On a Fuzzy Completely Closed Ideal of a BH-algebra

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Abstract

In this paper, we introduce the notion of a fuzzy completely closed ideal of a BH-algebra and study this notion on a BG-algebra. Also We stated and prove some theorems which determine the relationships between this notion and some types of fuzzy ideals of a BH-algebra.

1. INTRODUCTION

Thenotion of BCK-algebra introduced by Y. Imai and K. Iseki in 1966[1]. In the same year, K. Iseki introduced the notions of a BCI-algebra as a generalization of a BCK-algebra and the notion of ideal of a BCK-algebra[2]. In1983, Q.P.Hu and X.Li introduced the notion of a BCH-algebra which was a generalization of BCK/BCI-algebras [3]. In 1991, C. S. Hoo introduced the notion of an ideal , closed ideal and filter in a BCI-algebra[4]. In 1998, Y. B. Jun, E. H. Rogh and H. S. Kim introduced the notion of BH-algebra, which is a generalization of BCH-algebras[5]. In 2002, J. Neggers. and H. S. Kim introduced the notion of B-Algebras[6]. In 2008, C. B. Kim and H. S. Kim introduced the notion of BG-Algebras which is a generalization of a B-algebra [7]. In 2011, H. H. Abass and H. M. A. Saeed introduced the notion of a closed ideal with respect to an element of a BH-algebra[8]. In 2012, H H. Abass and H. A. Dahham introduced the notions of a completely closed ideal of a BH-algebra[9].

On the other hand, we will mention the development of a fuzzy set, fuzzy subalgebra, fuzzy ideals, fuzzy closed ideals and some other types of fuzzy ideals.

In 1965, L. A. Zadeh introduced the notion of a Fuzzy subset of a set as a method for representing uncertainty in real physical world[10]. In 1991, O. G. Xi applied the concept of fuzzy sets to the BCK-algebras[11]. In 1999, Y. B. Jun introduced the notion of Fuzzy closed ideals in BCH-algebras[12]. In 2009, A. B. Saeid and M. A. Rezvani introduced the notion of Fuzzy BF-algebras[13]. In 2011, T. Senapati, M. Bhowmik and M.Pal introduced the notion of Fuzzy closed ideals of B-algebras[14]. In 2011, H. H. Abass and H. M. A.Saeed introduced the notion of Fuzzy closed ideals with respect to an element of BH-algebras[8].

In this paper, we introduce the notions as we mentioned in the abstract.

2. PRELIMINARIES

In this section, we give some basic concepts about a BG-algebra , a BH-algebra, ideal of a BH-algebra , closed ideal of a BH-algebra, a completely closed ideal of a BH-algebra , closed ideal with respect to an element of a BH-algebra, completely closed ideal with respect to an element of a BH-algebra, a normal set , fuzzy set, fuzzy ideal, fuzzy closed ideal, fuzzy closed ideal with respect to an element of a BH algebra with some theorems and propositions which we needed in our work.

Definition (2.1) [7] :

A BG-algebra is a non-empty set X with a constant 0 and a binary operation " * " satisfying the following axioms:

1) x * x = 0,
2) x * 0 = x,
3) (x * y) * (0 * y) = x, for all x, y ∈ X.

Definition (2.2) [5] :

A BH-algebra is a nonempty set X with a constant 0 and a binary operation * satisfying the following conditions:

1) x * x = 0, ∀ x∈X.
2) x * y = 0 and y * x = 0 imply x = y, ∀ x, y∈ X.
3) x *0 = x, ∀ x∈X.

Proposition (2.3) [7]:

Every BG-algebra is a BH-algebra.

Definition (2.4) [15] :

A nonempty subset S of a BH-algebra X is called a BH-Subalgebra or Subalgebra of X if x * y ∈ S for all x, y ∈ S.

Definition (2.5) [5] :

Let I be a nonempty subset of a BH-algebra X. Then I is called an ideal of X if it satisfies:

1)0∈I. 2)x*y∈I and y ∈I imply x∈I

Definition (2.6) [16] :

An ideal I of a BH-algebra X is called a closed ideal of X if :for every x∈I, we have 0*x∈I.

Definition (2.7)[9]:

An ideal I of a BH-algebras is called a completely closed ideal if x*y∈I,∀x,y∈I.

Definition(2.8)[10]:

Let X be a non-empty set. A fuzzy set A in X (a fuzzy subset of X) is a function from X into the closed interval [0,1] of the real number.

