Applications on differential subordination involving linear operator

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Abstract

In this paper we introduce and investigate some subclasses of strongly close-to-convex functions associated with the linear operator ofmeromorphic p-valently functions and study several inclusion relationships with some properties ofthis operator.

Key Words: *linear operator, meromorphic functions, differential subordination, strongly Close-to-convex functions, p-valently functions.*

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1.Introduction

Let $\sum_p$ denote the class of meromorphic functions of the form

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_{k+p} z^{k+p}, \qquad (1-1)$$

which are analytic and p-valently in the punctured unit disk $\mathcal{U}^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\}$

$= \mathcal{U} - 0.$

If $f(z)$ and $g(z)$ are analytic in $\mathcal{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$ which (by definition) is analytic

in $\mathcal{U}$ such that $f(z) = g(w(z))$.

A function $f(z) \in \sum_p$ is said to be $p-$valent meromorphic starlike of order α $(0 \leq \alpha < p)$

if it satisfies

$$Re\left\{-\frac{zf'(z)}{f(z)}\right\} \geq \alpha \quad (z \in \mathcal{U}) \qquad (1-2)$$

and the class of such functions is defined by $MS^*(\alpha)$.

Also a function $f(z) \in \sum_p$ is said to be $p-$valent meromorphic convex of order

α $(0 \leq \alpha < p)$ if it satisfies

$$Re\left\{-(1+\frac{zf''(z)}{f'(z)})\right\} \geq \alpha (z \in \mathcal{U}) \quad (1-3)$$

and the class of such functions is defined by $MK(\alpha)$.

Let $f(z) \in \sum_p$ and $g(z) \in MS^*(\alpha)$. Then $f(z) \in MC(\alpha,\beta)$ if and only if

$$Re\left\{-\frac{zf'(z)}{g(z)}\right\} > \beta(z \in \mathcal{U}), \qquad (1-4)$$

where 0 ≤ α < p and 0 ≤ β < p. Such functions are called close-to-convex functions of

orderβ and type α in $\mathcal{U}$,[1,2].

Further, a function $f(z) \in \sum_p$ is called p-valentlymeromorphic strongly starlike of orderγ $(0 \le \gamma < p)$and type α $(0 \le \alpha < p)$in$\mathcal{U}$ if it satisfies

$$\left|\arg(-\frac{zf'(z)}{f(z)} - \alpha)\right| < \frac{\pi}{2}\gamma \quad (z \in \mathcal{U}). \quad (1-5)$$

If $f(z) \in \sum_p$ satisfies

$$\left|\arg(-\left(1+\frac{zf''(z)}{f'(z)}\right) - \alpha)\right| < \frac{\pi}{2}\gamma \quad (z \in \mathcal{U}),$$

for some γ $(0 \le \gamma < p)$ and α (0 ≤ α < p), then f is called p-valentlymeromorphic

strongly convex of order γ and type α in $\mathcal{U}$ and denoted by $MC(\gamma,\alpha)$. We note that the

classes mentioned above are the familiar classes which have been studied by many authors[3, 4]

For a function $f(z) \in \sum_p$ given by (1-1), we define a linear operator D^n by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = z^{-p}\left(z^{p+1}f(z)\right)' = z^{-p} + \sum_{k=0}^{\infty}(2p+k+1)a_{k+p}z^{k+p}$$

and

$$D^n f(z) = D\left(D^{n-1}f(z)\right) = z^{-p}\left(z^{p+1}D^{n-1}f(z)\right)'$$

$$= z^{-p} + \sum_{k=0}^{\infty}(2p+k+1)^n a_{k+p}z^{k+p} \quad .(N \in \mathbb{N}) \qquad (1-6)$$

Using the relation (1-6), it is easy to verify that

$$\left(D^n f(z)\right)' = D^{n+1}f(z) - (p+1)D^n f(z). \quad (1-7)$$

and denoted by $MS^*(\gamma,\alpha)$.

