On Generalization of Beta Operators

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Abstract

In this present paper, we introduce a generalization definition to Beta weight functions depend on a non-negative integer r called r −Beta. This definition restricts to the classical Lupas and to the classical Beta weight functions whenever $r = 0,1$ respectively. In addition, we used these weight functions to define two operators of summation and of summation-integral types. Surly, these operators restrict to the classical Lupas operators and to the classical Beta operators of both summation and summation-integral types whenever $r = 0,1$ respectively. In addition, we can get the mixed operators of Lupas-Beta and Beta-Lupas for a suitable chose of integer values. Furthermore, we derive a Voronovaskaja-type asymptotic formula for the new operators from which we can get the similar formulas for many operators of summation and summation-integral types of mixed Lupas-Beta (or Beta-Lupas) weight functions and more.

Key words: Linear positive operators, Beta operators, Lupas operators, Voronovaskaja-type asymptotic formula.

1. Introduction

For $n \in N := \{1,2,\dots\}$ and s is integer number the notation $(n)_s$ is defined as: [(1)]

$$(n)_s = \begin{cases} n(n+1)\dots(n+s-1), & s > 0 \\ 1 & , s = 0 \\ \dfrac{1}{(n-1)(n-2)\dots(n-s)} & , s < 0. \end{cases}$$

For $n \in N$, $r, k \in N^0 := N \cup \{0\}$, we define r −Beta weight functions as:

$$\beta_{n,k}^{\{r\}}(x) = \frac{(n+k)_r}{(1+x)^r} P_{n,k}(x), \qquad (1.1)$$

where $P_{n,k}(x)$ is the Lupas weight functions defined as: [(2)]

$$P_{n,k}(x) = \frac{(n)_k}{(k)!} x^k (1+x)^{-(n+k)}, x \in [0,\infty).$$

We define and study two generalization operators of summation and of summation-integral types as follows:

For

$$f \in C_\gamma[0,\infty) := \{f \in C[0,\infty): |f(t)| \le M(1+t)^\gamma \text{ for some } M > 0, \gamma > 0\},$$

the summation r-Beta operators are defined as:

$$B_n^{\{r\}}(f;x) = \frac{1}{(n)_r} \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x) f\left(\frac{k}{n+r}\right). \qquad (1.2)$$

The summation-integral r,s-Beta operators are defined as:

$$B_n^{\{r,s\}}(f;x) = \frac{1}{(n)_r (n)_{s-1}} \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t) f(t) dt. \qquad (1.3)$$

Our operators (1.2) and (1.3) are reduce to many operators which are studied from different authors. We refer here to the following operators:

Put $r=0$(or r = 1), the operators (1.2) reduce to the classical Lupas (or classical beta) operators discussed in [(2),(3)]respectively. Put $r=s=0$, the operators (1.3) reduce to the modified Lupas operators discussed in [(4)]. Put $r=0$and $s=1$, the operators (1.3) become the mixed Lupas-Beta type operators given in [(5)]. Put $r=s=1$, the operators (1.3) reduce to the modified beta operators studied in [(6)]. Put $r=1$ *and* $s=0$, the operators (1.3) reduce to the mixed Beta-Lupas type operator studied in [(7)] and so on.

The main object of the present paper is to study the basic pointwise convergence theorems for the operators (1.2) and (1.3) in simultaneous approximation and then proceed to introduceVoronoskaja type asymptotic formulas for the two operators. The basic pointwise convergence theorems and Voronoskaja-type asymptotic formulas for many papers in this field can getting by choose a suitable values of r and/or s. See [(2),(3),(4),(5),(6),(7)] and so on.

2. The operators $B_n^{\{r\}}(f;x)$.

In this section , we study the operators (1.2). We show that these operators converge to $f(x)$belongs to $C_\gamma[0,\infty)$ by applying Korovkin's theorem (see [(8)]). The first our lemma is giving us some properties of the weight functions $\beta_{n,k}^{\{r\}}(x)$.

Lemma 2.1.

For $x\in[0,\infty)$, $m\in N^0$, we get:

$$1)\ \sum_{k=0}^{\infty}\beta_{n,k}^{\{r\}}(x)=(n)_r; \qquad (2.1)$$

$$2)\ \sum_{k=0}^{\infty}k\beta_{n,k}^{\{r\}}(x)=(n)_{r+1}x; \qquad (2.2)$$

$$3)\ \sum_{k=0}^{\infty}k^2\beta_{n,k}^{\{r\}}(x)=(n)_{r+2}x^2+(n)_{r+1}x; \qquad (2.3)$$

$$4)\ x(1+x)\frac{d}{dx}\beta_{n,k}^{\{r\}}(x)=(k-(n+r)x)\beta_{n,k}^{\{r\}}(x); \qquad (2.4)$$

5) For $m\in N^0$, suppose that $\phi_{n,m}^{\{r\}}(x)=\sum_{k=0}^{\infty}k^m\beta_{n,k}^{\{r\}}(x)$, then

$$\phi_{n,m+1}^{\{r\}}(x)=x(1+x)\frac{d}{dx}\phi_{n,m}^{\{r\}}(x)+(n+r)\,x\phi_{n,m}^{\{r\}}(x). \qquad (2.5)$$

Proof.

Using the direct computation and the facts that$\sum_{k=0}^{\infty}P_{n,k}(x)=1$, $\sum_{k=0}^{\infty}kP_{n,k}(x)=nx$ and $\sum_{k=0}^{\infty}k^2P_{n,k}(x)=nx+n(n+1)x^2$ (see [(2)]) the consequences (1), (2) and (3) are easily follow.

