi_{\mathrm{r}}, z) = \left(\breve{\alpha}_{\mathrm{r}1}(\xi_{\mathrm{r}}, z), \ldots, \breve{\alpha}_{\mathrm{r}m}(\xi_{\mathrm{r}}, z) \right)^T$ is such that $\breve{\alpha}_{\mathrm{r}i}(\xi_{\mathrm{r}}, z)$ cancels the last component of $(\varphi_{\mathrm{r}})^* \mathrm{f}_{\mathrm{r}}^i$, to bring Σ_{r} into

$$\breve{\Sigma}_r = (\varphi_{\mathrm{r}})^* \mathrm{f}_{\mathrm{r}}^i : \begin{cases} \dot{\xi}_{\mathrm{r}} = \mathrm{f}_{\mathrm{r}}(\xi_{\mathrm{r}}, z) + \mathrm{g}_{\mathrm{r}1}(\xi_{\mathrm{r}}, z) u_{\mathrm{r}1} \\ \quad + \cdots + \mathrm{g}_{\mathrm{r}m}(\xi_{\mathrm{r}}, z) u_{\mathrm{r}m} \\ 0 = \sigma(\xi_{\mathrm{r}}, z) \end{cases}$$

where $\breve{\mathrm{f}}_{\mathrm{r}} = (\varphi_{\mathrm{r}})^* \mathrm{f}_{\mathrm{r}} = A_i^{\mathrm{r}-1} \xi_{\mathrm{r}}^i + \breve{F}_{\mathrm{r}}(\xi_{\mathrm{r}}, z)$ and $\breve{g}_{\mathrm{r}i} = \partial_{\xi_{\mathrm{r}}^i} = B_i$.

Each subsystem is of the form

$$\breve{\Sigma}_{\mathrm{r}}^i = (\varphi_{\mathrm{r}})^* \mathrm{f}_{\mathrm{r}}^i : \begin{cases} \dot{\xi}_{\mathrm{r}}^i = \breve{\mathrm{f}}_{\mathrm{r}}^i(\xi_{\mathrm{r}}, z) + B_i \breve{u}_{\mathrm{r}i} \\ 0 = \sigma(\xi_{\mathrm{r}}, z) \end{cases}$$

with the rth component of $\breve{\mathrm{f}}_{\mathrm{r}}^i$ zero, i.e., $\breve{\mathrm{f}}_{\mathrm{r}}^i(\xi_{\mathrm{r}}, z) = 0$. To normalize the $(r-1)$th component of $\breve{\mathrm{f}}_{\mathrm{r}}^i$ we apply the push-forward change of coordinates $\mathrm{x}_{\mathrm{r}-1} = \varphi_{\mathrm{r}}(\xi_{\mathrm{r}}, z)$ given by

$$\mathrm{x}_{\mathrm{r}-1} = \breve{\varphi}_{\mathrm{r}}(\xi_{\mathrm{r}}, z) = \begin{cases} \mathrm{x}_{\mathrm{r}-1j}^i = \breve{\varphi}_{\mathrm{r}j}^i(\xi_{\mathrm{r}}, z) = \xi_{\mathrm{r}j}^i, \\ \qquad 1 \le j \le r-1 \\ \mathrm{x}_{\mathrm{r}-1r}^i = \breve{\varphi}_{\mathrm{r}r}^i(\xi_{\mathrm{r}}, z) \\ \qquad = \breve{\mathrm{f}}_{\mathrm{r}r-1}^i(\hat{\xi}_{rr}, z) \end{cases}$$

followed by a feedback $u_{\mathrm{r}-1} = (u_{\mathrm{r}-1}^1, \ldots, u_{\mathrm{r}-1}^m)^T$ with

$$u_{\mathrm{r}-1}^i = M_{\breve{\mathrm{f}}} \breve{\varphi}_{\mathrm{r}r}^i(\xi_{\mathrm{r}}, z) + \sum_{j=1}^m M_{B_j} \breve{\varphi}_{\mathrm{r}r}^i(\xi_{\mathrm{r}}, z) \breve{u}_{\mathrm{r}j}, \quad 1 \le i \le m.$$