Also, we note that $D^n f(z)$ of another form of function studied by Liu and Srivastava[5],Srivastava and Patel[6]whointroduce several inclusion relationships by usingvarioussubclasses of meromorphic p-valent functions . A special cases of linear operator $D^n$for p =1 studied by Uralegaddi and Somanatha[7],Aouf and Hossen[8], and got interestingresultsby using the operator D^n.

For n ∈ℕ,let $MC_p^{n+1}(\alpha,\beta,\gamma.A,B)$be the class of functions $f(z) \in \sum_p$ satisfying the condition:

$$\left|-\arg(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \gamma)\right| \frac{\pi}{2}\delta \; (0 \le \gamma < p, 0 \le \delta < p;\, z \in \mathcal{U}), \qquad (1-8)$$

for some $g(z) \in S_p^{n+1}(\alpha,A,B), where$

$$S_p^{n+1}(\alpha,A,B) = \left\{g: \frac{1}{p+\alpha}(\frac{z(D^{n+1}g(z))'}{D^{n+1}g(z)} - \alpha) \prec \frac{1+Az}{1+Bz}\right\} \qquad (1-9)$$

$(0 \le \alpha < p, -1 \le B \le A \le 1, z \in U \; and \; g \in \sum_p)$ andthe functions $f$belonging to thisclass is called strongly close-to-convex function.

In this study and by using the technique of Cho,we find some argument propertiesof functions belonging to$\sum_p$ which include inclusion relationship and we obtain someinteresting results for the functions class $MC_p^{n+1}(\alpha,\beta,\gamma,A,B)$which we have defined here

by the operator D^n.

In order to obtain our results, we need each of the following lemmas.

Lemma 1.1[9]: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $Re\,\{\varepsilon h(z) + \eta\} >$

$0\ (\varepsilon, \eta \in C)$. If $p(z)$is analytic in $\mathcal{U}$ with $p(0) = 1$,then

$$p(z) + \frac{zp'(z)}{\varepsilon p(z) + \eta} \prec \mathrm{h(z)}(\mathrm{z} \in \mathcal{U}),$$

implies $p(z) \prec h(z)\ (z \in U)$.

Lemma 1.2[10]Let $h(z)$ be convex univalent in $\mathcal{U}$ and $w(z)$ be analytic in $\mathcal{U}$ with $Re\,\{w(z)\} \geq 0$. If $p(z)$is analytic in $\mathcal{U}$with $p(0) = h(0)$,then

$$p(z) + w(z)zp(z) \prec h(z)\ (z \in \mathcal{U}),$$

implies $p(z) \prec h(z)\ (z \in \mathcal{U})$.

Lemma 1.3[4]: Let $p(z)$ be analytic in $\mathcal{U}$ with $p(0) = 1$ and $p(z) \neq 0$ in $\mathcal{U}$. If there existstwo points z_1, z_2 in $\mathcal{U}$ such that

$$-\frac{\pi}{2}\alpha_1 = arg p(z_1) < arg p(z) < arg p(z_2) = \frac{\pi}{2}\alpha_2 \quad (1-10)$$

for some α_1, $\alpha_1(\alpha_1,\alpha_2 > 0)$and for all z(|z| < $|z_1| = |z_2|$), then we have

$$\frac{zp'(z_1)}{p(z_1)} = -i\frac{\alpha_1+\alpha_2}{2}m$$

and

$$\frac{zp'(z_2)}{p(z_2\ \ )} = -i\frac{\alpha_1+\alpha_2}{2}m, \qquad (1-11)$$

Where $m \geq \frac{1-|c|}{1+|c|}$and

$$c = i\,tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right). \qquad (1-12)$$

2.Main Results

We first derive the following with use of Lemma 1.1.

Proposition 2.1: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and $Re\,\{h(z)\} > 0$.

