Some Properties of a Class of Univalent Functions defined by Integral Operator

Rafid Habib Buti
Department of Mathematics and Computer Applications, College of Sciences
University of Al-Muthanna, E-mail Rafidhb@yahoo.com

Abstract

In this paper, we introduce some properties of a new subclass of univalent functions defined by integral operator, we obtain some closure theorems connected of the class.

1 . Introduction:

Let R denoted the class of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,(n\in\{1,2,...\})\quad(1)\ .$$

Which are analytic and univalent function in the unit disk :

$$U=\{z\in C:|z|<1\}$$

We defined SA the subclass of R consisting of the function defined by the form :

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,(a_n\geq 0,n\in\{1,2,...\})(2)$$

Let $f\in S$

$$g(z)=z-\sum_{n=2}^{\infty}b_nz^n,(b_n\geq 0,n\in\{1,2,...\})$$

Then the Hadamard Product ((or Convolution)) defined by

$$(f*g)(z)=z-\sum_{n=2}^{\infty}a_nb_nz^n=(g*f)(z)(3)$$

Lemma 1: The generalized integral operator is defined by the form

$$F_m^{\alpha,c}(z)=\frac{\alpha cm+1}{z^{\alpha cm}}\int_0^z\delta^{\alpha cm-1}f(\delta)d\delta\ (4)$$

$$.=z-\sum_{n=2}^{\infty}\left(\frac{\alpha cm+1}{\alpha cm+n}\right)a_nz^n$$

Where $\alpha cm>-1$, $f(z)$ defined by (2)

Definition 1: A function $f\in SA$ is said to be in the class $SA_m^{\alpha,c}(\alpha,\beta,\lambda,\mu,k,A,\theta)$

if and only if satisfies the condition

$$\left|\frac{\mu\beta\left[z^2\left(F_m^{\alpha,c}(z)\right)''-\theta z\left(F_m^{\alpha,c}(z)\right)'-z\right]}{k(1-\mathrm{A})+\lambda\left[z^2\left(F_m^{\alpha,c}(z)\right)''-\theta z\left(F_m^{\alpha,c}(z)\right)'-z\right]}\right|<1\,(5)$$

Where $\alpha cm>-1,k>0,0\leq \mathrm{A}<1,\lambda\geq 0,$
$\beta\geq 0,\mu>0,\theta\geq 0,z\in U$

Some anther classes study by[(1),(2),(3),(4),and (5)]

Main Result:

Theorem 1 : A function $f\in SA_m^{\alpha,c}(\mu,\beta,\lambda,k,\mathrm{A},\theta)$ if and only if

$$\sum_{n=2}^{\infty}\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)(\mu\beta+\lambda)a_n\leq k(1-\mathrm{A})\,(6)$$

.

(6)

Proof : Assume that the inequality (6)holds true and let $|z|=1$, then from (5),we have

$$\left|\mu\beta\left[z^2\left(F_m^{\alpha,c}(z)\right)''-\theta z\left(F_m^{\alpha,c}(z)\right)'+\theta z\right]\right|-$$

$$\left|k(1-\mathrm{A})+\lambda\left[z^2\left(F_m^{\alpha,c}(z)\right)''-\theta z\left(F_m^{\alpha,c}(z)\right)'+\theta z\right]\right|$$

$$=\left|\mu\beta\left[-\sum_{n=2}^{\infty}\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1)a_nz^n-\theta z+\sum_{n=2}^{\infty}\left(\frac{\alpha cm+1}{\alpha cm+n}\right)\theta na_nz^n+\theta z\right]\right|-$$

$$\left|k(1-\mathrm{A})+\lambda\left[-\sum_{n=2}^{\infty}\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1)a_nz^n-\theta z+\sum_{n=2}^{\infty}\left(\frac{\alpha cm+1}{\alpha cm+n}\right)\theta na_nz^n+\theta z\right]\right|$$

$$=\left|-\sum_{n=2}^{\infty}\mu\beta\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)a_nz^n\right|-$$

$$\left|k(1-A)-\sum_{n=2}^{\infty}\lambda\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)a_nz^n\right|$$

$$=\sum_{n=2}^{\infty}\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)(\mu\beta+\lambda)a_n-k(1-\mathrm{A})\le 0$$

.