The composition $\mathrm{x}_{\mathrm{r}-1} = \varphi_{\mathrm{r}}(\xi_{\mathrm{r}}, z) = \breve{\varphi}_{\mathrm{r}} \circ \varphi(\mathrm{x}_{\mathrm{r}}, z)$ and $u_{\mathrm{r}-1}$, in terms of u_{r}, form a transformation Γ_{r} such that $\Gamma_{\mathrm{r}}{}^* \Sigma_{\mathrm{r}} = \Sigma_{\mathrm{r}-1}$

$$\Sigma_{\mathrm{r}-1} : \begin{cases} \dot{\mathrm{x}}_{\mathrm{r}-1} = \mathrm{f}_{\mathrm{r}-1}(\mathrm{x}_{\mathrm{r}-1}, z) + \mathrm{g}_{\mathrm{r}-11}(\mathrm{x}_{\mathrm{r}-1}, z) u_{\mathrm{r}-11} \\ \quad + \cdots + \mathrm{g}_{\mathrm{r}-1m}(\mathrm{x}_{\mathrm{r}-1}, z) u_{\mathrm{r}-1m} \\ 0 = \sigma(\mathrm{x}_{\mathrm{r}-1}, z) \end{cases}$$

which is $(FB)_{\mathrm{k}-1}-$linear as it satisfies (7) with $\mathrm{k} = \mathrm{r}-1$.

Step k. Assume that Σ_{r} has been transformed, via explicit coordinates changes and feedback, into a $(FB)_{\mathrm{k}}-$linear system

$$\Sigma_{\mathrm{k}} : \begin{cases} \dot{\mathrm{x}}_{\mathrm{k}} = \mathrm{f}_{\mathrm{k}}(\mathrm{x}_{\mathrm{k}}, z) + \mathrm{g}_{\mathrm{k}1}(\mathrm{x}_{\mathrm{k}}, z) u_1 \\ \quad + \cdots + \mathrm{g}_{\mathrm{k}m}(\mathrm{x}_{\mathrm{k}}, z) u_m \\ 0 = \sigma(\mathrm{x}_{\mathrm{k}}, z) \end{cases}$$

with $\mathrm{f}_{\mathrm{k}}(\mathrm{x}_{\mathrm{k}}, z) = A^k \mathrm{x}_{\mathrm{k}} + F_{\mathrm{k}}(\hat{\mathrm{x}}_{\mathrm{k}k+1}, z)$ and $\mathrm{g}_{\mathrm{k}i}(\mathrm{x}_{\mathrm{k}}, z) = B_i$ for all $1 \le i \le m$. Since Σ_{r} (hence Σ_{k}) is feedback linearizable, then condition $(F'2)$ is satisfied, implying in particular,

$$\left[Mad_{\mathrm{f}_{\mathrm{k}}}^s(g_{\mathrm{k}i}), Mad_{\mathrm{f}_{\mathrm{k}}}^t(g_{\mathrm{k}j}) \right] = \sum_{l=0}^s \sum_{p=1}^m \Theta_p^l(\hat{\mathrm{x}}_{\mathrm{k}k+1}, z)\, Mad_{\mathrm{f}_{\mathrm{k}}}^l(g_{\mathrm{k}p})$$

$1 \le i,\ j \le m$ and $s \ge t \ge 0$, where $\Theta_p^l(\hat{\mathrm{x}}_{\mathrm{k}k+1}, z)$ are functions of the indicated variables. Since (7) of definition (3.1) holds, setting $s = r-k$ and $t = r-k-1$ implies

$$\frac{\partial^2 f_k(\hat{x}_{kk+1}, z)}{\partial x^i_{kk+1}\,\partial x^j_{kk+1}} = \sum_{p=1}^{m}\left[\Theta^l_p\left(\hat{x}_{kk+1}\frac{\partial f_k(\hat{x}_{kk+1}, z)}{\partial x^p_{kk+1}} + \sum_{l=0}^{s-1}\Theta^l_p A^l B_p\right)\right]$$

using (7) we see that all coefficients Θ^l_p are zero, i.e., the vector field $f_k(\hat{x}_{kk+1}, z)$ decomposes uniquely as