To prove (4), using the fact

$$x(1+x)\frac{d}{dx}P_{n,k}(x)=(k-nx)P_{n,k}(x)$$

(see[(4)]), we have:

$$\frac{d}{dx}\beta_{n,k}^{\{r\}}(x)=\frac{(n+k)_r}{(1+x)^r}\{x(1+x)\times\frac{d}{dx}P_{n,k}(x)-rxP_{n,k}(x)\}.$$

From which the consequence (4) is follow. Finally, the proof of (5) follows by using (4) and the direct computations. □

From above Lemma, we can get easily the following facts. Hence, the operators $B_n^{\{r\}}(f,x)$ converge to the function f.

$$1)B_n^{\{r\}}(1,x)=1; \qquad (2.6)$$

$$2)B_n^{\{r\}}(t,x)=x; \qquad (2.7)$$

$$2)B_n^{\{r\}}(t^2,x)=\frac{(n+r+1)}{(n+r)}x^2+\frac{1}{(n+r)}x. \qquad (2.8)$$

For $m\in N^0$, we define the m^{th} order moment $V_{n,m}^{\{r\}}(x)$ of the operators $B_n^{\{r\}}(.,x)$ as:

$$V_{n,m}^{\{r\}}(x)=\frac{1}{(n)_r}\sum_{k=0}^{\infty}\beta_{n,k}^{\{r\}}(x)$$

$$\times\left(\frac{k}{(n+r)}-x\right)^m.$$

Lemma 2.2.

For the functions $V_{n,m}^{\{r\}}(x)$, we have that $V_{n,0}^{\{r\}}(x)=1, V_{n,1}^{\{r\}}(x)=0$ and

$$(n+r)V_{n,m+1}^{\{r\}}(x)=x(1+x)$$
$$\times\left\{\frac{d}{dx}V_{n,m}^{\{r\}}(x)+mV_{n,m-1}^{\{r\}}(x)\right\}. \quad (2.9)$$

Consequently, the function $V_{n,m}^{\{r\}}(x)$ is a polynomial in x of degree m and for each $x\in[0,\infty)$, we have

$$V_{n,m}^{\{r\}}(x)=O\left(n^{-[m+1]/2}\right).$$

Proof.

By direct computation, we can find the values of $V_{n,0}^{\{r\}}(x)$ and $V_{n,1}^{\{r\}}(x)$

Next, we proof the recurrence relation (2.9). Using the equation $x(1+x)\frac{d}{dx}\beta_{n,k}^{\{r\}}(x)=\beta_{n,k}^{\{r\}}(x)(k-(n+r)x)$ (see Lemma (2.1)), we have:

$$x(1+x)\left\{\frac{d}{dx}V_{n,m}^{\{r\}}(x)-mV_{n,m-1}^{\{r\}}(x)\right\}$$
$$=\frac{1}{(n)_r}\sum_{k=0}^{\infty}(k-(n+r)x)\beta_{n,k}^{\{r\}}(x)(t-x)^m$$
$$=\frac{(n+r)}{(n)_r}\sum_{k=0}^{\infty}\left(\frac{k}{(n+r)}-x\right)$$
$$\times\beta_{n,k}^{\{r\}}(x)(t-x)^m.$$

Hence, the consequence (2.9) follows.

Finally, using the induction on m an (2.9), we can easily prove that $V_{n,m}^{\{r\}}(x)$ is a polynomial in x of degree m and

$$V_{n,m}^{\{r\}}(x)=O\left(n^{-[m+1]/2}\right)$$

for all $x\in[0,\infty)$. □

Lemma2.3.

For an integer, $m\geq 1$, we have:

$$B_n^{\{r\}}(t^m,x)=\frac{1}{(n)_r(n+r)^m}$$
$$\times\left[(n)_{r+m}x^m+\frac{m(m-1)}{2}(n)_{r+m-1}x^{m-1}\right]$$
$$+O(n^{-2}).$$

Proof.

Using induction on m and (2.5) our result follows immediately.

Lemma 2.4. [6]

There exist the polynomials $Q_{t,j.i}(x)$ independent of n and k such that

$$\{x(1+x)\}^iD^i\left(\beta_{n,k}^{\{r\}}(x)\right)=\sum_{\substack{2t+j\leq i\\ t,j\geq 0}}(n+r)^t$$
$$\times(k-(n+r)x)^jQ_{t,j.i}(x)\beta_{n,k}^{\{r\}}(x)$$

where $D=\frac{d}{dx}$.

Theorem 2.1.

Let $f\in C_\gamma[0,\infty),\gamma>0$ and $f^{(i)}$ exists at a point $x\in(0,\infty)$ where $i\in N$. Then we have:

$$\left(B_n^{\{r\}}(f;x)\right)^{(i)}\to f^{(i)}(x) \text{ as } n\to\infty.$$

Proof.

Using Taylor's expansion of f at x, we have

$$f(t)=\sum_{m=0}^{i}\frac{f^{(m)}(x)}{m!}(t-x)^m$$
$$+\varepsilon(t,x)(t-x)^i,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

Then,

$$\left(B_n^{\{r\}}(f;x)\right)^{(i)}$$
$$=\sum_{m=0}^{i}\frac{f^{(m)}(x)}{m!}\left(B_n^{\{r\}}((t-x)^m;x)\right)^{(i)}$$
$$+\left(B_n^{\{r\}}\left(\varepsilon(t,x)(t-x)^i;x\right)\right)^{(i)}$$
$$:=E_1+E_2.$$

Now, since $V_{n,m}^{\{r\}}(x)$ is a polynomial in x of degree m (Lemma 2.2) then

$$E_1=\frac{f^{(i)}(x)}{i!}\left(B_n^{\{r\}}\left((t-x)^i;x\right)\right)^{(i)}$$

$$= \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \left(B_n^{\{r\}}(t^j; x)\right)^{(i)}$$
$$= \frac{f^{(i)}(x)}{i!} \left(B_n^{\{r\}}(t^i; x)\right)^{(i)}$$
$$= f^{(i)}(x) \frac{(n)_{r+i}}{(n)_r (n+r)^i} \text{ (in view of Lemma 2.3)}$$
$$\to f^{(i)}(x) \text{as } n \to \infty.$$

