Fractional differential transform method for solving fuzzy integro-differential equations of fractional order

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Abstract

In this paper, we present an approximate solution for fuzzy integro-differential equations of fractional order of the form:

$$D^{q}_{*x_0}\tilde{y}(x) = \tilde{f}(x) + p(x)\tilde{y}(x) + \int_0^x K(x,s)\tilde{y}(s)\,ds, \qquad (1)$$

$$\text{with initial condition} \quad \tilde{y}(x_0) = \tilde{y}_{x_0}$$

where $D^{q}_{*x_0}$ denotes a differential operator with fractional order q in the Caputo sense, $\tilde{f}(x)$ is assumed to be fuzzy function and $\tilde{y}_{x_0}$ is assumed to be fuzzy number,using the differential transform method. The solution of our model equations are calculated in the form of convergent series with easily computable components. Two examples are solved as illustrations using symbolic computation. The numerical results show that the followed approach is easy to implement and accurate when applied to fuzzy integro-differential equations of fractional order.

Keywords: Fractional calculus, Fuzzy integro-differential equations, fractional differential transform method.

1. Introduction:

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical models, which has been applied to a wide variety of real problems, for instance, the golden mean(1), practical systems(2, 3), quantum optics and gravity, medicine (4)and engineering problems.

The concept of fuzzy sets which was originally introduced by Zadeh(5) led to the definition of the fuzzy number and its implementation in fuzzy control (1) and approximate reasoning problems (5, 6, 7). The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka(8), Nahmias(9), and Ralescu(10) all of which observed the fuzzy number as a collection of α-levels, $0 < \alpha \leq 1$(11).

The fractional integro-differential equations is a special kind of integral equations collecting integral equations and fractional calculus and in recent years, there has been a growing interest in the integro-differential equations, since many mathematical formulation of physical phenomena, such as nonlinear functional analysis and their applications in the theory of engineering, mechanics, physics, chemical kinetics, astronomy, biology, economics, potential theory and electrostatistics contain integro-differential equations, (12,13, 14).

The differential transform method was first applied in the engineering domain in (Zhou, J. K.)(15). In general, the differential transform method is applied to the solution of electric circuit problems. The differential transform method is a numerical method based on the Taylor series expansion, which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation.

However, the differential transform method obtains a polynomial series solution by means of iterative procedure. Recently, the application of differential transform method is successfully extended to obtain analytical approximate solutions to linear and nonlinear integro-differential equations of fractional order[16].

In this paper, the approximate solution of fuzzy integro-differential equation of fractional order will be discussed, in which fractional integro-differential equation could be considered as an important type of integro-differential equations, where the differentiation that appears in the equation is of non-integer order.

2. Basic Concepts of Fuzzy Sets Theory:

In this section, we present some basic definitions of fuzzy sets including the definition of fuzzy numbers and fuzzy functions.

Definition (1)[5]:

Let X be any set of elements. A fuzzy set $\tilde{A}$ is characterized by a membership function

$\mu_{\tilde{A}} : X \longrightarrow [0, 1]$, and may be written as the set of points

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \leq \mu_{\tilde{A}}(x) \leq 1\}.$$

Definition (2)[17]:

The crisp set of elements that belong to the set $\tilde{A}$ at least to the degree α is called the weak α-level set (or weak α-cut), and is defined by:

$$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha\}$$

while the strong α-level set (or strong α-cut) is defined by:

$$A'_\alpha = \{x \in X : \mu_{\tilde{A}}(x) > \alpha\}$$

Definition (3)[5]:

A fuzzy subset $\tilde{A}$ of a universal space X is convex if and only if the sets A_α are convex,

$\forall \alpha \in [0, 1]$.

Or equivalently, we can define convex fuzzy set directly by using its membership function to satisfy:

$$\mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \geq \text{Min}\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$$

for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$.

Definition (4)[17]:

A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line R, such that:

1. There exists exactly one $x_0 \in R$, with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$).
2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Now, the following two remarks illustrates the representation of a fuzzy number and fuzzy functions in terms of its α-level sets, because they are more convenient to use in applications.