Conversely, suppose that f is in the class $SA_m^{\alpha,c}(\mu,\beta,\lambda,k,\mathrm{A},\theta)$.Then from (5),we have

$$\left|\frac{\mu\beta\left[z^2\left(F_m^{\alpha,c}(z)\right)''-\theta\left(z\left(F_m^{\alpha,c}(Z)\right)'-z\right)\right]}{k(1-\mathrm{A})+\lambda\left[\left(F_m^{\alpha,c}(z)\right)''-\theta\left(z\left(F_m^{\alpha,c}(z)\right)'-z\right)\right]}\right|$$

$$=\left|\frac{\sum_{n=2}^{\infty}\mu\beta\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)a_nz^n}{k(1-\mathrm{A})-\sum_{n=2}^{\infty}\lambda\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)a_nz^n}\right|<1$$

Since $|\mathrm{Re}(z)|\le|z|$,We have

Choose the value of z on the real axis

$$\mathrm{Re}\left\{\frac{\sum_{n=2}^{\infty}\mu\beta\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)a_nz^n}{k(1-\mathrm{A})-\sum_{n=2}^{\infty}\lambda\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)a_nz^n}\right\}<1$$

and let $z\to 1$, we get

$$\sum_{n=2}^{\infty}\mu\beta\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)a_n\le$$

$$k(1-\mathrm{A})-\sum_{n=2}^{\infty}\lambda\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)a_n$$

So

$$\sum_{n=2}^{\infty}\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)(\mu\beta+\lambda)a_n\le k(1-\mathrm{A}).$$

This completes the proof .

Corollary 1 :Let $f\in SA_m^{\alpha,c}(\mu,\beta,\lambda,k,A,\theta)$, then

$$a_n\le\frac{k(1-A)}{\left(\frac{\alpha cm+1}{\alpha cm+n}\right)n(n-1-\theta)(\mu\beta+\lambda)}$$

.

Where $\theta>1$, $\mu\ge 0$, $\beta\ge 0$, $\lambda>0$, $0\le\alpha<1$.

Theorem 2 ::Let $f\in SA_m^{\alpha,c}(\mu,\beta,\lambda,k,A,\theta)$.then f is univalent convex in $|z|<R_2$ of order δ, $0\le\delta<1$, where

$$|z|\le\left\{\frac{(1-\delta)\left(\frac{\alpha cm+1}{\alpha cm+n}\right)(n-1-\theta)(\mu\beta+\lambda)}{(n-\delta)k(1-A)}\right\}^{\frac{1}{n-1}} \quad (7$$

Proof: f is univalent convex of order δ, $0\le\delta<1$ if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\delta .$$

Thus it is enough to show that

$$\left|\frac{zf''(z)}{f'(z)}\right|=\left|\frac{-\sum_{n=2}^{\infty}n(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}na_n z^{n-1}}\right|$$

$$\leq\frac{\sum_{n=2}^{\infty}n(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}na_n|z|^{n-1}} .$$

Thus

$$\left|\frac{zf''(z)}{f'(z)}\right|\leq 1-\delta \qquad \text{if}$$

$$\sum_{n=2}^{\infty}\frac{n(n-\delta)a_n|z|^{n-1}}{1-\delta}\leq 1 \quad (8) \quad .$$

Hence ,by the theorem 1 , if

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1}\leq\frac{n\left(\frac{\alpha cm+1}{\alpha cm+n}\right)(n-1-\theta)(\mu\beta+\lambda)}{k(1-A)}$$

or

$$|z|\leq\left\{\frac{(1-\delta)\left(\frac{\alpha cm+1}{\alpha cm+n}\right)(n-1-\theta)(\mu\beta+\lambda)}{(n-\delta)k(1-A)}\right\}^{\frac{1}{n-1}}$$

.

This completes the proof .

Theorem 3: Let $f\in SA_m^{\alpha,c}(\mu,\beta,\lambda,k,A,\theta)$. then f is univalent closed – to – convex in $|z|<R_3$ of order δ , $0\leq\delta<1$, where

$$|z|\leq\left\{\frac{\left(\frac{\alpha cm+1}{\alpha cm+n}\right)(n-1-\theta)(\mu\beta+\lambda)(1-\delta)}{k(1-A)}\right\}^{\frac{1}{n-1}} \quad (9)$$

Proof: f is univalent closed – to – convex of order δ , $0\leq\delta<1$ if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\delta$$

Thus it is enough to show that

$$|f'(z)-1|=\left|-\sum_{n=2}^{\infty}na_n z^{n-1}\right|\leq\sum_{n=2}^{\infty}na_n|z|^{n-1}$$

.