$$f_k(\hat{x}_{kk+1}, z) = A^k x_k + \breve{F}_k(\hat{x}_{kk}, z) + \sum_{i=1}^{m} x^i_{kk+1}\breve{g}_{ki}(\hat{x}_{kk}, z)$$

where $\hat{x}_{kk} = (x^1_{k1},\ldots,x^1_{kk}, x^2_{k1},\ldots,x^2_{kk},\ldots,x^m_{k1},\ldots,x^m_{kk})$. By Theorem 2.5 we construct a change of coordinates $\xi_k = \varphi_k(x_k, z)$ that rectifies the involutive distribution

$$\breve{D}_k = \mathrm{span}\left\{\begin{matrix}\breve{g}_{k1}(\hat{x}_{kk}, z), \breve{g}_{k2}(\hat{x}_{kk}, z),\ldots,\\ \breve{g}_{km}(\hat{x}_{kk}, z)\end{matrix}\right\}$$

Then we define a push-forward change of coordinates followed by an appropriate feedback transformation whose composition with $\xi_k = \varphi_k(x_k, z)$ yields a transformation Γ_k that maps Σ_k into Σ_{k-1}.

Example 3.4

Consider the index one multi-input nonlinear differential algebraic control systems

$$\Sigma_3 : \begin{cases} \dot{x}_3 = f_3(x_3, z) + g_{31}(x_3, z)u_1 \\ \quad + g_{32}(x_3, z)u_2 \\ 0 = \sigma(x_3, z) \end{cases}$$

Defined in the coordinates $x_3 = (x_{31},\ldots,x_{35})^T \in \Re^5$

$$\Sigma_3 : \begin{cases} \dot{x}_{31} = x_{32}(1+x_{33}) \\ \dot{x}_{32} = x_{33}(1+x_{31}) - x_{32}u_{31} \\ \dot{x}_{33} = x_{31} + x_{35} + z + (1+x_{33})u_{31} \\ \dot{x}_{34} = x_{35} + z \\ \dot{x}_{35} = u_{32} \\ 0 = x_{31}^2 - z \end{cases}$$

where

$$\left(\frac{\partial\sigma}{\partial z}\right)^{-1} = -1$$

$$\frac{\partial\sigma}{\partial x} = (2x_{31}\ \ 0\ \ 0\ \ 0\ \ 0)$$

$$\left(\frac{\partial\sigma}{\partial z}\right)^{-1}\frac{\partial\sigma}{\partial x} = (-2x_{31}\ \ 0\ \ 0\ \ 0\ \ 0)$$

$$f_3(x_3, z) = \begin{pmatrix} x_{32}(1+x_{33}) \\ x_{33}(1+x_{31}), x_{31}+x_{35}+z, x_{35}+z \\ 0 \end{pmatrix}$$

$g_{31}(x_3, z) = (0, -x_{32}, (1+x_{33}), 0, 0)^T$ and

$g_{32}(x_3, z) = (0, 0, 0, 0, 1)^T$

To rectifies the distribution $D_3 = \mathrm{span}\{g_{31}, g_{32}\}$ we look for a change of coordinates $y = \varphi_3(x_3, z)$. Apply Theorem (2.5) to $\nu = g_{31}(x_3, z)$ with $n = 5$ and $\sigma_3 = (1+x_{33})^{-1}$. Thus

$$\sigma_3\upsilon = \left(0, -x_{32}(1+x_{33})^{-1}, 1, 0, 0\right)^{-1}$$

Since $M^{s-1}_{\upsilon}\upsilon_1 = 0$ for all $s \geq 1$ we get

$$y_1 = x_{31} + \sum_{s=1}^{\infty}\frac{(-1)^s x_{33}^s}{s!}M^{s-1}_{\sigma_3\upsilon}(\sigma_3\upsilon_1)(x_3, z) = x_{31}$$