Remark (1)[18]:

A fuzzy number $\tilde{M}$ may be uniquely represented in terms of its α-level sets, as the following closed intervals of the real line:

$$M_\alpha = [m - \sqrt{1-\alpha}, m + \sqrt{1-\alpha}]$$

or

$$M_\alpha = [\alpha m, \frac{1}{\alpha} m]$$

where m is the mean value of $\tilde{M}$ and $\alpha \in (0, 1]$. This fuzzy number may be written as $M_\alpha = [\underline{M}, \overline{M}]$, where $\underline{M}$ refers to the greatest lower bound of $\tilde{M}$ and $\overline{M}$ to the least upper bound of $\tilde{M}$.

Remark (2)[18]:

Similar to the second approach given in remark (1), one can fuzzyfy any crisp or nonfuzzy function f, by letting:

$$\underline{f}(x) = \beta f(x),\ \overline{f}(x) = \frac{1}{\beta} f(x),\ x \in X,\ \beta \in (0, 1]$$

and hence the fuzzy function $\tilde{f}$ in terms of its β-levels is given by $f\beta = [\underline{f}, \overline{f}]$.

3. Riemann-Liouville and Caputo Fractional Order Derivatives

There are various types of definitions for the fractional order derivatives of order q > 0, the most commonly used definitions among various definitions of fractional order derivatives of order q > 0 are the Riemann-Liouville and Caputo formula.The difference between the two definitions in the order of evaluation. Riemann-Liouville fractional integration of order q is defined as:

$$J_{x_0}^{q} f(x) = \frac{1}{\Gamma(q)} \int_{x_0}^{x} (x-t)^{q-1} f(t)\,dt,$$

$q > 0,\ x > 0$

The following equations define Riemann-Liouville and Caputo fractional derivatives of order q, respectively:

$$D_{x_0}^{q} f(x) = \frac{d^m}{dx^m}\left[J_{x_0}^{m-q} f(x)\right]$$

$$D_{*x_0}^{q} f(x) = J_{x_0}^{m-q}\left[\frac{d^m}{dx^m} f(x)\right]$$

where $m-1 \le q < m$ and $m \in N$.

4. Analysis of the Differential Transform Method[19]

The differential transform of the k^{th} derivative of the function f, is defined by:

$$F(x) = \frac{1}{k!}\left(\frac{d^k f(x)}{dx^k}\right)\Bigg|_{x=x_0}$$

and the differential inverse transform of F(x), is defined as:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^k$$

From (3) and (4), we get:

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-x_0)^k}{k!}\left(\frac{d^k f(x)}{dx^k}\right)\Bigg|_{x=x_0}$$

which implies that the differential transform is derived from Taylor series expansion, but the method does not evaluate derivatives symbolically.

However, the corresponding derivatives are calculated recursively, and are defined by the transformed equation of the original functions. In practice, the function f is expressed by (4), so the differential transform method is a numerical method based on Taylor series expansion, which constructs a solution in terms of polynomials.

5. Fractional Differential Transform Method(FDTM): [20]

Let us expand the analytic function f as the fractional power series:

$$f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{k/\gamma}$$

where γ is the order of the fraction and F(k) is the fractional differential transform of f and in order to avoid the fractional initial and boundary conditions, we define the fractional derivative in the Caputo sense.

The relation between the Riemann-Liouville and Caputo operators is given by:

$$D_{*x_0}^{q} f(x) = D_{x_0}^{q}\left[f(x) - \sum_{k=0}^{\infty} \frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)\right]$$

Replacing f(x) by:

$$f(t)-\sum_{k=0}^{m-1}\frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)$$

in (2) and using (6) we obtain the fractional derivative in the Caputo sense, as:

$$D^q_{*x_0} f(x)=$$

$$\frac{1}{\Gamma(m-q)}\frac{d^m}{dx^m}\int_{x_0}^{x}\frac{f(t)-\sum_{k=0}^{m-1}\frac{1}{k!}(x-x_0)^k f^{(k)}(x_0)}{(x-t)^{1+q-m}}dt$$

Since the initial conditions are implemented by the integer-order derivative, the transformations of the initial conditions for $k = 0, 1, \ldots, (\gamma q-1)$, are defined by:

$$F(k)=\begin{cases}0, & \frac{k}{\gamma}\notin Z^+\\ \frac{1}{(k/\gamma)!}\left(\frac{d^{k/\gamma}}{dx^{k/\gamma}}f(x)\right)\Bigg|_{x=x_0}, & \frac{k}{\gamma}\in Z^+\end{cases}$$

where q is the order of the corresponding fractional equation .