Definition (2.9)[16]:

Let A and B be two fuzzy sets in X, then:

(A∩B)(x)=min{A(x),B(x)}, for all x∈X.

(A∪B)(x)=max{A(x),B(x)}, for all x∈X.

A∩B and A∪B are fuzzy sets in X.

In general, if $\{A_\alpha,\ \alpha\in\Lambda\}$ is a family of fuzzy sets in X, the :

$(\bigcap_{i\in\Gamma} A_i)$(x)=inf{Ai (x),i∈Γ}, for all x∈X and

$(\bigcup_{i\in\Gamma} A_i)$ (x)=sup{ Ai (x),i∈Γ}, for allx∈X.

which are also fuzzy sets in X.

Definition (2.10) [17]:

Let A be a fuzzy set in X, for all α∈[0,1]. The set A_α={x∈X, A(x)≥α} is called a level subset of A.

Definition (2.11)[15]:

A fuzzy set A in a BH-algebra X is said to be a fuzzy subalgebra of X if it satisfies: A(x*y)≥min{A(x),A(y)}, ∀x,y∈X.

Remark(2.12)[18]:

A fuzzy subset A of a BH-algebra X is said to be a fuzzy ideal if and only if:

1) For any x∈X, A(0) ≥ A(x).
2) For any x, y∈X, A(x) ≥ min{A(x*y), A(y)}.

Definition (2.13) [8] :

A fuzzy ideal A of a BH-algebra X is said to be closed if

A(0*x) ≥ A(x) for any x ∈X.

Theorem (2.14)[8]:

A fuzzy set A of a BH-algebra X is called a fuzzy p-ideal of X if it satisfies:

1) A(0) ≥ A(x), For any x∈X.
2) A(x) ≥ min{A((x *z) *(y *z)), A(y)}, for all x, y, z ∈ X.

Definition (2.15)[8]:

A fuzzy set A of a BH-algebra X is called a fuzzy a-ideal of X if it satisfies

A (0) ≥ A (x), For any x∈X.

A(y * x) ≥ min{A((x * z) * (0 * y)), A(z)}, for all x, y, z ∈ X.

Definition (2.16)[19]:

A fuzzy set M in a B-algebra X is said to be fuzzy normal if it satisfies the inequality M((x*a)*(y*b)) ≥ min{M(x*y),M(a*b)}, for all a, b, x, y ∈X.

Lemma (2.17)[7]:

Let (X,*,0) be a BG-algebra. Then

1) The right cancellation law holds in X, i.e., x*y = z*y implies x=z,

2) $0*(0*x)=x$ for all $x\in X$,
3) If $x*y=0$, then $x=y$ for any $x,y\in X$,
4)If $0*x=0*y$, then $x=y$ for any $x,y\in X$,
5) $(x*(0*x))*x=x$ for all $x\in X$.

Proposition (2.18)[8]:

Let X be a BH-algebra. Then every fuzzy p-ideal of X is a fuzzy ideal of X.

Proposition (2.19)[9]:

A BH-algebra X is called an associative BH-algebra if:

$(x*y)*z=x*(y*z)$, for all $x,y,z\in X$.

Proposition (2.20)[9]:

Let X be an associative BH-algebra. Then the following properties are hold :

1) $0*x=x \quad \forall x\in X$.
2) $x*y=y*x \quad \forall x,y\in X$.

3. THE MAIN RESULTS

In this section, we define the notion of a fuzzy completely closed ideal of a BH-algebra. For our discussion , we shall link this notion with other notions which mentioned in preliminaries.

Definition (3.1):

Let X be a BH-algebra and A be a fuzzy ideal of X. Then A is called a fuzzy completely closed ideal ,if

$A(x*y) \geq \min\{A(x),A(y)\},\forall x,y\in X$.

Example(3.2):

Consider the BH-algebra X = {0, 1, 2, 3} with the following operation table.

*	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

The fuzzy ideal A which is defined by

$$A(x) = \begin{cases} 0.5 & x = 0,1 \\ 0.4 & x = 2,3 \end{cases}$$

is a fuzzy completely closed ideal

Theorem (3.3):

Let X be a BH-algebra. If $x*y=z$, $z\in\{0, x, y\}$, $\forall x,y\in X$, then every fuzzy ideal is a fuzzy completely closed ideal.

Proof:

Let A be a fuzzy ideal and $x, y\in X$.