Now, we estimate E_2 as follows:

$$E_2 = \left(B_n^{\{r\}}\left(\varepsilon(t,x)(t-x)^i; x\right)\right)^{(i)}$$
$$= \frac{1}{(n)_r} \sum_{k=0}^{\infty} \left(\beta_{n,k}^{\{r\}}(x)\right)^{(i)} \varepsilon(t,x)(t-x)^i$$

From lemma (2.4), we get.

$$E_2 = \sum_{\substack{2t+j\le i \\ t,j\ge 0}} \frac{Q_{t,j.i}(x)}{(n)_r} \frac{(n+r)^t}{\{x(1+x)\}^i}$$
$$\times \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x) \varepsilon(t,x)$$
$$\times (k-(n+r)x)^j (t-x)^i$$
$$\le \sum_{\substack{2t+j\le i \\ t,j\ge 0}} \frac{Q_{t,j.i}(x)}{(n)_r} \frac{(n+r)^{t+j}}{\{x(1+x)\}^i}$$
$$\times \left(\varepsilon \sum_{|t-x|<\delta} \beta_{n,k}^{\{r\}}(x) |t-x|^{i+j} \right.$$
$$\left. + \sum_{|t-x|<\delta} \beta_{n,k}^{\{r\}}(x) |t-x|^{i+j} \varepsilon(t,x) \right)$$
$$E_2 := E_3 + E_4.$$

Since $\varepsilon(t,x) \to 0$ as $t \to x$ for a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$ whenever $0 < |t-x| < \delta$. Further if $\gamma \ge \max\{\alpha, i\}$, where γ isany integer, then we can find a constant $\mathcal{M}_1 > 0$ such that $\left|\varepsilon(t,x)(t-x)^i\right| \le \mathcal{M}_1 (1+t)^\gamma$ for $|t-x| \ge \delta$. Since

$$\sup_{\substack{2t+j\le i \\ t,j\ge 0}} \frac{Q_{t,j.i}(x)}{\{x(1+x)\}^i} := \mathcal{M}(x) = \mathcal{M}_2$$
$$\forall x \in (0,\infty).$$

Thus, applying Schwarz inequality for summation

$$E_3 \le \varepsilon \mathcal{M}_2$$
$$\times \sum_{\substack{2t+j\le i \\ t,j\ge 0}} (n+r)^{t+j} \left(\frac{1}{(n)_r} \sum_{|t-x|<\delta} \beta_{n,k}^{\{r\}}(x) \right)^{\frac{1}{2}}$$
$$\times \left(\frac{1}{(n)_r} \sum_{|t-x|<\delta} \beta_{n,k}^{\{r\}}(x) (t-x)^{2(i+j)} \right)^{1/2}$$
$$\le \mathcal{M}_2 \, \varepsilon \sum_{\substack{2t+j\le i \\ t,j\ge 0}} (n+r)^{t+j} O\left(n^{-\frac{(i+j)}{2}}\right)$$
$$= \varepsilon O(1)$$
$$= \mathrm{o}(1) \text{since } \varepsilon \text{ is arbitrary.}$$
$$E_4 \le \mathcal{M}_1 \mathcal{M}_2 \sum_{\substack{2t+j\le i \\ t,j\ge 0}} \frac{(n+r)^{t+j}}{(n)_r}$$
$$\times \sum_{|t-x|\ge\delta} \beta_{n,k}^{\{r\}}(x)(1+t)^\gamma$$

Applying Schwarz inequality for summation

$$E_4 \le \mathcal{M}_1 \mathcal{M}_2$$
$$\times \sum_{\substack{2t+j\le i \\ t,j\ge 0}} (n+r)^{t+j} \left(\frac{1}{(n)_r} \sum_{|t-x|\ge\delta} \beta_{n,k}^{\{r\}}(x) \right)^{1/2}$$
$$\times \left(\frac{1}{(n)_r} \sum_{|t-x|\ge\delta} \beta_{n,k}^{\{r\}}(x) \, (1+t)^{2\gamma} \right)^{1/2}$$
$$= \mathcal{M} \sum_{\substack{2t+j\le i \\ t,j\ge 0}} (n+r)^{t+j} O(n^{-\mathcal{S}})$$
$$= O\left(n^{t+j-\mathcal{S}}\right).$$

Therefore

$$E_2 = o(1) + O\left(n^{t+j-\mathcal{S}}\right)$$
$$= O\left(n^{t+j-\mathcal{S}}\right)$$
$$= o(1) \text{for some } s > t+j.$$

Then $\left(B_n^{\{r\}}(f;x)\right)^{(i)} \to f^{(i)}(x)$.

Theorem 2.2.

Let$f \in C_\gamma[0,\infty)$, $\gamma > 0$. If$f^{(i+2)}$exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n\left[\left(B_n^{\{r\}}(f;x)\right)^{(i)} - f^{(i)}(x)\right]$$
$$= \frac{i(i-1)}{2} f^{(i)}(x) + ri f^{(i+1)}(x)$$

$$+\frac{x(1+x)}{2}f^{(i+2)}(x)$$

Proof.

By Taylor expansion of f, we have

$$f(t)=\sum_{m=0}^{i+2}\frac{f^{(m)}(x)}{m!}(t-x)^m$$
$$+\varepsilon(t,x)(t-x)^{i+2}$$

where $\varepsilon(t,x)\ \to\ 0$ as $t\ \to\ x$.

$$\left(B_n^{\{r\}}(f;x)\right)^{(i)}$$
$$=\sum_{m=0}^{i+2}\frac{f^{(m)}(x)}{m!}\left(B_n^{\{r\}}((t-x)^m;x)\right)^{(i)}$$
$$+\left(B_n^{\{r\}}\left(\varepsilon(t,x)(t-x)^{i+2};x\right)\right)^{(i)}$$

$:=Q_1+Q_2$.