Thus

$$|f'(z)-1|\leq 1-\delta \qquad \text{if}$$

$$\sum_{n=2}^{\infty}\frac{na_n|z|^{n-1}}{1-\delta}\leq 1$$

(10)

(10)

Hence, by the theorem 1 and (10),will be true if

$$\frac{n|z|^{n-1}}{1-\delta}\leq\frac{n\left(\frac{\alpha cm+1}{\alpha cm+n}\right)(n-1-\theta)(\mu\beta+\lambda)}{k(1-A)}$$

,

or if

$$|z|\leq\left\{\frac{\left(\frac{\alpha cm+1}{\alpha cm+n}\right)(n-1-\theta)(\mu\beta+\lambda)(1-\delta)}{k(1-A)}\right\}^{\frac{1}{n-1}}$$

.

This completes the proof .

Definition(H.Slivarman): For every $f \in SA$, we define the convolution operator $RS_{a,b,r}(f)(z)$ as below

$$RS_{a,b,r}(f)(z) = {}_2F_1(a,b;r;z) * f(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{(a)_n (b)_n}{(r)_n n!} a_n z^n \quad (*) \quad ,$$

Where ${}_2F_1(a,b;c=r;z)$ is Gaussin hypergeometric function .

Theorem 4 : Let $f \in SA_m^{\alpha,c}(\mu,\beta,\lambda,k,A,\theta)$. then the convolution operator $RS_{a,b,r}(f)(z)$ defined by $(*)$ be in the class $f \in SA_m^{\alpha,c}(\mu,\beta,\lambda,k,A,\theta)$

Where $\frac{(a)_n (b)_n}{(r)_n 2} \le 1$.

Proof: Since $f \in SA_m^{\alpha,c}(\mu,\beta,\lambda,k,A,\theta)$, then from theorem 1 , we have

$$\sum_{n=2}^{\infty} \left(\frac{\alpha cm+1}{\alpha cm+n}\right) n(n-1-\theta)(\mu\beta+\lambda) a_n \le k(1-\mathrm{A})$$

To proof the convolution operator defined by $(*)$, we must to show that

$$\sum_{n=2}^{\infty} \left(\frac{\alpha cm+1}{\alpha cm+n}\right) n(n-1-\theta)(\mu\beta+\lambda) \left[\frac{(a)_n (b)_n}{(r)_n n!} a_n\right] \le k(1-\mathrm{A})$$

.

$$\sum_{n=2}^{\infty} \left(\frac{\alpha cm+1}{\alpha cm+n}\right) n(n-1-\theta)(\mu\beta+\lambda) \left[\frac{(a)_n (b)_n}{(r)_n n!}\right] a_n$$

$$\le \sum_{n=2}^{\infty} \left(\frac{\alpha cm+1}{\alpha cm+n}\right) n(n-1-\theta)(\mu\beta+\lambda) \left[\frac{(a)_n (b)_n}{(r)_n n!}\right] a_n$$

Since $\frac{(a)_n (b)_n}{(r)_n 2} \le 1$, then

$$\sum_{n=2}^{\infty} \left(\frac{\alpha cm+1}{\alpha cm+n}\right) n(n-1-\theta)(\mu\beta+\lambda) \left[\frac{(a)_n (b)_n}{(r)_n 2}\right] a_n$$

$$\le \sum_{n=2}^{\infty} \left(\frac{\alpha cm+1}{\alpha cm+n}\right) n(n-1-\theta)(\mu\beta+\lambda) \left[\frac{(a)_n (b)_n}{(r)_n n!} a_n\right] \le k(1-\mathrm{A})$$

.

Then the convolution operator $RS_{a,b,r}(f)(z)$ belong the class $SA_m^{\alpha,c}(\mu,\beta,\lambda,k,A,\theta)$.

This completes the proof .

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بعض الخصائص لصنف من الدوال احاديه التكافؤ معرفة بواسطة المؤثر التكاملي

رافد حبيب بطي
قسم الرياضيات وتطبيقات الحاسوب
كلية العلوم
جامعة المثنى

الخلاصة

في هذا البحث نحن قدمنا خصائص لصنف جزئي جديد للدوال احادية التكافؤ المعرفة بواسطه المؤثر التكاملي ، نحن حصلنا بعض النظريات المغلقة مرتبطة بواسطة هذا الصنف .