Simultaneous Approximation By Generalization of Szãsz Type Operators

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Abstract

In the present paper, we introduce a generalization of the Summation-Integral Szãsz type operators denoted by $\tilde{S}_{n,p}(f(t);x)$. First, we prove the convergence theorem for the operator. Then, we find a recurrence relation of the m-th order moment for the operator $\tilde{S}_{n,p}(f(t);x)$. Finally, we give a Voronovskaja-type asymptotic formula and an error estimate in terms of modulus of continuity of the function being approximate.

Keywords and phrases: Linear positive operator, Simultaneous approximation, Voronovskaja-type asymptotic formula, Degree of approximation, Modulus of continuity.

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1. Introduction:

The Szász-Mirakyan operators are defined by

$$L_n(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x) f\left(\frac{k}{n}\right),$$

$x \in R_0 := [0,\infty), n \in N := \{1,2,\dots\}$

where

$$q_{n,k}(x) = \frac{(nx)^k}{e^{nx} k!}, k \in N^0 := N \cup \{0\}$$

The operators $L_n(f;x)$ knew by the classical Szãsz operators (Szãsz). After that, Kasana et. al. (H. S. Kasana et al), proposed a modification of the classical Szãsz operators to approximate a space of functions integrable on R_0 as:

$$R_n(f(t);x) = n \sum_{k=0}^{\infty} q_{n,k}(x) \int_0^{\infty} q_{n,k}(t) f(t)\, dt.$$

Rempulska and Graczyk in the paper [3] introduced a modification of the Szãsz operators and studied some direct results in ordinary approximation as:

$$M_{n,r}(f,x) = \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} f\left(\frac{rk}{n}\right), \quad x \in R_0, n \in N,$$

where $A_r(t) = \sum_{k=o}^{\infty} \frac{t^{rk}}{(rk)!}$ for $t \in R_0$.

In this paper, we introduced a new sequence of linear positive operators $\tilde{S}_{n,p}(f(t);x)$ given as follows:

$$\tilde{S}_{n,p}(f(t);x) = \frac{1}{G_x}\left(n\sum_{k=1}^{\infty} q_{n,pk}(x)\int_0^{\infty} q_{n,pk-1}(t)\, f(t)\, dt + f(0)\, q_{n,0}(x)\right) \quad (1)$$

2. Preliminaries :

We give some lemmas which help us in the proofs of main theorems.

Lemma 1: [2]

For $n, r \in N$, $x \in R_0$ and

$$A_r^{(m)}(nx) = n^m \sum_{k=m}^{\infty} \frac{(nx)^{rk-m}}{(rk-m)!} \quad ; rk \geq m,$$

where $A_r(nx) = \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!}$, we get :

(1) $\sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} rk = xA_r'(nx)$;

(2) $\sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} (rk)^2 = x^2 A_r''(nx) + xA_r'(nx)$;

(3) $\sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} (rk)^3 = x^3 A_r'''(nx) + 3x^2 A_r''(nx) + x\, A_r'(nx)$;

(4) $\sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} (rk)^m = x^m A_r^{(m)}(nx) + \frac{m(m-1)}{2} x^{m-1} A_r^{(m-1)}(nx) +$ terms of form $Cx^l A_r^{(l)}(nx)$, where $0 < l < m-1$.

Lemma 2: [3]

For $n, r \in N$ and $x \in (0, \infty)$, we get:

$$\frac{A_r^{(m)}(nx)}{n^m A_r(nx)} \to 1\, , \frac{A_r^{(m)}(nx)}{n^s A_r(nx)} = o(1)$$

as $n \to \infty, s = r + m, \quad r \geq 1$.

And $\frac{A_r^{(m)}(nx)}{n^s A_r(nx)} = o(n^{-r})$.

Lemma 3: [1]

There exist polynomials $Q_{i,j,r}(x)$ independent of n and k for sufficiently large n, such that:

$$x^r q_{n,k}^{(r)}(x) = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\, (k - nx)^j Q_{i,j,r}(x)\, q_{n,k}(x).$$

3.The Convergence Theorem of $\tilde{S}_{n,p}(f(t);x)$:

We show that the operator $\tilde{S}_{n,p}(f;x)$ converges to $f(x)$ in $C_h[0,\infty)$ by applying the conditions of Korovkin's theorem. Where $C_h[0,\infty)$ the space of all real valued continuous functions on interval $[0,\infty)$ such that $|f(t)| \leq Ce^{ht}$ for some constant $C > 0$ and $h > 0, t \geq 0$.