Next, we shall give some theorems regarding the fractional differential transform method, for the details of proofs see[16, 21],

Theorem (1):

If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) \pm H(k)$, where F, G and H are the differential transforms of f, g and h, respectively.

Theorem (2):

If $f(x) = g(x)h(x)$, then $F(k) = \sum_{\ell=0}^{k} G(\ell)H(k-\ell)$, where F, G and H are the differential transforms of f, g and h, respectively.

Theorem (3):

If $f(x) = g_1(x)g_2(x)\ldots g_{n-1}(x)g_n(x)$, then:

$$F(k) = \sum_{k_{n-1}=0}^{k}\sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2} G_1(k_1)G_2(k_2-k_1)\cdots G_{n-1}(k_{n-1}-k_{n-2})G_n(k-k_{n-1})$$

where $G_1, G_2, \ldots, G_n$ are the differential transforms of $g_1, g_2, \ldots, g_n$; respectively.

Theorem (4):

If $f(x) = (x-x_0)^p$, then $F(k) =\delta(k-\gamma p)$, where:

$$\delta(k)=\begin{cases}1, & \text{if } k=0\\ 0, & \text{if } k\neq 0\end{cases} \qquad (7)$$

Theorem (5):

If $f(x) = D^q_{*x_0}[g(x)]$, then

$$F(x) = \frac{\Gamma\left(q+1+\frac{k}{\gamma}\right)}{\Gamma\left(1+\frac{k}{\gamma}\right)}G(k+\gamma q). \qquad (8)$$

Theorem (6):

If $f(x)=\int_{x_0}^{x} g(t)dt$, then $F(k)=\gamma\frac{G(k-\gamma)}{k}$, where $k\geq\gamma$

Theorem (7):

If $f(x) = g(x)\int_{x_0}^{x} h(t)dt$, then $F(k)=\gamma\sum_{k_1=\alpha}^{k}\frac{H(k_1-\gamma)}{k_1}G(k-k_1)$, where $k\geq\gamma$

Theorem (8):

If $f(x) = \int_{x_0}^{x} h_1(t)h_2(t)\ldots h_{n-1}(t)h_n(t)dt$,

then

F(k)=

$$\frac{\gamma}{k}\sum_{k_{n-1}=0}^{k-\gamma}\sum_{k_{n-2}=0}^{k_{n-1}}\cdots\sum_{k_2=0}^{k_3}\sum_{k_1=0}^{k_2}H_1(k_1)H_2(k_2-k_1)\ldots H_{n-1}(k_{n-1}-k_{n-2})H_n(k-k_{n-1}-\gamma)$$

, where $k\ge\gamma$.

6. Approximate Solution of Fuzzy integro-differential Equations of Fractional Orderusing FDTM:

Now, as an application of the fractional differential transform method for solving fuzzy integro-differential equations of fractional order, consider the fuzzy integro-differential equations of fractional order:

$$D^{q}_{*x_0}\tilde{y}(x)=\tilde{f}(x)+p(x)\tilde{y}(x)+\int_0^x K(x,s)\tilde{y}(s)\,ds,$$

$$\text{with initial condition}\quad \tilde{y}(x_0)=\tilde{y}_{x_0}$$

where $D^{q}_{*x_0}$ denotes a differential operator with fractional order q in the Caputo sense, $\tilde{f}(t)$ is assumed to be fuzzy function and may be represented as $\tilde{f}(t)=[\underline{f},\overline{f}]$,and therefore the solution of Eq.(9) which may be given by the form $\tilde{y}=[\underline{y},\ \overline{y}]$, where $\underline{y}$ and $\overline{y}$ refers to the lower and upper solutions of $\tilde{y}$, respectively.