If a function $f(z) \in \sum_p$ satisfies the following condition:

$$\frac{1}{p+\alpha}\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - \alpha\right) \prec h(z),$$

then

$$\frac{1}{p+\alpha}\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\right) \prec h(z).$$

$$(0 \leq \alpha < p; z \in \mathcal{U})$$

Proof. Let

$$p(z) = \frac{1}{p+\alpha}\left(\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\right). \quad (2-1)$$

Then $p(z)$ is analytic function in $\mathcal{U}$with$p(0) = 1$. By using (1-7), we obtain

$$p+1+\alpha+(p+\alpha)p(z) = \frac{D^{n+1}f(z)}{D^n f(z)}. \qquad (2-2)$$

Differentiating Logarithmically with respect to z and multiplying by , we get

$$\mathrm{p}(z)+\frac{zp'(z)}{p+1+\alpha+(p+\alpha)p(z)}=\frac{1}{p+\alpha}\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha\right).$$

Now, by using Lemma 1.1, we obtain that

$$\frac{1}{p+\alpha}\left(\frac{z(D^{n}f(z))'}{D^{n}f(z)}-\alpha\right)\prec h(z),$$

deduce that $p(z)\prec h(z)$.

Setting $h(z)=\frac{1+Az}{1+Bz}(-1\le B\le A\le 1)$, in Lemma 2.1 , we obtain

Corollary 2.1: For $n\in\mathbb{N}$ and $p\in\{1,2,\dots\}$ we have

$$S_p^{n+1}(\alpha,A,B)\subset S_p^{n}(\alpha,A,B).$$

Proposition 2.2: Let $h(z)$ be convex univalent in $\mathcal{U}$ with $h(0)=1$ and $Re\,\{h(z)\}>0$.If

$$\frac{1}{p+\alpha}\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha\right)\prec h(z),$$

then

$$\frac{1}{p+\alpha}\left(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta f(z)}-\alpha\right)\prec h(z),$$

$$(0\le\alpha<p;z\in\mathcal{U})$$

where

$$\mathbb{I}_\theta \mathrm{f}(z)=\frac{\theta-p}{z^\theta}\int_0^z t^{\theta-1}f(t)dt\quad(\theta\ge 0)\quad(2-3)$$

$$z(D^{n+1}\mathbb{I}_\theta f(z))'=(\theta-p)((D^{n+1}f(z))-\theta((D^{n+1}f(z)).\qquad(2-4)$$

$$p(z)=\frac{1}{p+\alpha}\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha\right),$$

$p(z)$is analytic function in $\mathcal{U}$ with $p(0)=1$. Then, by (2-4), we get

$$\theta+\alpha+(\mathrm{p}+\alpha)\mathrm{p}(z)=(\theta-p)\frac{D^{n+1}f(z)}{D^{n+1}\mathbb{I}_\theta f(z)}.\qquad(2-5)$$

By differentiating (2-5) logarthmically with respect to z and multiplying by z,we have

$$p(z)+\frac{zp'(z)}{\theta+\alpha+(\mathrm{p}+\alpha)\mathrm{p}(z)}=\frac{1}{p+\alpha}\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha\right).$$

Thus, by Lemma 1.1, we get

$$\frac{1}{p+\alpha}\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)}-\alpha\right)\prec h(z).$$

$f(z)\in\sum_p$ satisfies

Taking $h(z)=\frac{1+Az}{1+Bz}(-1\le B\le A\le 1)$, in Proposition 2.2 , we obtain

Corollary 2.2:

If $f(z)\in S_p^{n+1}(\alpha,\mathrm{A},\mathrm{B})$, then $\mathbb{I}_\theta(f)\in S_p^{n+1}(\alpha,\mathrm{A},\mathrm{B})$

Hence on Applying Proposition 2.2 we prove the following theorem

Theorem 2.1: Let$f(z)\in\sum_p$ and $0\le\delta_1,\delta_2\le p,0\le\alpha<p$. If

$$-\frac{\pi}{2}\delta_1<\arg(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)}-\gamma)<\frac{\pi}{2}\delta_2$$

for some $g(z)\in S_p^{n+1}(\alpha,\mathrm{A},\mathrm{B})$, then

Proof. From (2-3),we have

$$-\frac{\pi}{2}\beta_1<\arg\left(-\frac{z(D^{n}f(z))'}{D^{n}g(z)}-\gamma\right)<\frac{\pi}{2}\beta_2,$$