Theorem 1:

For $n, p \in N$ and $x \in R_0$, the following conditions are hold:

(1) $\tilde{S}_{n,p}(1;x) = 1;$

(2) $\tilde{S}_{n,p}(t;x) = \dfrac{xA'_p(nx)}{nA_p(nx)} \to x \ \ as \ \ n \to \infty\,;$

(3) $\tilde{S}_{n,p}(t^2;x) = \dfrac{x^2A''_p(nx)}{n^2A_p(nx)} + \dfrac{2xA'_p(nx)}{n^2A_p(nx)} \to x^2 \ \ as \ \ n \to \infty.$

Therefore,

$\tilde{S}_{n,p}(f(t);x) \ \to f(x) \ as \ \ n \to \infty$ for all $f \in C_h[0,\infty)$.

Proof:

Using **Lemma's 1** and **2** the consequence (1) is easily verified, hence the details are omitted.

The proof of (2) is given below:

$$\tilde{S}_{n,p}(t;x)$$

$$= \frac{1}{G_x}\left(n\sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t)\, t\, dt + 0\,.\right)$$

$$= \frac{1}{n\,G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\ (pk)$$

$$= \frac{xA'_p(nx)}{nA_p(nx)} \quad \text{(in view of } \textbf{Lemma 2} \text{)}$$

$\to x \ \ as \ \ n \to \infty\,.$ (in view of **Lemma 2**)

Using the same technique and **Lemma's 1** and **2** the value of $\tilde{S}_{n,p}(t^2;x)$ is followed immediately as:

$$\tilde{S}_{n,p}(t^2;x) = \frac{x^2A''_p(nx)}{n^2A_p(nx)} + \frac{2xA'_p(nx)}{n^2A_p(nx)} \to x^2 \ \ as \ \ n \to \infty.$$

Therefore, by Korovkin's, we get:

$\tilde{S}_{n,p}(f(t);x) \to f(x) \ \ as \ \ n \to \infty.$ ■

4. The *m*-th Order Moment for $\tilde{S}_{n,p}(f(t);x)$:

In this part, we give the definition of the *m*-th order moment for the operators $\tilde{S}_{n,p}(f;x)$ and prove a recurrence relation for this moment.

Definition 1:

The m-th order moment $(m \in N^0)$ for the operators $\tilde{S}_{n,p}(f(t);x)$ is defined by:

$$\tilde{T}_{n,m}(x) = \tilde{S}_{n,p}((t-x)^m;x)$$

$$= \frac{1}{G_x}\left(n\sum_{k=1}^{\infty} q_{n,pk}(x)\int_0^{\infty} q_{n,pk-1}(t)(t - x)^m\, dt + (-x)^m e^{-nx}\right)$$

$$= U_{n,m}(x) + \frac{(-x)^m e^{-nx}}{G_x}; \qquad (2)$$

Lemma 4:

For the function $U_{n,m}(x)$ which is defined above, we have:

$$U_{n,0}(x) = 1 - e^{-nx}, U_{n,1}(x) = x\left(\frac{G_x'}{nG_x} + e^{-nx}\right),$$

$$U_{n,2}(x) = \frac{x^2G_x'' + 2xG_x'}{n^2G_x} - \frac{2x}{n} - x^2e^{-nx},$$

and

$$nU_{n,m+1}(x) = x\left\{U_{n,m}'(x) + 2mU_{n,m-1}(x) + \frac{G_x'}{G_x}U_{n,m}(x)\right\} + mU_{n,m}(x), \qquad m \geq 1,$$

where $G_x = e^{-nx}A_p(nx)$ and $G_x' = e^{-nx}\left(A_p'(nx) - nA_p(nx)\right)$.

Proof:

By direct computation, we easily have the values of $U_{n,0}(x)$, $U_{n,1}(x)$ and $U_{n,2}(x)$.