Now, to find the lower solution $\underline{y}$, we must solve using the fractional differential transform method, the nonfuzzy fractional integro-differential equation :

$$D^{q}_{*x_0}\underline{y}(x)=\underline{f}(x)+p(x)\underline{y}(x)+\int_0^x k(x,s)\underline{y}(s)\,ds$$

$$\underline{y}(x_0)=\underline{y}_{x_0}$$

according to theorems 1-8 and Eq. (8),

$$\underline{Y}(k+\gamma q)=\frac{\Gamma\left(\frac{\gamma+k}{\gamma}\right)}{\Gamma\left(\frac{\gamma+k+1}{\gamma}\right)}\left[\underline{F}(k)+\sum_{k_1=0}^{k}P(k_1)\underline{Y}(k-k_1)+\frac{\gamma}{k}\sum_{k_1=0}^{k-\gamma}K(k_1)\underline{Y}(k-k_1-\gamma)\right]$$

(11)

where $\underline{F}$, P, $\underline{Y}$ and K are the differential transforms of $\underline{f}$, p , $\underline{y}$, and k respectively and the transformations of the initial conditionsaccording to Eq. (8) for k = 0, 1, …, (γq− 1), are defined by:

$$\underline{Y}(x)=\begin{cases}0, & \frac{k}{\gamma}\notin Z^{+}\\ \frac{1}{(k/\gamma)!}\left(\frac{d^{k/\gamma}}{dx^{k/\gamma}}\underline{y}(x)\right)\Big|_{x=x_0}, & \frac{k}{\gamma}\in Z^{+}\end{cases}$$

And therefore by using the series (5) the lower solution $\underline{y}(x)$ will be (9)

$$\underline{y}(x)=\sum_{k=0}^{\infty}\underline{Y}(k)(x-x_0)^{k/\gamma}$$

Similarly,to find the upper solution $\overline{y}$, we must solve usingthe fractional differential transform method, the nonfuzzy fractional integro-differential equation:

$$D^{q}_{*}\overline{y}(x)=\overline{f}(x)+p(x)\overline{y}(x)+\int_0^x k(x,s)\overline{y}(s)\,ds$$

$$\overline{y}(a)=\overline{y}_0$$

according to theorems 1-8 and eq. (8), .(10)

$$\bar{Y}(k+\gamma q)=\frac{\Gamma\left(\frac{\gamma+k}{\gamma}\right)}{\Gamma\left(\frac{\gamma+k+1}{\gamma}\right)}\Bigg[\bar{F}(k)$$

$$+\sum_{k_1=0}^{k}P(k_1)\bar{Y}(k-k_1)$$

$$+\frac{\gamma}{k}\sum_{k_1=0}^{k-\gamma}K(k_1)\bar{Y}(k-k_1-\gamma)\Bigg] \quad (15)$$

where $\bar{F}$, $P, \bar{Y}$ and K are the differential transforms of $\bar{f}$, p , $\bar{y}$ and k respectively and the transformations of the initial conditions for k = 0, 1, …, (γq– 1), are defined by:

$$\bar{Y}(x)=\begin{cases}0, & \frac{k}{\gamma}\notin Z^{+}\\ \frac{1}{(k/\gamma)!}\left(\frac{d^{k/\gamma}}{dx^{k/\gamma}}\bar{y}(x)\right)\Bigg|_{x=x_0}, & \frac{k}{\gamma}\in Z^{+}\end{cases}$$

And therefore by using the series (5) the upper solution $\bar{y}(x)$ will be

$$\bar{y}(x)=\sum_{k=0}^{\infty}\bar{Y}(k)(x-x_0)^{k/\gamma}$$

7. Illustrative Examples:

In this section, we present two fuzzy integro-differential equations of fractional order, linear and nonlinear, and we use the approach given in the section six in order to find the approximate solution.