If$x*y=0$,

$\Rightarrow A(x*y)=A(0) \geq \min\{A(x),A(y)\}$

If $x*y=x$,

$\Rightarrow A(x*y) =A(x) \geq \min\{A(x),A(y)\}$

If $x*y=y$,

$\Rightarrow A(x*y) =A(y) \geq \min\{A(x),A(y)\}$

$\therefore$ A is a fuzzy completely closed ideal.■

proposition (3.4):

Let X be an associative BH-algebra. Then every fuzzy ideal is a fuzzy closed ideal.

Proof:

Let A be a fuzzy ideal, and $x \in X$.

$A(0*x)=A(x) \geq A(x)$

[By proposition(2.20)(1)]

$\therefore$ A is a fuzzy closed ideal.■

Theorem(3.5):

Let X be a BH-algebra. If A is a fuzzy completely closed ideal. Then A_α is a completely closed ideal for all $\alpha\in[0,A(0)]$.

Proof:

To prove A_α is an ideal,

1)Since $\alpha\in[0,A(0)]$

$\Rightarrow A(0)\geq \alpha \forall\alpha\in[0,A(0)]$.

$\Rightarrow 0\in A_\alpha \forall\alpha\in[0,A(0)]$.

2) Let $x*y \in A_\alpha$,$y\in A_\alpha$

$\Rightarrow A(x*y) \geq\alpha, A(y) \geq\alpha$

$\Rightarrow \min\{A(x*y),A(y)\} \geq\alpha$

but $A(x) \geq \min\{A(x*y),A(y)\}$

$\Rightarrow A(x) \geq \alpha \Rightarrow x\in A_\alpha$

$\therefore A_\alpha$ is an ideal.

Now,let$x,y\in A\alpha$

$\Rightarrow A(x)\geq\alpha$, $A(y)\geq\alpha$

$\Rightarrow \min\{A(x),A(y)\}\geq\alpha \Rightarrow A(x*y)\geq\alpha$

$\Rightarrow x*y\in A_\alpha$

$\therefore A_\alpha$ is a completely closed ideal.■

Proposition (3.6):

Let X be an associative BH-algebra. If A is a fuzzy P-ideal, then A is a fuzzy closed ideal.

Proof:

Since A is a fuzzy a P-ideal,
$\therefore$ A is a fuzzy ideal
[By proposition(2.18)]
Now, let $x \in X$
$\Rightarrow A(0*x)=A(x) \geq A(x)$
[By Proposition(2.20)(1)]
$\therefore$ A is a fuzzy closed ideal.■

Theorem (3.7):
Let X be BH-algebra. If A is a fuzzy ideal, then the set $X_A=\{x \in X: A(x)=A(0)\}$ is an ideal.
Proof:
Let A be a fuzzy ideal.
1)Since $A(0)=A(0)$, $\Rightarrow 0 \in X_A$
2)Let $x*y$, $y \in X_A \Rightarrow A(x*y)=A(y)=A(0)$
$\Rightarrow \min\{A(x*y),A(y)\}=A(0)$
but $A(x) \geq \min\{A(x*y),A(y)\}=A(0)$
$\therefore$ $A(x) \geq A(0)$
but $A(0) \geq A(x)$ $\Rightarrow A(x)=A(0)$ $\Rightarrow x \in X_A$
$\therefore X_A$ is an ideal.■

Theorem (3.8):
Let X be BH-algebra and A be a fuzzy completely closed ideal. Then the set $X_A=\{x \in X: A(x)=A(0)\}$ is a completely closed ideal.
Proof:
Let A be a fuzzy completely closed ideal.
$\Rightarrow$A is a fuzzy ideal,
$\Rightarrow X_A$ is an ideal, [By theorem(3.7)]
Now, let $x, y \in X_A$
$\Rightarrow A(x)=A(y)=A(0)$
$\Rightarrow \min\{A(x),A(y)\}=A(0)$
but $A(x*y) \geq \min\{A(x),A(y)\}=A(0)$
$\Rightarrow A(x*y) \geq A(0)$
but $A(0) \geq A(x*y) \Rightarrow A(x*y)=A(0)$
$\therefore x*y \in X_A$
$\therefore X_A$ is a completely closed ideal.■