Now,

$$U_{n,m}'(x) = \frac{n}{G_x^2}\left\{G_x\left(\sum_{k=1}^{\infty} q_{n,pk}'(x)\int_0^{\infty} q_{n,pk-1}(t)(t - x)^m\,dt - m\sum_{k=1}^{\infty} q_{n,pk}(x)\int_0^{\infty} q_{n,pk-1}(t)(t - x)^{m-1}\,dt\right) - G_x'\sum_{k=1}^{\infty} q_{n,pk}(x)\int_0^{\infty} q_{n,pk-1}(t)(t - x)^m\,dt\right\}.$$

Then

$$U_{n,m}'(x) + mU_{n,m-1}(x) + \frac{G_x'}{G_x}U_{n,m}(x) = \frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}'(x)\int_0^{\infty} q_{n,pk-1}(t)(t - x)^m\,dt.$$

Since $xq_{n,pk}'(x) = (pk - nt + n(t - x))q_{n,pk}(x)$ we get:

$$x\left\{U_{n,m}'(x) + mU_{n,m-1}(x) + \frac{G_x'}{G_x}U_{n,m}(x)\right\} = \frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\int_0^{\infty} q_{n,pk-1}(t)(pk - nt)(t - x)^m\,dt + nU_{n,m+1}(x).$$

$$= \frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\int_0^{\infty} q_{n,pk-1}(t)\big((pk-1)$$

$$- nt\big)(t-x)^m\,dt$$

$$+ \frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\int_0^{\infty} q_{n,pk-1}(t)(t$$

$$- x)^m\,dt + nU_{n,m+1}(x)$$

we obtain:

$$x\left\{U'_{n,m}(x) + mU_{n,m-1}(x) + \frac{G'_x}{G_x}U_{n,m}(x)\right\}$$

$$- nU_{n,m+1}(x) - U_{n,m}(x)$$

$$= \frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\int_0^{\infty} tq'_{n.pk-1}(t)(t$$

$$- x)^m\,dt.$$

Using the identity $t = (t-x) + x$ and integration by parts, we have:

$$nU_{n,m+1}(x) = x\left\{U'_{n,m}(x)\right.$$

$$+ 2mU_{n,m-1}(x)$$

$$\left. + \frac{G'_x}{G_x}U_{n,m}(x)\right\}$$

$$+ mU_{n,m}(x)\,; \qquad m \geq 1. \blacksquare$$

Lemma 5:

For the function $\tilde{T}_{n,m}(x)$ which is defined in (2), we have

$$\tilde{T}_{n,0}(x) = 1, \qquad \tilde{T}_{n,1}(x) = \frac{xG'_x}{nG_x} \quad \text{and} \quad \tilde{T}_{n,2}(x) = \frac{x^2G''_x}{n^2G_x} + \frac{2xG'_x}{n^2G_x} + \frac{2x}{n}.$$

The recurrence relation is:

$$n\tilde{T}_{n,m+1}(x) = x\left\{\tilde{T}'_{n,m}(x) + 2m\tilde{T}_{n,m-1}(x) + \frac{G'_x}{G_x}\tilde{T}_{n,m}(\,x)\right\} + m\,\tilde{T}_{n,m}(x\,), m \geq 1. \quad (3)$$

where $G_x = e^{-nx}A_p(nx)$ and $G'_x = e^{-nx}\big(A'_p(nx) - nA_p(nx)\big)$.

Further, we can determine for sufficiently large n the following:

(i) $\tilde{T}_{n,m}(x)$ as a polynomial in x of degree m.

(ii) for every $x \in R_0$, $\tilde{T}_{n,m}(x) = O\left(n^{-\left[\frac{(m+1)}{2}\right]}\right)$.

Proof:

It is easy to show that

$$\tilde{T}_{n,0}(x) = 1, \qquad \tilde{T}_{n,1}(x) = \frac{xG'_x}{nG_x} \quad \text{and} \quad \tilde{T}_{n,2}(x) = \frac{x^2G''_x}{n^2G_x} + \frac{2xG'_x}{n^2G_x} + \frac{2x}{n}.$$

From (2) and **Lemma 4**, we have:

$$n\tilde{T}_{n,m+1}(x) = n\left\{U_{n,m+1}(x) + \frac{(-x)^{m+1}e^{-nx}}{G_x}\right\}$$

$$= x\left\{U'_{n,m}(x) + 2mU_{n,m-1}(x) + \frac{G'_x}{G_x}U_{n,m}(x)\right\} + mU_{n,m}(x) + \frac{n(-x)^{m+1}e^{-nx}}{G_x}$$