Example (1):

Consider the following linear fuzzy integro-differential equations of fractional order:

$$D_*^{0.5}\tilde{y}(x)=\tilde{y}(x)+\tilde{f}(x)+\int_0^x\tilde{y}(s)\,ds$$

, $x\geq 0$,

with initial condition $\tilde{y}(0)=\tilde{0}$, and $\tilde{f}$ will be given as $\tilde{f}=[\underline{f},\bar{f}]$, where

$$\underline{f}(x)=\beta[\frac{8}{3\sqrt{\pi}}x^{1.5}-x^2-\frac{1}{3}x^3] \quad ,$$

$$\bar{f}(x)=\frac{1}{\beta}[\frac{8}{3\sqrt{\pi}}x^{1.5}-x^2-\frac{1}{3}x^3] \quad , \quad 0<\beta<1$$

The solution then will be of the form [$\underline{y}$, $\bar{y}$], where $\underline{y}(x)$ represent the solution of the equation:

$$D_*^{0.5}\underline{y}(x)=\underline{y}(x)+\underline{f}(x)+\int_0^x\underline{y}(s)\,ds \quad (16)$$

with initial condition

$\underline{y}(0)=-\sqrt{1-\alpha}$, $0<\alpha\leq 1$

and $\bar{y}(x)$ represent the solution of the equation: (17)

$$D_*^{0.5}\bar{y}(x)=\bar{y}(x)+\bar{f}(x)+\int_0^x\bar{y}(s)\,ds$$

with initial condition $\bar{y}(0)=\sqrt{1-\alpha}$, $0<\alpha\leq 1$

Using equations (11) and (12) and by choosing γ =2, thus we have

$$\underline{Y}(k+1)=\frac{\Gamma\left(\frac{2+k}{2}\right)}{\Gamma\left(\frac{3+k}{2}\right)}\Bigg[\underline{Y}(k)+\frac{8\beta}{3\sqrt{\pi}}\delta(k-4)$$

$$-\beta\delta(k-3)-\frac{\beta}{3}\delta(k-6)+\frac{2\underline{Y}(k-2)}{k}\Bigg]$$

with $\underline{Y}(0)=-\sqrt{1-\alpha}$, $0<\alpha\leq 1$

similarlyto find $\bar{y}(x)$ we shall use equations (15) and (16), hence we get:

$$\bar{Y}(k+1)=\frac{\Gamma\left(\frac{2+k}{2}\right)}{\Gamma\left(\frac{3+k}{2}\right)}\left[\bar{Y}(k)+\frac{8}{3\beta\sqrt{\pi}}\delta(k-3)-\frac{1}{\beta}\delta(k-3)-\frac{1}{3\beta}\delta(k-6)+\frac{2\bar{Y}(k-2)}{k}\right]$$

with $\bar{Y}(0)=\sqrt{1-\alpha}$, $0<\alpha\le 1$

Following Figures (1) – (4) represent the approximate solution of example one using different values of α and β by using inverse differential transform up to certain terms.

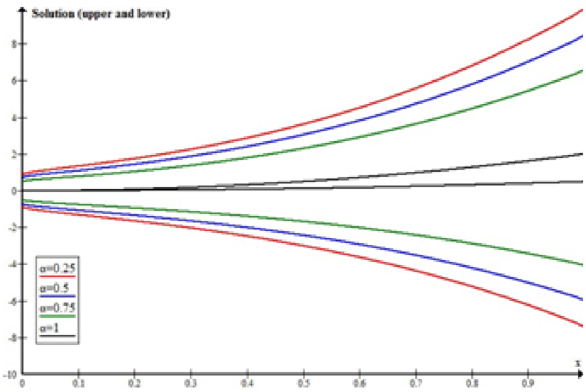

Fig.(1) Upper and lower solutions of example (1) for β=0.25 and different values of α.

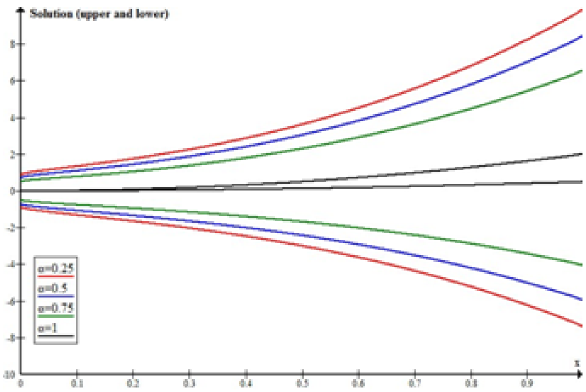

Fig.(2) Upper and lower solutions of example (1) for β=0.5 and different values of α.