Theorem (3.9) :
Let X be BH-algebra and let A be a fuzzy set.Then A is a fuzzy ideal if and only if $A'(x)=A(x)+1-A(0)$ is a fuzzy ideal.
Proof:
Let A be a fuzzy ideal,
$A'(0)=A(0)+1-A(0)$,
$\Rightarrow A'(0)=1 \Rightarrow A'(0) \geq A'(x)$ $\forall x \in X$
2)$A'(x)=A(x)+1-A(0)$
$\geq \min\{A(x*y),A(y)\}+1-A(0)$
$\geq \min\{A(x*y)+1-A(0),A(y)+1-A(0)\}$
$\geq \min\{A'(x*y),A'(y)\}$
$\therefore$ $A'(x) \geq \min\{A'(x*y),A'(y)\}$
$\therefore$ A' is a fuzzy ideal.
Conversely
Let A' be a fuzzy ideal.
1)$A(0)=A'(0)-1+A(0)$,
$\Rightarrow A(0) \geq A'(x)-1+A(0)$
$\Rightarrow A(0) \geq A(x)$ $\forall x \in X$
2)$A(x)=A'(x)-1+A(0)$
$\geq \min\{A'(x*y),A'(y)\}-1+A(0)$
$\geq \min\{A'(x*y)-1+A(0),A'(y)-1+A(0)\} \geq \min\{A(x*y),A(y)\}$
$\therefore$ $A(x) \geq \min\{A(x*y),A(y)\}$
$\therefore$ A is a fuzzy ideal.■

Theorem (3.10) :
Let X be BH-algebra and A be a fuzzy set of X. Then A is a fuzzy completely closed ideal if and only if $A'(x)=A(x)+1-A(0)$ is a fuzzy completely closed ideal.
Proof:
Let A be a fuzzy completely closed ideal,
$\Rightarrow$ A is a fuzzy ideal. $\Rightarrow$ A' is a fuzzy ideal.
[By theorem(3.9)]
Now,
Let $x,y \in X$
$A'(x*y)=A(x*y)+1-A(0)$
$\geq \min\{A(x),A(y)\}+1-A(0)$
$\geq \min\{A(x)+1-A(0),A(y)+1-A(0)\}$
$\geq \min\{A'(x),A'(y)\}$
$\therefore$ $A'(x) \geq \min\{A'(x),A'(y)\}$
$\therefore$ A' is a fuzzy completely closed ideal
Conversely
Let A' be a fuzzy completely closed ideal,
$\Rightarrow$ A' is a fuzzy ideal. $\Rightarrow$ A is a fuzzy ideal.
[By theorem(3.9)]
Now,
Let $x,y \in X$
$A(x*y)=A'(x*y)-1+A(0)$
$\geq \min\{A'(x),A'(y)\}-1+A(0)$
$\geq \min\{A'(x)-1+A(0),A'(y)-1+A(0)\}$
$\geq \min\{A(x),A(y)\}$

∴ A(x) ≥ min{A(x),A(y)}

∴ A' is a fuzzy completely closed ideal.■

Proposition(3.11) :

Let X be a BH-algebra. Then every fuzzy normal set is a fuzzy subalgebra.

Proof :

Let M be a fuzzy normal set and x,y∈X,

M(x*y)=M((x*y)*(0*0))

≥ min {M(x*0),M(y*0)}

≥ min {M(x),M(y)}

∴M is a fuzzy subalgebra.■

Proposition(3.12) :

Let X be a BH-algebra. If M is a fuzzy normal set, then M(0) ≥M(x) ∀x∈X .

Proof :

Let M be a fuzzy normal set and x∈X.

M(0)=M((x*x)*(0*0))

≥min{M(x*0),M(x*0)}

≥min{M(x),M(x)} ≥ M(x)

∴ M(0) ≥ M(x).■

Proposition(3.13) :

Let X be an associative BH-algebra. Then every fuzzy normal set is a fuzzy ideal .

Proof :

Let M be a fuzzy normal set.

1)M(0)=M((x*x)*(0*0))

≥min{M(x*0),M(x*0)}

≥min{M(x),M(x)}

≥ M(x)

2) M(x)=M(x*0)

=M(x*(y*y))

=M((x*0)*(y*y))

≥ min{M(x*y),M(0*y)}

≥min{M(x*y),M(y)}

[By proposition(2.20)(1)]

∴M is a fuzzy ideal.■

Proposition (3.14):

Let X be an associative BH-algebra. Then every fuzzy normal set is a fuzzy completely closed ideal.

Proof :

Let M be a fuzzy normal set.