$$= x\left\{\tilde{T}'_{n,m}(x) + \frac{n(-x)^m e^{-nx} + m(-x)^{m-1}e^{-nx}}{G_x} + \frac{G'_x(-x)^m e^{-nx}}{G_x^2}\right\} + 2mx\left\{\tilde{T}_{n,m-1}(x) - \frac{(-x)^{m-1}e^{-nx}}{G_x}\right\} + x\frac{G'_x}{G_x}\left\{\tilde{T}_{n,m}(x) - \frac{(-x)^m e^{-nx}}{G_x}\right\} + m\left\{\tilde{T}_{n,m}(x) - \frac{(-x)^m e^{-nx}}{G_x}\right\} + \frac{n(-x)^{m+1}e^{-nx}}{G_x},$$

then

$$n\tilde{T}_{n,m+1}(x) = x\left\{\tilde{T}'_{n,m}(x) + 2m\tilde{T}_{n,m-1}(x) + \frac{G'_x}{G_x}\tilde{T}_{n,m}(x)\right\} + m\,\tilde{T}_{n,m}(x), \qquad m \geq 1.$$

From the values of $\tilde{T}_{n,0}(x)$ and $\tilde{T}_{n,1}(x)$, it is clear that the consequences (i) and (ii) are hold for $m = 0$ and $m = 1$. The consequence (i) can be proved easily by using (3) and the induction on m.
Using the same technique in ***Lemma 1.5*** (ii) [2], the proof of consequence (ii) is holds. ■

Lemma 6:

For $m \geq 1$,we have:

$$\tilde{S}_{n,p}(t^m;x)$$

$$= \frac{1}{n^m A_p(nx)}\Big(x^m A_p^{(m)}(nx)$$

$$+ m(m-1)x^{m-1}A_p^{(m-1)}(nx)$$

$$+ \text{terms of form } C\, x^l A_p^{(l)}(nx),\ \ l$$

$$< m-1\Big)$$

Proof:

By direct computation , **Lemma 1** and using (1.1) [2] , we get:

$$\tilde{S}_{n,p}(t^m;x)$$

$$= \frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t)\, t^m\, dt$$

$$+ 0$$

$$= \frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x) \frac{\big((pk-1)+m\big)!}{(pk-1)!\ n^{m+1}}$$

$$= \frac{1}{n^m G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\Big((pk-1)$$

$$+ m\big)\big((pk-1)+m$$

$$- 1\big)\dots pk\ \Big)$$

$$= \frac{1}{n^m G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\ \Bigg(\ (pk)^m$$

$$+ \frac{m(m-1)}{2}(pk)^{m-1}$$

$$+\ \text{T.L.P.}(pk)\Bigg)$$

Where T.L.P. the terms in lower power of x .

$$= \frac{1}{n^m\, A_p(nx)}\Bigg(x^m A_p^{(m)}(nx)$$

$$+ \frac{m(m-1)}{2}x^{m-1}A_p^{(m-1)}(nx)$$

$$+ \frac{m(m-1)}{2}x^{m-1}A_p^{(m-1)}(nx)$$

$$+ \text{terms of form } C\, x^l A_p^{l}(nx),\ \ l$$

$$< m-1\Bigg)$$

$$= \frac{1}{n^m A_p(nx)}\Big(x^m A_p^{(m)}(nx)$$

$$+ m(m-1)x^{m-1}A_p^{(m-1)}(nx)$$

$$+ \text{terms of form } C\, x^l A_p^{(l)}(nx),\ \ l$$

$$< m-1\Big). \qquad \blacksquare$$

Now, we prove that the derivatives of the operator (1) are approximation processes for corresponding order derivatives of the function being approximated.

Theorem 2:

Suppose that $r \in N, f \in C_h[0,\infty)$ for some $h > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} \tilde{S}_{n,p}^{(r)}(f(t);x) = f^{(r)}(x). \qquad (4)$$

Further, if $f^{(r)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (4) holds uniformly on $[a,b]$.