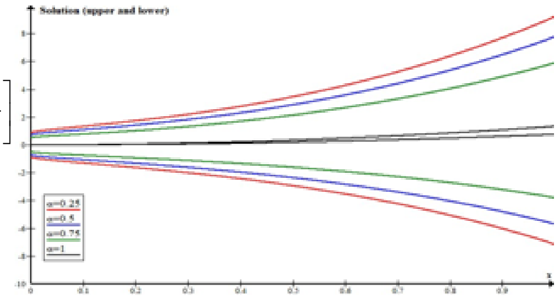

Fig.(3) Upper and lower solutions of example (1) for β=0.75 and different values

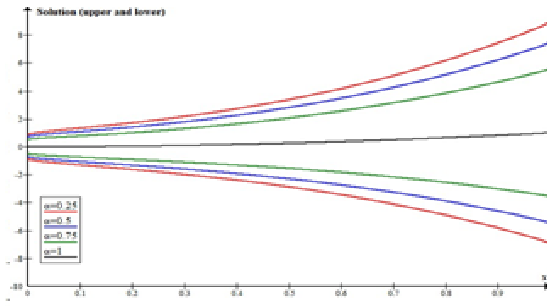

Fig.(4) Upper and lower solutions of example (1) for β=1 and different values of

Example (2):

Consider the following nonlinear fuzzy integro-differential equations of fractional order:

$$D_{*x_0}^{0.75}\tilde{y}(x)=\tilde{f}(x)-\tilde{y}(x)+\int_0^x[\tilde{y}(s)]^2\,ds$$

, x>0 ,

with initial condition $\tilde{y}(0)=\tilde{0}$, and $\tilde{f}$ will be given as $\tilde{f}=[\underline{f},\bar{f}]$, where

$$\underline{f}(x)=\beta[\frac{1}{\Gamma(1.25)}x^{0.25}+x-\frac{1}{3}x^3] \ ,$$

$$\bar{f}(x)=\frac{1}{\beta}[\frac{1}{\Gamma(1.25)}x^{0.25}+x-\frac{1}{3}x^3] \ , \quad 0<\beta\le 1$$

The solution then will be of the form $[\underline{y},\ \bar{y}]$, where $\underline{y}(x)$ represent the solution of the equation:

$$D_*^{0.75}\underline{y}(x)=\underline{f}(x)-\underline{y}(x)+\int_0^x[\underline{y}(s)]^2\,ds \quad ,$$

with initial condition $\underline{y}(0) = -\sqrt{1-\alpha}$, $0 < \alpha \le 1$.

and $\bar{y}(x)$ represent the solution of the equation:

$$D_*^{0.75}\bar{y}(x) = \bar{f}(x) - \bar{y}(x) + \int_0^x [\bar{y}(s)]^2\, ds$$ 3)

with initial condition $\underline{y}(0) = \sqrt{1-\alpha}$, $0 < \alpha \le 1$

Using equations (11) and (12) and by choosingγ=4, thus we have

$$\underline{Y}(k+3) = \frac{\Gamma\left(\frac{4+k}{4}\right)}{\Gamma\left(\frac{7+k}{4}\right)}\left[\frac{\beta}{\Gamma(1.25)}\delta(k-1) - \beta\delta(k-4) - \frac{\beta}{3}\delta(k-12)\right.$$
$$\left. -\underline{Y}(k) + \frac{4}{k}\sum_{k_1=0}^{k-4} \underline{Y}(k_1)\underline{Y}(k-k_1-4)\right]$$

(24)

with $\underline{Y}(0) = -\sqrt{1-\alpha}$, $\underline{Y}(1) = 0$, $\underline{Y}(2) = 0$.