Let

whereβ_1 and β_2 $(0<\beta_1,\beta_2\le\mathrm{p})$ are the solution of the equations:

$$\delta_1 = \begin{cases} \beta_1 + \frac{\pi}{2} tan^{-1} \left\{ \frac{(\beta_1 + \beta_2)(1 - |c|) cos \frac{\pi}{2} t_1}{2\left(\frac{(p+\alpha)(1+A)}{(1+B)} + p + 1 + \alpha\right)(1 + |c|) + (\beta_1 + \beta_2) sin \frac{\pi}{2} t_1} \right\} & B \neq -1, \\ \beta_1 & B = -1, \end{cases}$$

$(2-6)$

$$\delta_2 = \begin{cases} \beta_2 + \frac{\pi}{2} tan^{-1} \left\{ \frac{(\beta_1+\beta_2)(1-|c|) cos \frac{\pi}{2} t_1}{2\left(\frac{(p+\alpha)(1+A)}{(1+B)} + p + 1 + \alpha\right)(1+|c|) + (\beta_1 + \beta_2) sin \frac{\pi}{2} t_1} \right\} & B \neq -1, \\ \beta_2 & B = -1, \end{cases}$$

$(2-7)$

where c is given by (1-12) and $t_1 = \frac{\pi}{2} sin^{-1}\left(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(P+1+\alpha)(1-B^2)}\right)$.

Proof. Let

$$p(z) = \frac{1}{p+\alpha}\left(\frac{z(D^{n+1} f(z))'}{D^{n+1} f(z)} - \gamma\right). \qquad (2-8)$$

By using (1-7) and simple calculations, we have

$$[(p + \gamma)(p(z) - \gamma] D^n g(z) = D^{n+1} f(z) - (p+1) D^n f(z). \qquad (2-9)$$

Take the derivative of both sides (2-9) and multiply by z, we get

$(p+\gamma) z p'(z) D^n g(z) + [(p+\gamma)p(z) - \gamma] z\left(D^n g(z)\right)' = z\left(D^{n+1} f(z)\right)' - (p+1)\left(D^n f(z)\right)'$ (2-10)

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$,thenwe have$g(z) \in S_p^n(\alpha, A, B)$. (by Corollary 2.1)

Now, put

$$q(z) = p \frac{1}{p+\alpha}\left(\frac{z(D^n g(z))'}{D^n g(z)} - \alpha\right).$$ and *By using* (1 − 7), *we get*

$$(p + \alpha) q(z) + \alpha + p + 1 = \frac{D^{n+1} g(z)}{D^n g(z)}. \qquad (2-11)$$

Therefore, by (2-10),(2-11), we have

$$\frac{1}{p+\alpha}\left(\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} - \gamma\right) = p(z) + \frac{z p'(z)}{(p+\alpha) q(z) + \alpha + p + 1}.$$

Making use the result of Silverman and Silvia [11],we obtain

$$\left|q(z) - \frac{1 - AB}{1 - B^2}\right| < \frac{A - B}{1 - B^2} \quad (z \in \mathcal{U};\ B \neq -1)\ (2-12)$$

and

$$Re\,\{q(z)\} > \frac{1-A}{2} (z \in \mathcal{U};\ B \neq -1). \qquad (2-13)$$

By (2-12),(2-13), we get

$(p + \alpha) q(z) + p + \alpha + 1 = r e^{i \frac{\pi\theta}{2}}$,

where ,if $B \neq -1$,we have

$$\frac{(p+\alpha)(1-A)}{1-B}\mathrm{P}+\alpha+1<r<\frac{(p+\alpha)(1+A)}{1+B}\mathrm{P}+\alpha+1$$

$-t_1 < \varphi < t_1$, t_1 is defined in $(2-8)$

and if $B = -1$,we have

$$\frac{(p+\alpha)(1-A)}{2}\mathrm{P}+\alpha+1<r<\infty, -1<\emptyset<1.$$

Now by Lemma 1.2 and assumption with $w = \frac{1}{(p+\alpha)q(z)+p+\alpha+1}$,we note that p(z) is analytic .with $p(0) = 1$and $Re\,\{p(z)\} > 0$ in $\mathcal{U}$