Proof:

By using Taylor's expansion of f, we get

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^r; \quad \varepsilon(t,x) \to 0 \; as \; t \to x.$$

Hence,

$$\tilde{S}_{n,p}^{(r)}(f(t);x) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \tilde{S}_{n,p}^{(r)}\big((t-x)^i;x\big) + \tilde{S}_{n,p}^{(r)}(\varepsilon(t,x)(t-x)^r;x) := \Sigma_1 + \Sigma_2.$$

By using **Lemma 6,** if $i < r$ we have $\tilde{S}_{n,p}^{(r)}\big(t^i;x\big) \to 0.$

$$\Sigma_1 = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \tilde{S}_{n,p}^{(r)}\big((t-x)^i;x\big)$$

$$= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{r} \binom{r}{j} (-x)^{r-j}\, \tilde{S}_{n,p}^{(r)}(t^j;x)$$

$$\to \frac{f^{(r)}(x)}{r!} \tilde{S}_{n,p}^{(r)}(t^r;x)$$

$$\to f^{(r)}(x) \qquad as \; n \to \infty.$$

$$\Sigma_2 = \frac{d^r}{dx^r}\left(\frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x) \int_0^{\infty} q_{n,pk-1}(t)\varepsilon(t,x)(t-x)^r dt + \frac{1}{G_x}\, \varepsilon(0,x)(-x)^r e^{-nx}\right):$$

$$= I_1 + I_2.$$

Next, making use of **Lemma 3** and replace k by pk and the fact $\frac{1}{G_x} \geq \frac{1}{\left|G_x^{(r)}\right|}$, $r \in N^0$, we have:

$$|I_1| \leq \frac{1}{G_x}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i+1}\frac{|Q_{i,j,r(x)}|}{x^r}\sum_{k=1}^{\infty} q_{n,pk}(x)|pk - nx|^j \int_0^{\infty} q_{n,pk-1}(t)\,|\varepsilon(t;x)(t-x)^r|dt.$$

Since $\varepsilon(t,x) \to 0$ *as* $t \to x$, for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$, whenever $0 < |t-x| < \delta$. For $|t-x| \geq \delta$, there exists a constant $C > 0$ such that $|\varepsilon(t,x)(t-x)^r| \leq Ce^{ht}$. Hence,

$$|I_1| \leq C\frac{n}{G_x}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^{i}\sum_{k=1}^{\infty} q_{n,pk}(x)|pk - nx|^j \left(\varepsilon\int_{|t-x|<\delta} q_{n,pk-1}(t)|(t-x)^r|dt + \int_{|t-x|\geq\delta} q_{n,pk-1}(t)|\varepsilon(t;x)(t-x)^r|dt\right)$$
$$:= J_1 + J_2$$

Since $\sup\limits_{\substack{2i+j\leq r\\ i,j\geq 0}} \dfrac{|Q_{i,j,r}(x)|}{x^r} = M(x), x \in (0,\infty)$ but fixed.

Now, using Cauchy-Schwarz inequality for integration and then for summation, we are led to

$$J_1 \leq C\frac{\varepsilon}{G_x}\sum_{\substack{2i+j\leq r\\ i,j0}} n^{i+1}\sum_{k=1}^{\infty} q_{n,pk}(x)|pk - nx|^j \left(\int_{|t-x|<\delta} q_{n,pk-1}(x)dt\right)^{\frac{1}{2}} \times\left(\int_{|t-x|<\delta} q_{n,pk-1}(x)(t-x)^{2r}\,dt\right)^{\frac{1}{2}}.$$

$$\leq C\varepsilon \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i\left(\frac{n^{2j}}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\left(\frac{pk}{n} - x\right)^{2j}\right)^{\frac{1}{2}} \times\left(\frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\int_{|t-x|<\delta} q_{n,pk-1}(t)(t - x)^{2r}\,dt\right)^{\frac{1}{2}}.$$

We have:

$$\frac{n^{2j}}{G_x}\left(\sum_{k=1}^{\infty} q_{n,pk}(x)\left(\frac{pk}{n} - x\right)^{2j} + q_{n,0}(x)(-x)^{2j} - q_{n,0}(x)(-x)^{2j}\right)$$

$$= \frac{n^{2j}}{G_x}\left(\sum_{k=0}^{\infty} q_{n,pk}(x)\left(\frac{pk}{n} - x\right)^{2j} - e^{-nx}(-x)^{2j}\right)$$

$= n^{2j}\left(O(n^{-j}) + O(n^{-s})\right) = O(n^j)$ for any $s > 0$,

Since,

$$\frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\int_{|t-x|<\delta} q_{n,pk-1}(t)(t-x)^{2r}\,dt = U_{n,2r}(x) = O(n^{-r})$$

Hence,

$$J_1 = C\varepsilon \sum_{\substack{2i+j\le r\\ i,j0}} n^i\, O\left(n^{\frac{j}{2}}\right) O\left(n^{-\frac{r}{2}}\right) = \varepsilon\, O\left(n^{\frac{2i+j}{2}-\frac{r}{2}}\right) = \varepsilon O(1).$$