Similarlyto find $\bar{y}(x)$ we use equations (15) and (16), hence we get

$$\bar{Y}(k+3) = \frac{\Gamma\left(\frac{4+k}{4}\right)}{\Gamma\left(\frac{7+k}{4}\right)}\left[\frac{1}{\beta\Gamma(1.25)}\delta(k-1) + \frac{1}{\beta}\delta(k-4) - \frac{1}{3\beta}\delta(k-12)\right.$$
$$\left. -\bar{Y}(k) + \frac{4}{k}\sum_{k_1=0}^{k-4} \bar{Y}(k)\bar{Y}(k-k_1-4)\right]$$

(25)

with $\bar{Y}(0) = \sqrt{1-\alpha}$, $\bar{Y}(1) = 0$, $\bar{Y}(2) = 0$.

Following Figures (5) – (8) represent the approximate solution of example one using different values of α and β by using inverse differential transform up to certain terms.

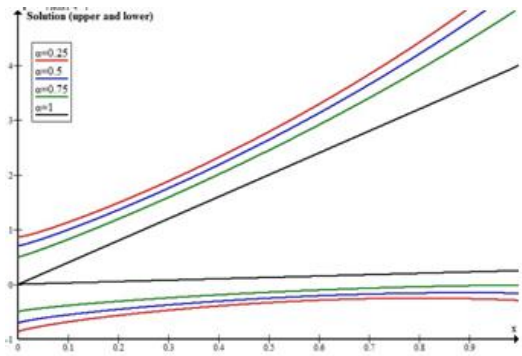

Fig.(5) Upper and lower solutions of example (2) for β=0.25 and different values of α.

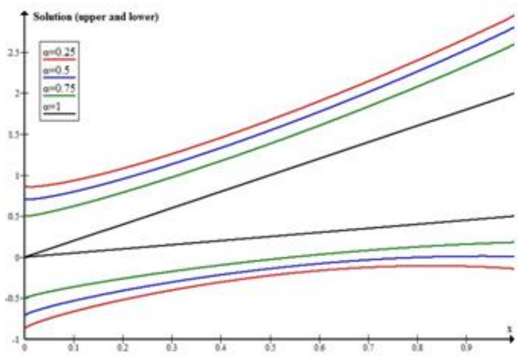

Fig.(6) Upper and lower solutions of example (2) for β=0.5 and different values of α.

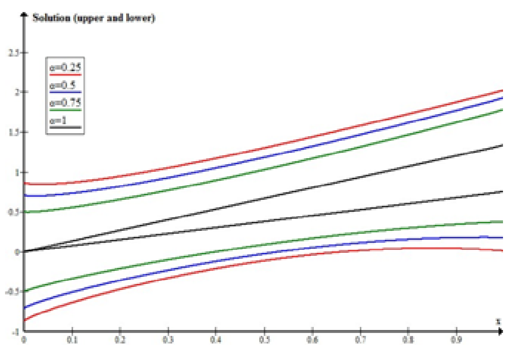

Fig.(7) Upper and lower solutions of example (2) for β=0.75 and different values of α.

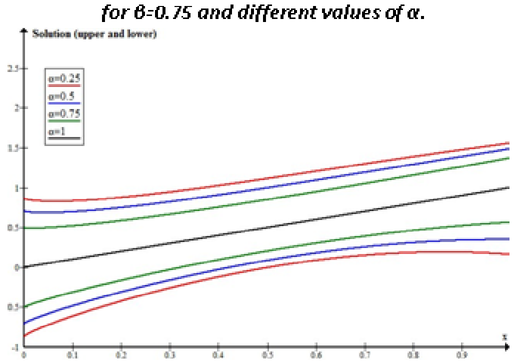

Fig.(8) Upper and lower solutions of example (2) for β=1 and different values of α.

8. Conclusions:

1. The differential transform method proved its validity and accurate results in solving fuzzy fractional integro-differential equations.
2. crisp solution, i.e., the solution of nonfuzzy fractional integro-differential equations, may be considered as a special case of the solution of the fuzzy fractional differential equations with α=1and β = 1.
3. The validity of the results may be achieved from the equality of the upper and lower solutions at α=1 and β= 1.

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