⇒ M is a fuzzy ideal

[By proposition(3.13)]

Now,

Let x,y∈X,

M(x*y)=M((x*y)*(0*0))

≥min {M(x*0),M(y*0)}

≥ min {M(x),M(y)}

∴ M is a fuzzy completely closed ideal.■

Proposition(3.15) :

Let X be an associative BH-algebra. Then every fuzzy a-ideal is a fuzzy ideal.

Proof:

Let A be a fuzzy a-ideal,

1)A(0) ≥ A(x),∀x∈X

2) Let x,z∈X

A(x)=A(x*0)

≥ min{A(x*z)*(0*0),A(z)}

Now, Let z=y

≥ min{A(x*y)*(0*0),A(y)}

≥ min{A(x*y),A(y)}

∴A is a fuzzy ideal.■

Theorem(3.16) :

Let X be an associative BH-algebra. Then every fuzzy a-ideal is a fuzzy completely closed ideal.

Proof:

Let A be a fuzzy a-ideal,

⇒ A is a fuzzy ideal, [By Proposition(3.15)]

Now,Let x,y∈X

A(x*y)=A(y*x) [By Proposition(2.20)(2)]

≥ min{A(x*z)*(0*y),A(z)}

Now, when y=z

≥min{A(x*y)*(0*y),A(y)}

≥ min{A(x), A(y)}

∴A is a fuzzy completely closed ideal.■

Proposition (3.17):

Let {Ai:i∈Γ}be a family of fuzzy completely closed ideals of a BH-algebra X. Then $(\bigcap_{i\in\Gamma} A_i)$ is a fuzzy completely closed ideal of X.

Proof:

To prove that $(\bigcap_{i\in\Gamma} A_i)$ is a fuzzy ideal,

Let x∈X.

$(\bigcap_{i\in\Gamma} A_i)$ (0) = inf{ Ai(0), i∈Γ}

≥inf{Ai(x), i∈Γ}

$=(\bigcap_{i\in\Gamma} A_i)$ (x)

$\Rightarrow(\bigcap_{i\in\Gamma} A_i)$ (0) $\geq(\bigcap_{i\in\Gamma} A_i)$ (x)

(2) Let x, y∈X

$(\bigcap_{i\in\Gamma} A_i)$ (x) = inf{Ai(x), i∈Γ}

≥inf{min{Ai(x*y),

Ai(y)}, i∈Γ }

=min{inf{Ai(x*y),i∈Γ},inf{Ai(y),i∈Γ}}

=min{$(\bigcap_{i\in\Gamma} A_i)$ (x*y),$(\bigcap_{i\in\Gamma} A_i)$(y) }

$\Rightarrow(\bigcap_{i\in\Gamma} A_i)$ (x) ≥min{$(\bigcap_{i\in\Gamma} A_i)$ (x*y), $(\bigcap_{i\in\Gamma} A_i)$ (y) }

Therefore,

$(\bigcap_{i\in\Gamma} A_i)$ is a fuzzy ideal of X.

To prove that $(\bigcap_{i\in\Gamma} A_i)$ is a fuzzy completely closed ideal of X

Let x,y∈X

$(\bigcap_{i\in\Gamma} A_i)$ ((x*y)) = inf{ Ai (x*y) , i∈Γ}

≥inf{min{Ai(x),

Ai(y)}, i∈Γ }

≥min{infAi(x),infAi(y)}, i∈Γ }

≥min{$(\bigcap_{i\in\Gamma} A_i)$ (x), $(\bigcap_{i\in\Gamma} A_i)$ (y) }

$\Rightarrow(\bigcap_{i\in\Gamma} A_i)$ (x*y)

≥min{$(\bigcap_{i\in\Gamma} A_i)$(x),$(\bigcap_{i\in\Gamma} A_i)$ (y) } ∀x,y∈X

Therefore, $(\bigcap_{i\in\Gamma} A_i)$ is a fuzzy completely closed ideal of X. ■

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حول المثالية الضبابية المغلقة تماما في جبر−BH

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<u>الخلاصة</u>

قدمنا في هذا البحث مفهوم المثالية الضبابية المغلقة تماما في جبر−BH,كما درسنا هذا المفهوم في جبر−BG.
كما وضعنا وأثبتنا بعض المبرهنات ذات العلاقة بين هذا المفهوم و بعض انواع المثاليات الضبابية في جبر BH و جبر BG.