Next, again using Cauchy-Schwarz inequality for integration and then for summation, we have:

$$J_2 \le \frac{C}{G_x}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^{i+1}\sum_{k=1}^{\infty} q_{n,pk}(x)(pk-nx)^j \left(\int_{|t-x|\ge\delta} q_{n,pk-1}(t)dt\right)^{\frac{1}{2}} \times \left(\int_{|t-x|\ge\delta} q_{n,pk-1}(t)e^{2ht}\,dt\right)^{\frac{1}{2}}$$

$$\le C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left(\frac{1}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)(pk-nx)^{2j}\right)^{\frac{1}{2}} \times\left(\frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\int_{|t-x|\ge\delta} q_{n,pk-1}(t)e^{2ht}\,dt\right)^{\frac{1}{2}}$$

Making use of Taylor's expansion, Cauchy-Schwarz inequality for integration and then for summation and **Lemma 5** , we have:

$I= O(n^{-s})$; $s \ge 0$,
where $I := \frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)\int_{|t-x|\ge\delta} q_{n,pk-1}(t)e^{2ht}\,dt.$
Therefore,

$$= C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\, O\left(n^{\frac{j}{2}}\right) O(n^{-s}) = o(1),$$

$$\text{for } s > \frac{r}{2}.$$

Now, since $\varepsilon > 0$ is arbitrary, it follows that $I_1 \to 0$ as $n \to \infty$. Also $I_2 \to 0$ $as\, n \to \infty$ and hence $\Sigma_2 = o(1)$. Combining the estimates Σ_1 and Σ_2, (4) is immediate.

■

The next theorem is a Voronovskaja-type asymptotic formula for the operators $\tilde{S}_{n,p}^{(r)}(f(t);x)$, $r \in N$.

Theorem 3:

Let $f \in C_h[0,\infty)$ for some $h > 0$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n\left\{\tilde{S}_{n,p}^{(r)}(f(t);x) - f^{(r)}(x)\right\} = rf^{(r+1)}(x) + x\,f^{(r+2)}(x).$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then (5) holds uniformly on $[a,b]$.

Proof:

By using Taylor's expansion of f,

we have:

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2}; \quad \varepsilon(t,x) \to 0 \ as \ t \to x.$$

Then

$$\tilde{S}_{n,p}^{(r)}(f(t);x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}\tilde{S}_{n,p}^{(r)}\big((t-x)^i;x\big) + \tilde{S}_{n,p}^{(r)}(\varepsilon(t,x)(t-x)^{r+2};x).$$

Using the same technique of **Theorem 2,**

we get:

$$\tilde{S}_{n,p}^{(r)}(\varepsilon(t,x)(t-x)^{r+2};x) \to 0 \ as \ n \to \infty.$$

Then

$$\tilde{S}_{n,p}^{(r)}(f(t);x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}\tilde{S}_{n,p}^{(r)}((t-x)^i;x)$$

$$= \frac{f^{(r)}(x)}{r!}\tilde{S}_{n,p}^{(r)}(t^r;x) + \frac{f^{(r+1)}(x)}{(r+1)!}((-x)(r+1)\tilde{S}_{n,p}^{(r)}(t^r;x) + \tilde{S}_{n,p}^{(r)}(t^{r+1};x) + \frac{f^{(r+2)}(x)}{(r+2)!}\left(\frac{(r+1)(r+2)}{2}x^2\tilde{S}_{n,p}^{(r)}(t^r;x) + (r+2)(-x)\tilde{S}_{n,p}^{(r)}(t^{r+1};x)+\tilde{S}_{n,p}^{(r)}(t^{r+2};x)\right).$$

For sufficiently large n and using **Lemma 2**, we get:

$$\lim_{n\to\infty} n\left\{\frac{f^{(r)}(x)}{r!}\ r! + \frac{f^{(r+1)}(x)}{(r+1)!}\left((-x)(r+1)r! + (r+1)!\,x + \frac{r(r+1)r!}{n}\right) + \frac{f^{(r+2)}(x)}{(r+2)!} \times\left(\frac{(r+1)(r+2)r!}{2}x^2 + (r+2)(-x)\left((r+1)!\,x + \frac{r(r+1)r!}{n}\right) + \frac{(r+2)!}{2}x^2 + \frac{(r+1)(r+2)(r+1)!}{n}x\right) - f^{(r)}(x)\right\}$$
$$= rf^{(r+1)}(x) + x\,f^{(r+2)}(x).$$

$$\lim_{n\to\infty} n\left\{\tilde{S}_{n,p}^{(r)}(f(t);x) - f^{(r)}(x)\right\} = rf^{(r+1)}(x) + x\,f^{(r+2)}(x).$$ ■

Finally, we give an estimate of the degree of approximation by the operators $\tilde{S}_{n,p}^{(r)}(f(t);x)$.