Then,

$$Q_1-f^{(i)}(x)$$
$$=\sum_{m=0}^{i+2}\frac{f^{(m)}(x)}{m!}\left(B_n^{\{r\}}((t-x)^m;x)\right)^{(i)}$$
$$-f^{(i)}(x)$$
$$=\sum_{m=0}^{i+2}\frac{f^{(m)}(x)}{m!}\sum_{j=0}^{m}\binom{m}{j}(-x)^{m-j}\left(B_n^{\{r\}}(t^j;x)\right)^{(i)}$$
$$-f^{(i)}(x)$$
$$=\sum_{m=i}^{i+2}\frac{f^{(m)}(x)}{m!}\sum_{j=i}^{m}\binom{m}{j}(-x)^{m-j}\left(B_n^{\{r\}}(t^j;x)\right)^{(i)}$$
$$-f^{(i)}(x)$$
$$=\left\{n\frac{f^{(i)}(x)}{i!}\left(B_n^{\{r\}}(t^i;x)\right)^{(i)}\right.$$
$$+n\frac{f^{(i+1)}(x)}{(i+1)!}$$
$$\times\left[-(i+1)x\left(B_n^{\{r\}}(t^i;x)\right)^{(i)}\right.$$
$$\left.+\left(B_n^{\{r\}}(t^{i+1};x)\right)^{(i)}\right]$$
$$+n\frac{f^{(i+2)}(x)}{(i+2)!}$$
$$\times\left[\frac{(i+1)(i+2)}{2}x^2\left(B_n^{\{r\}}(t^i;x)\right)^{(i)}\right.$$
$$-(i+2)x\left(B_n^{\{r\}}(t^{i+1};x)\right)^{(i)}$$
$$\left.\left.\times\left[\left(B_n^{\{r\}}(t^{i+2};x)\right)^{(i)}\right]\right]\right\}-f^{(i)}(x)$$
$$=nf^{(i)}(x)\left\{\frac{(n)_{r+i}}{(n)_r(n+r)^i}-1\right\}$$
$$+nf^{(i+1)}(x)\left\{-\frac{(n)_{r+i}}{(n)_r(n+r)^i}x\right.$$
$$\left.+\frac{(n)_{r+i+1}}{(n)_r(n+r)^{i+1}}x+\frac{(n)_{r+i}}{2(n)_r(n+r)^{i+1}}i\right\}$$
$$+nf^{(i+2)}(x)\left\{\frac{(n)_{r+i}}{2(n)_r(n+r)^i}x^2\right.$$
$$-\frac{(n)_{r+i+1}}{(n)_r(n+r)^{i+1}}x^2-\frac{(n)_{r+i}}{2(n)_r(n+r)^{i+1}}ix$$
$$+\frac{(n)_{r+i+2}}{2(n)_r(n+r)^{i+2}}x^2$$
$$\left.+\frac{(n)_{r+i+1}}{2(n)_r(n+r)^{i+2}}(i+1)x\right\}+O(n^{-2})$$
$$=\frac{i(i-1)}{2}f^{(i)}(x)+rif^{(i+1)}(x)$$
$$+\frac{x(1+x)}{2}f^{(i+2)}(x)\text{as } n\to\infty.$$

The uniformity assertion follows as in the proof of Theorem 2.1. □

3. The operator $B_n^{\{r,s\}}(f;x)$.

In this section, we study the summation-Integral r-Beta (s-Beta) type operators (1.3)

It can be easily verify that the operators defined above are linear positive operators and that $B_n^{\{r,s\}}(1;x)=1$.

Now, we show that the operators $B_n^{\{r,s\}}(f;x)$ converge to $f\ (x)$ in $C_\alpha[0,\infty)$ by applying Korovkin's theorem[8]:

Lemma 3.1.

For $x\in[0,\infty)$, if the following conditions hold:

$$(1)B_n^{\{r,s\}}(1;x)=1; \quad (3.1)$$
$$(2)B_n^{\{r,s\}}(t;x)=\frac{(n)_{s-2}}{(n)_{s-1}}(1+(n+r)x)$$
$$=x\text{as } n\to\infty; \quad (3.2)$$
$$(3)B_n^{\{r,s\}}(t^2;x)$$
$$=\frac{(n)_{s-3}}{(n)_{s-1}}(2+4(n+r)x$$
$$+(n+r)_2x^2)$$

$$= x^2 \text{ as } n \to \infty \qquad (3.3)$$

Proof.

By using the integration by parts, we can easily get the result.

$$\int_0^\infty \beta_{n,k}^{\{s\}}(t)\, t^m dt = (k+1)_m (n)_{s-m-1}. \quad (3.4)$$

1)When $m = 0$, by the formulas (2.1), (3.4), we can get:

$$B_n^{\{r,s\}}(1;x) = \frac{(n)_{s-1}}{(n)_r(n)_{s-1}} \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x) = 1.$$

2)When $m = 1$, by the formulas (2.1),(2.2) and (3.4), we can get:

$$B_n^{\{r,s\}}(t;x)$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x)\,(k+1)_1 (n)_{s-2}$$
$$= \frac{(n)_{s-2}}{(n)_r(n)_{s-1}} \left(\sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x) + \sum_{k=0}^{\infty} k\beta_{n,k}^{\{r\}}(x) \right)$$
$$= \frac{(n)_{s-2}}{(n)_r(n)_{s-1}} ((n)_r + (n)_{r+1} x)$$
$$= \frac{(n)_{s-2}}{(n)_{s-1}} (1 + (n+r)x).$$

3)When $m = 3$, by the formulas (2.1),(2.2), (2.3) and (3.4), we can get:

$$B_n^{\{r,s\}}(t^2;x)$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x)\,(k+1)_2 (n)_{s-3}$$
$$= \frac{(n)_{s-3}}{(n)_r(n)_{s-1}} \left\{ 2 \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x) + 3 \sum_{k=0}^{\infty} k\beta_{n,k}^{\{r\}}(x) + \sum_{k=0}^{\infty} k^2 \beta_{n,k}^{\{r\}}(x) \right\}$$
$$= \frac{(n)_{s-3}}{(n)_r(n)_{s-1}} \{2(n)_r + 3(n)_{r+1} x + (n)_{r+1} x + (n)_{r+2} x^2\}$$
$$= \frac{(n)_{s-3}}{(n)_{s-1}} \{2 + 4(n+r)x + (n+r)_2 x^2\}. \quad \square$$

Lemma 3.2.