On the other side $\upsilon_2(x_3, z) = -x_{32}$ implies $M_{\sigma_3\upsilon}\sigma_3\upsilon_2 = 2x_{32}(1+x_{33})^{-2}$, $M^2_{\sigma_3\upsilon}\sigma_3\upsilon_2 = -6x_{32}(1+x_{33})^{-3}$ which gives $M^{s-1}_{\sigma_3\upsilon}\sigma_3\upsilon_2 = (-1)^s s!\, x_{32}(1+x_{33})^{-s}$. Thus

$$y_2 = x_{32} + \sum_{s=1}^{\infty} \frac{(-1)^s x_{33}^s}{s!} M_{\sigma_3 \upsilon}^{s-1}(\sigma_3 \upsilon_2)(x_3, z)$$
$$= x_{32}(1+x_{33})$$

Notice that $M_{\sigma_3 \upsilon}\sigma_3 = -(1+x_{33})^{-2}$, $M_{\sigma_3 \upsilon}^2 \sigma_3 = 2(1+x_{33})^{-3}$ which gives $M_{\sigma_3 \upsilon}^{s-1}\sigma_3 = (-1)^{s-1}(s-1)!(1+x_{33})^{-s}$. Thus

$$y_3 = \sum_{s=1}^{\infty} \frac{(-1)^{s+1} x_{33}^s}{s!} M_{\sigma_3 \upsilon}^{s-1}(\sigma_3)(x_3, z)$$
$$= \sum_{s=1}^{\infty} \frac{1}{s}\left(\frac{x_{33}}{1+x_{33}}\right)^s$$
$$= \ln(1+x_{33})$$

We apply the change of coordinates

$$y = \varphi_3(x_3, z) = \begin{cases} y_1 = x_{31} \\ y_2 = x_{32}(1+x_{33}) \\ y_3 = \ln(1+x_{33}) \\ y_4 = x_{34} \\ y_5 = x_{35} \\ 0 = x_{31}^2 - z \end{cases}$$

Whose inverse is given by

$$x_3 = \varphi_3^{-1}(y, z) = \begin{cases} x_{31} = y_1 \\ x_{32} = y_2 e^{-y_3} \\ x_{32} = e^{y_3} - 1 \\ x_{32} = y_4 \\ x_{32} = y_5 \\ 0 = y_1^2 - z \end{cases}$$

To transform the original system into

$$\breve{\Sigma}_3 = \begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = (1+y_1)e^{y_3}(e^{y_3}-1) \\ \quad + y_2 e^{-y_3}(y_1 + y_5 + z) \\ \dot{y}_3 = e^{-y_3}(y_1 + y_5 + z) + u_{31} \\ \dot{y}_4 = y_5 + z \\ \dot{y}_5 = u_{32} \\ 0 = \breve{\sigma}(y, z) = y_1^2 - z \end{cases}$$

$$\left(\frac{\partial \breve{\sigma}}{\partial z}\right)^{-1} = -1$$

$$\frac{\partial \breve{\sigma}}{\partial y} = (2y_1 \ \ 0 \ \ 0 \ \ 0 \ \ 0)$$

$$\left(\frac{\partial \breve{\sigma}}{\partial z}\right)^{-1} \frac{\partial \breve{\sigma}}{\partial y} = (-2y_1 \ \ 0 \ \ 0 \ \ 0 \ \ 0)$$

This is (FB)-form and can be put into

$$x_2 = \breve{\varphi}(y, z) = \begin{cases} x_{21} = \breve{\varphi}_1(y, z) = y_1 \\ x_{22} = \breve{\varphi}_2(y, z) = y_2 \\ x_{23} = \breve{\varphi}_3(y, z) \\ \quad = (1+y_1)e^{y_3}(e^{y_3}-1) \\ \quad\quad + y_2 e^{-y_3}(y_1 + y_5 + z) \\ x_{24} = \breve{\varphi}_4(y, z) = y_4 \\ x_{25} = \breve{\varphi}_5(y, z) = y_5 + z \\ 0 = y_1^2 - z \end{cases}$$