Hence by Lemma 1.3 for z_1 , $z_2 \in \mathcal{U}$,such that the condition (1-10) is satisfied, then we

have (1-11) under the restriction (1-12).Now

if B $\neq -1$, we get

$$arg\left(-\left(p(z_1)+\frac{zp'(z_1)}{(p+\alpha)q(z_1)+\alpha+p+1}\right)\right)=-\frac{\pi}{2}\beta_1+\arg\left(1-i\frac{\beta_1+\beta_2}{2}m\left(re^{i\frac{\pi}{2}}\right)^{-1}\right)$$
$$\le\frac{-\pi}{2}\beta_2-tan^{-1}\left(\frac{(\beta_1+\beta_2)m\,sin\frac{\pi}{2}(1-\emptyset)}{2r+(\beta_1+\beta_2)m\,cos\frac{\pi}{2}(1-\emptyset)}\right)$$
$$\le\frac{-\pi}{2}\beta_2-tan^{-1}\left(\frac{(\beta_1+\beta_2)(1-|c|)cos\frac{\pi}{2}t_1}{2\left(\frac{(p+\alpha)(1+A)}{(1+B)}+p+1+\alpha\right)(1+|c|)+(\beta_1+\beta_2)sin\frac{\pi}{2}t_1}\right)$$
$$=\frac{-\pi}{2}\delta_1,$$

and

$$arg\left(-\left(p(z_2)+\frac{zp'(z_2)}{(p+\alpha)q(z_2)+\alpha+p+1}\right)\right)\ge\frac{-\pi}{2}\beta_2$$
$$-tan^{-1}\left(\frac{(\beta_1+\beta_2)(1-|c|)cos\frac{\pi}{2}t_1}{2\left(\frac{(p+\alpha)(1+A)}{(1+B)}+p+1+\alpha\right)(1+|c|)+(\beta_1+\beta_2)sin\frac{\pi}{2}t_1}\right)$$
$$=\frac{-\pi}{2}\delta_2.$$

If $B = -1$,we get

$$arg\left(-\left(p(z_1)+\frac{zp'(z_1)}{(p+\alpha)q(z_1)+\alpha+p+1}\right)\right)\le\frac{-\pi}{2}\beta_1$$

and

$$arg\left(-\left(p(z_2)+\frac{zp'(z_2)}{(p+\alpha)q(z_2)+\alpha+p+1}\right)\right)\ge\frac{\pi}{2}\beta_2.$$

There are contradictions to the assumption. Hence the proof is complete

Corollary 2.3:

$$MC_p^{n+1}(\alpha,\beta,\gamma,A,B)\subset MC_p^{n}(\alpha,\beta,\gamma,A,B)$$

Setting $n = 0, \delta_1 = \delta_2 = \delta$ in Theorem 2.1 , we get:

Corollary 2.4: Let $f(z)\in\sum_p$ .If

$$\left|\frac{z(z^{-p}(z^{p+1}f(z)')'}{(z^{-p}(z^{p+1}g(z))'}-\gamma\right|\le\frac{\pi}{2}\delta$$

$$\left|\arg\left(-\frac{z\,f(z))'}{g(z)}-\gamma\right)\right|<\frac{\pi}{2}\beta\,,$$

whereβ $(0<\beta\le p)$is the solution of equation:

$$\delta=\left\{\beta+\frac{\pi}{2}tan^{-1}\left\{\frac{\beta\,cos\frac{\pi}{2}t_1}{\frac{(p+\alpha)(1+A)}{1+B}+p+1+\alpha+\beta sin\frac{\pi}{2}t_1}\right\}\right.$$
$$\beta$$

and $t_1 = \frac{\pi}{2} sin^{-1}\left(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(P+1+\alpha)(1-B^2)}\right)$.