The m-th order moment $T_{n,m}^{\{r,s\}}(x)$ for the summation-Integral of mixed r-Beta and s-Beta type operators defined as:

$$T_{n,m}^{\{r,s\}}(x) = B_n^{\{r,s\}}((t-x)^m; x)$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt.$$

And has the following properties:

$$T_{n,0}^{\{r,s\}}(x) = 1,$$
$$T_{n,1}^{\{r,s\}}(x) = \frac{(n)_{s-2}}{(n)_{s-1}} + \left(\frac{(n)_{s-2}(n+r)}{(n)_{s-1}} - 1 \right) x$$

and there the recurrence relation holds:

$$(n+s-m-2) T_{n,m+1}^{\{r,s\}}(x) = (n+r)x T_{n,m}^{\{r,s\}}(x) - (n+s)x T_{n,m}^{\{r,s\}}(x) + (1+2x)(m+1) T_{n,m}^{\{r,s\}}(x) + x(1+x)\frac{d}{dx} T_{n,m}^{\{r,s\}}(x) + 2mx(1+x) T_{n,m-1}^{\{r,s\}}(x). \qquad (3.5)$$

Further, we have:

1) $T_{n,m}^{\{r,s\}}(x)$is a polynomial in x of degree $\leq m$ and is a polynomial in $n^{-1}$of degree m.

2)for every $x \in [0,\infty)$,

$$T_{n,m}^{\{r,s\}}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right),$$

where$\left[\frac{m+1}{2}\right]$ denotes the integer part of the value $\frac{m+1}{2}$.

Proof:

Using Lemma (3.1).

$$T_{n,0}^{\{r,s\}}(x) = B_n^{\{r,s\}}(1;x) = 1;$$
$$T_{n,1}^{\{r,s\}}(x) = B_n^{\{r,s\}}((t-x);x)$$
$$= B_n^{\{r,s\}}(t;x) - x B_n^{\{r,s\}}(1;x)$$
$$= \frac{(n)_{s-2}}{(n)_{s-1}} + \left(\frac{(n)_{s-2}(n+r)}{(n)_{s-1}} - 1 \right) x;$$
$$T_{n,m}^{\{r,s\}}(x) = \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^{\infty} \beta_{n,k}^{\{r\}}(x)$$

$$\times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$

$$\frac{d}{dx}T_{n,m}^{\{r,s\}}(x)$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \frac{d}{dx}\beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$- \frac{m}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^{m-1} dt$$

$$x(1+x)\left(\frac{d}{dx}T_{n,m}^{\{r,s\}}(x) + mT_{n,m-1}^{\{r,s\}}(x)\right)$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty x(1+x)\frac{d}{dx}\beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x)(k-(n+r)x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty (k-(n+s)t)\beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$+ \frac{(n+s)}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty t\beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$- \frac{(n+r)x}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty t(1+t)\frac{d}{dx}\beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$+ \frac{(n+s)}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^{m+1} dt$$
$$+ \frac{(n+s)x}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$-(n+r)xT_{n,m}^{\{r,s\}}(x)$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \{(t-x+x)+(t-x+x)^2\} \times \frac{d}{dx}\beta_{n,k}^{\{s\}}(t)(t-x)^m dt$$
$$-(n+r)xT_{n,m}^{\{r,s\}}(x) +(n+s)T_{n,m+1}^{\{r,s\}}(x) +(n+s)xT_{n,m}^{\{r,s\}}(x)$$
$$= \frac{1}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \frac{d}{dx}\beta_{n,k}^{\{s\}}(t)\,(t-x)^{m+2} dt$$
$$+ \frac{(1+2x)}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \frac{d}{dx}\beta_{n,k}^{\{s\}}(t)\,(t-x)^{m+1} dt$$
$$+ \frac{x(1+x)}{(n)_r(n)_{s-1}} \sum_{k=0}^\infty \beta_{n,k}^{\{r\}}(x) \times \int_0^\infty \frac{d}{dx}\beta_{n,k}^{\{s\}}(t)\,(t-x)^m dt$$
$$-(n+r)xT_{n,m}^{\{r,s\}}(x) +(n+s)T_{n,m+1}^{\{r,s\}}(x) +(n+s)xT_{n,m}^{\{r,s\}}(x)$$
$$= -(m+2)T_{n,m+1}^{\{r,s\}}(x) -(1+2x)(m+1)T_{n,m}^{\{r,s\}}(x)$$

$$-mx(1+x)T_{n,m-1}^{\{r,s\}}$$
$$-(n+r)xT_{n,m}^{\{r,s\}}(x)$$
$$+(n+s)T_{n,m+1}^{\{r,s\}}(x)$$
$$+(n+s)xT_{n,m}^{\{r,s\}}(x).$$

Hence,

$$(n+s-m-2)T_{n,m+1}^{\{r,s\}}(x)$$
$$=(s+r)xT_{n,m}^{\{r,s\}}(x)$$
$$+(1+2x)(m+1)T_{n,m}^{\{r,s\}}(x)$$
$$+x(1+x)\frac{d}{dx}T_{n,m}^{\{r,s\}}(x)$$
$$+2mx(1+x)T_{n,m-1}^{\{r,s\}}(x). \quad \square$$

Lemma 3.3.

Thenthe recurrence relation is

$$(n+s-m-2)B_n^{\{r,s\}}(t^{m+1};x)$$
$$=x(1+x)\frac{d}{dx}B_n^{\{r,s\}}(t^m;x)+((n+r)x$$
$$+(m+1))B_n^{\{r,s\}}(t^m;x) \quad (3.6). \quad \square$$

Proof:

By direct computation and using Lemma 3.2.