Theorem 4:

Let $f \in C_h[0,\infty)$ for some $h > 0$ and $r \le v \le r+2$. If $f^{(v)}$ exists and continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then for sufficiently large n,

$$\left\|\tilde{S}_{n,p}^{(r)}(f(t);x) - f^{(r)}(x)\right\|_{C[a,b]} \le C_1 n^{-1} \sum_{i=r}^{v} \|f^{(i)}\|_{C[a,b]} + C_2 n^{\frac{-1}{2}} \omega_{f^{(v)}}\left(n^{\frac{-1}{2}}; (a-\eta, b+\eta)\right) + O(n^{-2}),$$

where C_1, C_2 are constants independent of f and $n, \omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$, and $\|.\|_{C[a,b]}$denotes the sup-norm on $[a,b]$.

proof:

By our hypothesis

$$f(t) = \sum_{i=0}^{v} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(v)}(\xi) - f^{(v)}(x)}{v!}(t-x)^v \chi(t) + h(t,x)\big(1-\chi(t)\big),$$

where ξ lies between t, x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta, b+\eta)$.

For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, we get:

$$f(t) = \sum_{i=0}^{v} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(v)}(\xi) - f^{(v)}(x)}{v!}(t-x)^v.$$

For $t \in [0,\infty)\backslash(a-\eta, b+\eta)$ and $x \in [a,b]$, we define

$$h(t,x) = f(t) - \sum_{i=0}^{v} \frac{f^{(i)}}{i!}(t-x)^i.$$

Now,

$$\tilde{S}_{n,p}^{(r)}(f(t);x) - f^{(r)}(x) = \left(\sum_{i=0}^{v} \frac{f^{(i)}(x)}{i!} \tilde{S}_{n,p}^{(r)}\big((t-x)^i; x\big) - f^{(r)}(x)\right)$$

$$+\tilde{S}_{n,p}^{(r)}\left(\frac{f^{(v)}(\xi) - f^{(v)}(x)}{v!}(t-x)^v \chi(t); x\right) + \tilde{S}_{n,p}^{(r)}\big(h(t,x)\big(1-\chi(t)\big); x\big)$$

$$:= \textstyle\sum_1 + \sum_2 + \sum_3.$$

By using **Lemma 6**, we get

$$\textstyle\sum_1 = \displaystyle\sum_{i=r}^{v} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r} \left(\frac{1}{n^j A_p(nx)}\left(x^j A_p^{(j)}(nx) + j(j-1)x^{j-1} A_p^{(j-1)}(nx) + O(n^{-2})\right)\right.$$

$$- f^{(r)}(x).$$

Consequently,

$$\|\textstyle\sum_1\|_{C[a,b]} \le C_1 n^{-1}\left[\displaystyle\sum_{i=r}^{v} \|f^{(i)}\|_{C[a,b]}\right] + O(n^{-2}),$$ uniformly on $[a,b]$.

To estimate $\sum_2$ we proceed as follows:

$$|\sum_2| \le \tilde{S}_{n,p}^{(r)}\left(\frac{|f^{(v)}(\xi)-f^{(v)}(x)|}{v!}|t-x|^v\chi(t);x\right)$$

$$\le \frac{\omega_{f^{(v)}}(\delta;(a-\eta,b+\eta))}{v!}\tilde{S}_{n,p}^{(r)}\left(\left(1+\frac{|t-x|}{\delta}\right)|t-x|^v;x\right)$$

$$\le \frac{\omega_{f^{(v)}}(\delta;(a-\eta,b+\eta))}{v!}\left[\frac{n}{G_x}\sum_{k=0}^{\infty}\left|q_{n,pk}^{(r)}(x)\right|\int_0^{\infty} q_{n,pk}(t)(|t-x|^v+\delta^{-1}|t-x|^{v+1})dt + e^{-nx}(x^v+\delta^{-1}x^{v+1})\right], \delta > 0.$$