Theorem 2.2: Let $f(z) \in \sum_p$ and $0 < \delta_1, \delta_2 \leq 1, 0 \leq \gamma < 1$. If

$$-\frac{\pi}{2}\delta_1 < \arg\left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right) < \frac{\pi}{2}\delta_2$$

for some $g(z) \in S_p^{n+1}(\alpha, A, B)$, then

$$-\frac{\pi}{2}\beta_1 < \arg\left(-\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \gamma\right) < \frac{\pi}{2}\beta_2,$$

where $\mathbb{I}_\theta$ is defined by(2-3),and β_1 , β_2 are the solutions of

$$\delta_1 = \begin{cases} \beta_1 + \frac{\pi}{2} tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)cos\frac{\pi}{2}t_1}{2\left(\frac{(p+\alpha)(1+A)}{(1+B)}+p+1+\alpha\right)(1+|c|)+(\beta_1+\beta_2)sin\frac{\pi}{2}t_1}\right\} & B \neq -1, \\ \beta_1 & B = -1, \end{cases} \quad (2-14)$$

and

$$\delta_2 = \begin{cases} \beta_2 + \frac{\pi}{2} tan^{-1}\left\{\frac{(\beta_1+\beta_2)(1-|c|)cos\frac{\pi}{2}t_1}{2\left(\frac{(p+\alpha)(1+A)}{(1+B)}+p+1+\alpha\right)(1+|c|)+(\beta_1+\beta_2)sin\frac{\pi}{2}t_1}\right\} & B \neq -1, \\ \beta_2 & B = -1, \end{cases}$$

(2-15)

here c is given by (1-12) and $t_2 = \frac{\pi}{2} sin^{-1}\left(\frac{(p+\alpha)(1-B)}{(p+\alpha)(1-AB)+(\theta+\alpha)(1-B^2)}\right)$.

Proof. Let

$$p(z) = \frac{1}{p+\alpha}\left(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \gamma\right).$$

Since $g(z) \in S_p^{n+1}(\alpha, A, B)$, thenwe have by Corollary 2.2 that$\mathbb{I}_\theta g(z) \in S_p^{n+1}(\alpha, A, B)$.

By (2-5),we get

$$[(p+\gamma)p(z)+\gamma]D^{n+1}\mathbb{I}_\theta g(z) = (\theta - p)(D^{n+1} f(z)) - \theta(D^{n+1}\mathbb{I}_\theta f(z)$$

and simplifying, we obtain

$$(p+\gamma)zp'^{(z)} + [(p+\gamma)p(z) + \gamma][(p+\alpha)q(z)+\theta+\alpha] = (\theta - p)\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)},$$

$$q(z) = \frac{1}{p+\alpha}\left(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \alpha\right).$$

Therefore,

$$\frac{1}{p+\alpha}\left(\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \alpha\right) = p(z) + \frac{zp'(z_2)}{(p+\alpha)q(z)+\alpha+\theta}.$$

Thus, by the similar way of proof Theorem 2.1, we obtain the required result and the proof is complete.

Setting $\delta_1 = \delta_2 = \delta$ in Theorem 2.2, we obtain

Corollary 2.5: Let $f(z) \in \sum_p$ and $0 \leq \gamma < p, 0 < \delta \leq 1$. If

$$\left|arg\left(-\frac{z(D^{n+1} f(z))'}{D^{n+1} g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta$$

fo

$$\left|arg\left(-\frac{z(D^{n+1}\mathbb{I}_\theta f(z))'}{D^{n+1}\mathbb{I}_\theta g(z)} - \gamma\right)\right| < \frac{\pi}{2}\beta,$$

Where$\mathbb{I}_\theta$is given by(2-5),and $\beta(0 < \beta \leq 1)$ is the solution of the equation

$$\delta = \begin{cases} \beta + \frac{\pi}{2}\tan^{-1}\left\{\frac{\beta\cos\frac{\pi}{2}t_1}{\frac{(p+\alpha)(1+A)}{1+B}+p+1+\alpha+\beta\sin\frac{\pi}{2}t_1}\right\} & B \neq -1, \\ \beta & B = -1. \end{cases}$$

,

Corollary 2.6: If $f(z) \in MC_p^{n+1}(\gamma,\delta,\alpha,A,B)$, then $\mathbb{I}_\theta f(z) \in MC_p^{n+1}(\gamma,\delta,\alpha,A,B)$.

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