Lemma 3.4. For $m \geq 1$, we have:

$$B_n^{\{r,s\}}(t^m;x)=\frac{(n)_{s-(m+1)}}{(n)_{s-1}}\{(n+r)_m x^m$$
$$+m^2(n+r)_{m-1}x^{m-1}\}+O(n^{-2}).$$

Proof:

The result is true for $m = 1$. We prove the result by induction method. Suppose that the result is true for m, then

$$B_n^{\{r,s\}}(t^m;x)=\frac{(n)_{s-(m+1)}}{(n)_{s-1}}\{(n+r)_m x^m$$
$$+m^2(n+r)_{m-1}x^{m-1}\}+O(n^{-2})$$

Thus using the identities (3.6), we have

$$(n+s-m-2)B_n^{\{r,s\}}(t^{m+1};x)$$
$$=x(1+x)\frac{d}{dx}B_n^{\{r,s\}}(t^m;x)$$
$$+\big((n+r)x+(m+1)\big)B_n^{\{r,s\}}(t^m;x);$$
$$=\frac{(n)_{s-(m+1)}}{(n)_{s-1}}[(x+x^2)\{m(n+r)_m x^{m-1}$$
$$+m^2(m-1)(n+r)_{m-1}x^{m-2}\}$$
$$+\big((n+r)x+(m+1)\big)\{(n+r)_m x^m$$
$$+m^2(n+r)_{m-1}x^{m-1}\}]+O(n^{-2})$$
$$=\frac{(n)_{s-(m+1)}}{(n)_{s-1}}[m(n+r)_m x^m$$
$$+m(n+r)_m x^{m+1}$$
$$+m^2(m-1)(n+r)_{m-1}x^m$$
$$+(n+r)(n+r)_m x^{m+1}$$
$$+m^2(n+r)(n+r)_{m-1}x^m$$
$$+(m+1)(n+r)_m x^m]+O(n^{-2})$$
$$=\frac{(n)_{s-(m+1)}}{(n)_{s-1}}[(m+n+r)(n+r)_m x^{m+1}$$
$$+(2m+1)(n+r)_m x^m$$
$$+(n+r+m-1)m^2(n+r)_{m-1}x^m]$$
$$+O(n^{-2})$$
$$B_n^{\{r,s\}}(t^{m+1};x)$$
$$=\frac{(n)_{s-(m+2)}}{(n)_{s-1}}[(n+r)_{m+1}x^{m+1}$$
$$+(2m+1)(n+r)_m x^m+m^2(n+r)_m x^m]$$
$$+O(n^{-2})$$
$$B_n^{\{r,s\}}(t^{m+1};x)$$
$$=\frac{(n)_{s-(m+2)}}{(n)_{s-1}}[(n+r)_{m+1}x^{m+1}$$
$$+(m+1)^2(n+r)_m x^m]+O(n^{-2}).$$

This completes the proof of the lemma. $\square$

Next, study the rate of point wise convergence, an asymptotic formula and errorestimation in terms of higher order modulus of continuity in simultaneous approximation for the operators (1.3).

Theorem 3.1.

Let $f \in C_\gamma[0,\infty), \gamma > 0,$ and $f^{(i)}$exists at a point $x \in (0,\infty)$. Then we have

$$\left(B_n^{\{r,s\}}(f;x)\right)^{(i)} = f^{(i)}(x) \text{as} n \to \infty.$$

Proof. By Taylor expansion of f, we have

$$f(t)=\sum_{m=0}^{i}\frac{f^{(m)}(x)}{m!}(t-x)^m+\varepsilon(t,x)(t-x)^i,$$
where $\varepsilon(t,x)\to 0 \; as \; t\to x$.
Now, we get
$$B_n^{\{r,s\}}(f;x)=B_n^{\{r,s\}}\left(\sum_{m=0}^{i}\frac{f^{(m)}(x)}{m!}(t-x)^m+\varepsilon(t,x)(t-x)^i;x\right)$$
$$\left(B_n^{\{r,s\}}(f;x)\right)^{(i)}=\sum_{m=0}^{i}\frac{f^{(m)}(x)}{m!}\left(B_n^{\{r,s\}}((t-x)^m;x)\right)^{(i)}+\left(B_n^{\{r,s\}}(\varepsilon(t,x)(t-x)^i;x)\right)^{(i)}$$
$$:=U_1+U_2$$
First, to estimate U_1, using binomial expansion of $(t-x)^m$, and Lemmas(2.2),(3.2), we have:
$$U_1=\sum_{m=0}^{i}\frac{f^{(m)}(x)}{m!}\left(B_n^{\{r,s\}}((t-x)^m;x)\right)^{(i)},$$
$$=\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\left(B_n^{\{r,s\}}(t^j;x)\right)^{(i)}$$
$$=\frac{f^{(i)}(x)}{i!}\left(B_n^{\{r,s\}}(t^i;x)\right)^{(i)}$$
$$=\frac{f^{(i)}(x)}{i!}\frac{(n)_{s-(i+1)}}{(n)_{s-1}}\{(n+r)_i x^i+\text{terms in lower powers of } x\}^{(i)}$$
$$\to f^{(i)}(x) \; as \; n\to\infty.$$
$$U_2=\left(B_n^{\{r,s\}}(\varepsilon(t,x)(t-x)^i;x)\right)^{(i)}$$
$$=\frac{1}{(n)_r(n)_{s-1}}\sum_{k=0}^{\infty}\left(\beta_{n,k}^{\{r\}}(x)\right)^{(i)}\times\int_0^\infty\beta_{n,k}^{\{s\}}(t)\,\varepsilon(t,x)(t-x)^i dt$$
Next, using Lemma (2.4), we obtain
$$U_2=\sum_{\substack{2t+j\le i\\ t,j\ge 0}}\frac{Q_{t,j,i}(x)}{(n)_r(n)_{s-1}}\frac{(n+r)^t}{\{x(1+x)\}^i}\times\sum_{k=0}^{\infty}[k-(n+r)x]^j\,\beta_{n,k}^{\{r\}}(x)\times\int_0^\infty\beta_{n,k}^{\{s\}}(t)\,\varepsilon(t,x)(t-x)^i dt$$
Since $\varepsilon(t,x)\to 0$ as $t\to x$ for a given $\varepsilon>0$ there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. Further if $\gamma\ge\max\{\alpha,i\}$, where γ isany integer, then we can find a constant $M_1>0$ such that $\left|\varepsilon(t,x)(t-x)^i\right|\le M_1(1+t)^\gamma$ for $|t-x|\ge\delta$.