$$u_{21} = M_{f_3}\varphi_3(y,z) + M_{B_1}\varphi_3(y,z)u_{31} + M_{B_2}\varphi_3(y,z)u_{32}$$
$$= \left[e^{y_3}\left(e^{y_3}-1\right) + y_2 e^{-y_3} + 2y_1 y_2 e^{-y_3}\right] y_2$$
$$+ 2\left[(1+y_1)(e^{y_3}-1)(y_1 + y_5 + y_1^2)\right]$$
$$+ (1+y_1)e^{y_3}(y_1 + y_5 + y_1^2) + y_2 e^{-y_3} u_{32}$$
$$+ \left[(1+y_1)e^{y_3}\left((e^{y_3}-1) + e^{y_3}\right) + y_2 e^{-y_3}(y_1 + y_5 + y_1^2)\right] u_{31}$$

$$u_{22} = M_{f_3}\varphi_5(y,z) + M_{B_1}\varphi_5(y,z)u_{31}$$
$$+ M_{B_2}\varphi_5(y,z)u_{32} = 2y_1 y_2 + u_{32}$$

The composition $x_2 = \breve{\varphi} \circ \varphi_3(x_3, z)$ gives

$$\begin{cases} x_{21} = x_{31} \\ x_{22} = x_{32}(1+x_{33}) \\ x_{23} = (1+x_{31})x_{33}(1+x_{33}) + x_{32}(x_{31} + x_{35} + x_{31}^2) \\ x_{24} = x_{34} \\ x_{25} = x_{35} + x_{31}^2 \end{cases}$$

Brings Σ_3 into linear form

$$\Sigma_2 : \begin{cases} \dot{x}_{21} = x_{22} \\ \dot{x}_{22} = x_{32} \\ \dot{x}_{23} = u_{21} \\ \dot{x}_{24} = x_{25} \\ \dot{x}_{25} = u_{22} \end{cases}$$

References

1- N.H. NcClamroch, "On Control Described by a Class of Nonlinear Differential-Algebraic Equations: State Realizations and Local Control", Proceeding of the American Control Conference, P. 1701-1706, 1990.

2- N.H. NcClamroch, "Feedback Stabilization of Control Systems Described by a Class of nonlinear Differential-Algebraic Equations", Systems and Control Letter, Vol. 15, P. 53-60, 1990.

3- S. Kaprielian and K. Clements, "Feedback stabilization for an AC/DC power system model" , proceedings of the 29th Conference on Decision and control, P. 3367-3372, 1990.

4- S. Kaprielian, K. Clements and J. Turi, "Vector Input-Output Linearization for a Class of Descriptor Systems", P. 1949-1954, 1991.

5- Z. Jiandong and C. Zhaolin, " Exact Linearization for a Class of Nonlinear Differential-Algebraic Systems", Proceeding of the 4th World Congress on Intelligent Control and Automation, P. 211-214, 2002.

6- J. Wang and C. Chen, "Exact Linearization of Nonlinear Differential-Algebraic Systems", Proceedings 2001International Conference on Information Techology and Information Networks, Beijing, Vol. 4, P. 284-290, 2001.

7- Nada K. Mahdi, "State and Feedback Linearization of Single-Input Nonlinear Differential Algebraic Control Systems", *MSc Thesis*, College of Science-University of Basrah, 2012.

8- Ayad R. Khudair, Nada K. Mahdi, "Feedback Linearization of Single-Input Nonlinear Differential Algebraic Control Systems", International Mathematical Forum, Vol. 8, no.4, P. 167-179, 2013.

9- K. E. Brenan, S. L. Campbell and L. R. Petzold, "Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations", SIAM, 1996.

خطيه التغذية العكسية لأنظمة السيطرة الجبرية التفاضلية اللاخطية
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المستخلص

مسألة خطية التغذية العكسية لأنظمة السيطرة الجبرية التفاضلية اللاخطية ذات الدليل الواحد باستخدام تحويلات التغذية العكسية. بالرغم من أن الشروط الضرورية والكافية للمسألة درست في بداية سنة 2000. أنجز الحل الكامل للمسألة بتعريف خوارزمية مكنتنا من حساب (Linearizing Feedback Coordinates) بشكل واضح لأنظمة السيطرة الجبرية التفاضلية ذات الدليل الواحد بدون حل معادلات تفاضلية جزئية. الخوارزمية تحتوي على $n-1$ من الخطوات (n بعد النظام).