Now, for $s = 0,1,2,\ldots$, we have:

$$\frac{n}{G_x}\sum_{k=1}^{\infty} q_{n,pk}(x)|pk-nx|^j\int_0^{\infty} q_{n,pk}(t)|t-x|^s\,dt \qquad (6)$$

$$\le \left(\frac{1}{G_x}\sum_{k=0}^{\infty} q_{n,pk}(x)(pk-nx)^{2j}\right)^{\frac{1}{2}} \times \left(\frac{n}{G_x}\sum_{k=0}^{\infty} q_{n,pk}(x)\int_0^{\infty} q_{n,pk}(t)(t-x)^{2s}\,dt\right)^{\frac{1}{2}}$$

$$= O\left(n^{\frac{j}{2}}\right)O\left(n^{\frac{-s}{2}}\right)$$

$$= O\left(n^{\frac{(j-s)}{2}}\right) \text{ uniformly on } [a,b].$$

Therefore , by using **Lemma 3**, (6) and substitute k by pk, we get:

$$\frac{n}{G_x}\sum_{k=0}^{\infty}\left|q_{n,pk}^{(r)}(x)\right|\int_0^{\infty} q_{n,pk}(t)|t-x|^s\,dt$$

$$\le \frac{n}{G_x}\sum_{k=0}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i|pk-nx|^j\,\frac{|Q_{i,j,r}(x)|}{x^r}q_{n,pk}(x)\int_0^{\infty} q_{n,pk}(t)(t-x)^s\,dt \qquad (7)$$

$$\le \left(\sup_{\substack{2i+j\le r\\ i,j\ge 0}}\sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^r}\right)\left(\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\,(\frac{n}{G_x}\sum_{k=0}^{\infty} q_{n,pk}(x)\ |pk-nx|^j\int_0^{\infty} q_{n,pk}(t)|t-x|^s\ dt\right)$$

$$= C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O\left(n^{\frac{(j-s)}{2}}\right)$$

$$= O\left(n^{\frac{(r-s)}{2}}\right), \text{uniformly on } [a,b],$$

since

$$\sup_{\substack{2i+j\le r\\ i,j\ge 0}}\sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^r} = M(x), \text{ but fixed.}$$

Choosing $\delta = n^{\frac{-1}{2}}$ and applying (7), we are led to

$$\|\sum_2\|_{C[a,b]} \le \frac{\omega_{f^{(v)}}\left(n^{\frac{-1}{2}};(a-\eta,b+\eta)\right)}{v!}\left[O\left(n^{\frac{(r-v)}{2}}\right)+n^{\frac{1}{2}}O\left(n^{\frac{(r-v-1)}{2}}\right)+O(n^{-m})\right],$$

for any $m > 0$,

$$\leq C_2 n^{\frac{-(r-v)}{2}} \omega_{f^{(v)}} \left(n^{\frac{-1}{2}}; (a-\eta, b+\eta)\right).$$

Since $t \in [0,\infty)\backslash(a-\eta, b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$.

Thus,

$$|\Sigma_3| \leq \frac{n}{G_x} \sum_{k=0}^{\infty} \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} n^i |pk - nx|^j \frac{|Q_{i,j,r}(x)|}{x^r} q_{n,pk}(x) \int_{|t-x|\geq\delta} q_{n,pk}(t)|h(t,x)|\, dt + e^{-nx}|h(0,x)|.$$

For $|t-x| \leq \delta$, we can find a constant $C > 0$ such that $|h(t,x)| \leq Ce^{ht}$.

Finally using Schwarz inequality for integration and then for summation , we

get:
$|\Sigma_3| = O(n^{-s}), s > 0$ uniformly on $[a,b]$.

Combining the estimates of $\sum_1, \sum_2, \sum_3$, the required result is immediate.
■

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التقريب المتعدد بأستخدام تعميم لمؤثر من النمط زاز

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الخلاصة

في هذا البحث، نقدم تعميما للمؤثر من النمط مجموع-تكامل Szãsz ونرمز له $\tilde{S}_p(f(t);x)$. أولا، ندرس تقارب المؤثر وبعد ذلك نجد صيغة تكرارية لهذا المؤثر وفي النهاية نثبت صيغة ڤرونوفسكي للتقارب (Voronovskaja-type asymptotic formula) ونحصل على الخطأ المخمن في حدود مقياس الاستمرارية للدوال المقربة.