Since,
$$\sup_{\substack{2t+j\le i\\ t,j\ge 0}}\frac{Q_{t,j,i}(x)}{\{x(1+x)\}^i}:=M(x)=M_2\;\forall x\in(0,\infty).$$
Thus,
$$U_2\le M_2\sum_{\substack{2t+j\le i\\ t,j\ge 0}}\frac{(n+r)^t}{(n)_r(n)_{s-1}}\times\sum_{k=0}^{\infty}|k-(n+r)x|^j\,\beta_{n,k}^{\{r\}}(x)$$
$$\times\left[\varepsilon\int_{|t-x|<\delta}\beta_{n,k}^{\{s\}}(t)\,|t-x|^i dt+\int_{|t-x|\ge\delta}\beta_{n,k}^{\{s\}}(t)\left|\varepsilon(t,x)(t-x)^i\right|dt\right]$$
$$:=U_3+U_4$$
$$U_3\le\varepsilon M_2\sum_{\substack{2t+j\le i\\ t,j\ge 0}}(n+r)^t\times\sum_{k=0}^{\infty}\frac{\beta_{n,k}^{\{r\}}(x)}{(n)_r(n)_{s-1}}|k-(n+r)x|^j\times\int_{|t-x|<\delta}\beta_{n,k}^{\{s\}}(t)\,|t-x|^i dt$$
Applying Schwarz inequality for integration
$$U_3\le\varepsilon M_2\sum_{\substack{2t+j\le i\\ t,j\ge 0}}(n+r)^t\times\sum_{k=0}^{\infty}\frac{\beta_{n,k}^{\{r\}}(x)}{(n)_r(n)_{s-1}}|k-(n+r)x|^j$$

$$\times\left(\int_{|t-x|<\delta}\beta_{n,k}^{\{s\}}(t)\,dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_{|t-x|<\delta}\beta_{n,k}^{\{s\}}(t)\,(t-x)^{2i}dt\right)^{1/2}.$$
Applying Schwarz inequality for summation
$$U_3 \le \varepsilon M_2 \sum_{\substack{2t+j\le i\\ t,j\ge 0}}(n+r)^t$$
$$\times\left((n)_{s-1}\sum_{k=0}^{\infty}\frac{\beta_{n,k}^{\{r\}}(x)}{(n)_r(n)_{s-1}}\right.$$
$$\left.\times(k-(n+r)x)^{2j}\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{k=0}^{\infty}\frac{\beta_{n,k}^{\{r\}}(x)}{(n)_r(n)_{s-1}}\right.$$
$$\left.\times\int_{|t-x|<\delta}\beta_{n,k}^{\{s\}}(t)\,(t-x)^{2i}dt\right)^{\frac{1}{2}}$$
by using Lemmas (2.2)and (3.2), we obtain
$$U_3 \le \varepsilon M_2\left(n^{-\frac{i}{2}}\right)\sum_{\substack{2t+j\le i\\ t,j\ge 0}}(n+r)^t\left(n^{\frac{j}{2}}\right)$$
$$=\varepsilon O(1)=\mathrm{o}(1).$$
Since $\varepsilon>0$ and arbitrariness then,
$$U_4 \le M_1\,M_2\sum_{\substack{2t+j\le i\\ t,j\ge 0}}(n+r)^t$$
$$\times\sum_{k=0}^{\infty}\frac{\beta_{n,k}^{\{r\}}(x)}{(n)_r(n)_{s-1}}|k-(n+r)x|^j$$
$$\times\int_{|t-x|\ge\delta}\beta_{n,k}^{\{s\}}(t)\,(1+t)^{\gamma}dt$$
Using the Schwarz inequality of integral and summation
$$U_4 \le M_1\,M_2\sum_{\substack{2t+j\le i\\ t,j\ge 0}}(n+r)^t$$
$$\times\sum_{k=0}^{\infty}\frac{\beta_{n,k}^{\{r\}}(x)}{(n)_r(n)_{s-1}}|k-(n+r)x|^j$$
$$\times\left(\int_{|t-x|\ge\delta}\beta_{n,k}^{\{s\}}(t)\,dt\right)^{\frac{1}{2}}$$
$$\times\left(\int_{|t-x|\ge\delta}\beta_{n,k}^{\{s\}}(t)\,(1+t)^{2\gamma}dt\right)^{1/2}$$
$$\le M_1\,M_2\sum_{\substack{2t+j\le i\\ t,j\ge 0}}(n+r)^t$$
$$\times\left((n)_{s-1}\sum_{k=0}^{\infty}\frac{\beta_{n,k}^{\{r\}}(x)}{(n)_r(n)_{s-1}}(k\right.$$
$$\left.-(n+r)x)^{2j}\right)^{\frac{1}{2}}$$
$$\left(\sum_{k=0}^{\infty}\frac{\beta_{n,k}^{\{r\}}(x)}{(n)_r(n)_{s-1}}\right.$$
$$\left.\times\int_{|t-x|\ge\delta}\beta_{n,k}^{\{s\}}(t)\,(1+t)^{2\gamma}dt\right)^{\frac{1}{2}}$$
$$U_4 \le M_1\,M_2$$
$$\times\sum_{\substack{2t+j\le i\\ t,j\ge 0}}(n+r)^t\left(n^{\frac{j}{2}}\right)O\left(n^{-\frac{m}{2}}\right)$$
$$=\mathrm{o}(1), when\ m>i.$$
Finally collection the estimates of U_3&U_4, we get the required result. □

Theorem 3.2.

Let$f\in C_\gamma[0,\infty)$,$\gamma>0$. If$f^{(i+2)}(x)$exists at a point $x\in(0,\infty)$, then
$$\lim_{n\to\infty} n\left[\left(B_n^{\{r,s\}}(f;x)\right)^{(i)}-f^{(i)}(x)\right]$$
$$=\{\quad\}\left\{\left[ir+\frac{i(i-1)}{2}\right]\right.$$
$$\left.-\left[is-\left(\frac{(i+1)(i+2)}{2}-1\right)\right]\right\}f^{(i)}(x)$$
$$+\{x(2i+r-s+2)+(i+1)\}f^{(i+1)}(x)$$
$$+x(x+1)f^{(i+2)}(x)$$
Proof:

By Taylor expansion of f, we have
$$f(t)=\sum_{m=0}^{i+2}\frac{f^{(m)}(x)}{m!}(t-x)^m$$
$$+\varepsilon(t,x)(t-x)^{i+2}$$
where$\varepsilon(t,x)\to 0$ as $t\to x$.

$$\left(B_n^{\{r,s\}}(f;x)\right)^{(i)}$$
$$=\sum_{m=0}^{i+2}\frac{f^{(m)}(x)}{m!}\left(B_n^{\{r,s\}}((t-x)^m;x)\right)^{(i)}$$
$$+\left(B_n^{\{r,s\}}\left(\varepsilon(t,x)(t-x)^{i+2};x\right)\right)^{(i)}$$
$$:=W_1+W_2$$
$$W_1-f^{(i)}(x)=\sum_{m=0}^{i+2}\frac{f^{(m)}(x)}{m!}$$
$$\times\left(B_n^{\{r,s\}}((t-x)^m;x)\right)^{(i)}-f^{(i)}(x)$$
$$=\sum_{m=0}^{i+2}\frac{f^{(m)}(x)}{m!}$$
$$\times\sum_{j=0}^{m}\binom{m}{j}(-x)^{m-j}\left(B_n^{\{r,s\}}(t^j;x)\right)^{(i)}$$
$$-f^{(i)}(x)$$
$$=\sum_{m=i}^{i+2}\frac{f^{(m)}(x)}{m!}$$
$$\times\sum_{j=i}^{m}\binom{m}{j}(-x)^{m-j}\left(B_n^{\{r,s\}}(t^j;x)\right)^{(i)}$$
$$-f^{(i)}(x)$$
$$=\left\{\quad\frac{f^{(i)}(x)}{i!}\left(B_n^{\{r,s\}}(t^i;x)\right)^{(i)}\right.$$
$$+\frac{f^{(i+1)}(x)}{(i+1)!}\left[-(i+1)x\left(B_n^{\{r,s\}}(t^i;x)\right)^{(i)}\right.$$
$$\left.+\left(B_n^{\{r,s\}}(t^{i+1};x)\right)^{(i)}\right]$$
$$+\frac{f^{(i+2)}(x)}{(i+2)!}\left[\frac{(i+1)(i+2)}{2}x^2\left(B_n^{\{r,s\}}(t^i;x)\right)^{(i)}\right.$$
$$-(i+2)x\left(B_n^{\{r,s\}}(t^{i+1};x)\right)^{(i)}$$
$$\left.\left.+\left(B_n^{\{r,s\}}(t^{i+2};x)\right)^{(i)}\right]\quad\right\}$$
$$-f^{(i)}(x)$$
$$=f^{(i)}(x)\left\{\frac{(n)_{s-(i+1)}}{(n)_{s-1}}(n+r)_i-1\right\}$$
$$+f^{(i+1)}(x)\left\{-\frac{(n)_{s-(i+1)}}{(n)_{s-1}}x\,(n+r)_i\right.$$
$$+\{(n+r)_{i+1}x+(i+1)(n+r)_i\}$$
$$\left.\times\frac{(n)_{s-(i+2)}}{(n)_{s-1}}\right\}$$
$$+f^{(i+2)}(x)\left\{\frac{x^2}{2}\frac{(n)_{s-(i+1)}}{(n)_{s-1}}(n+r)_i\right.$$
$$-\frac{(n)_{s-(i+2)}}{(n)_{s-1}}\{(n+r)_{i+1}x^2+(n+r)_ix\}$$
$$+\left\{\frac{(n)_{s-(i+3)}}{(n)_{s-1}}\left\{\frac{(n+r)_{i+2}}{2}x^2\right.\right.$$
$$\left.+(i+2)(n+r)_{i+1}x\right\}$$
$$+O(n^{-2}).$$

The uniformity assertion follows as in the proof of Theorem (3.1). □

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حول تعميم مؤثر بيتا

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الخلاصة

في بحثنا هذا، قدمنا تعميم لدوال وزن بيتا معتمدا عدد صحيحا غير سالب r سمي $r-$Beta. هذا التعريف يمكن قصره الى دوال وزن Lupas العادية و دوال وزن Beta العادية متى ما كان $r = 0,1$ على الترتيب. كذلك استخدمنا دوال الوزن هذه مؤثرين من النمط مجموع و مجموع-تكامل. بالتأكيد هذين المؤثرين يقتصران الى المؤثرات الاعتيادية لـ Lupasو Beta ولكلا النمطين مجموع ومجموع –تكامل متى ما كان $r = 0,1$ على الترتيب. كذلك يمكن إيجاد مؤثرات أخرى ممزوجة مثل Lupas-Betaو Beta-Lupas من المؤثرات الجديدة باعطاء قيم مناسبة صحيحة. واكثر من ذلك، قمنا باشتقاق الصيغة المشابهه لـ Voronovaskaja لهذه المؤثرات الجديدة ومنها يمكننا إيجاد الصيغ المشابهه لكثير من المؤثرات من النمطين مجموع ومجموع –تكامل ولمزج من دوال وزن (Lupas-Beta (or Beta-Lupas واكثر من ذلك.