Double weighted distribution

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Abstract

In this paper, we present the Double Weighted Inverse Weibull (DWIW) using different weight functions. In particular, we derive the pdf, cdf, reliability, hazard and reverse hazard functions.

Keywords: Double Weighted distribution;Inverse Weibull; Double Weighted Inverse Weibull.

1.Introducation

The concept of weighted distributions can be traced to the work of Fisher (1934)[1]related to his studies on how methods of ascertainment can influence the form of distribution of recorded observations. Rao (1965)[2], introduced and formulated it in general terms by in connection with modeling, statistical data where the usual practice of using standard distributions for the purpose was not found to be appropriate.He identified various situations which refer to instances where the recorded observations cannot be considered as a random sample from the original distributions but that can be modeled by weighted distributions. These situations may occur due to non-observability of some events or damage caused to the original observation resulting in a reduced value, or adoption of a sampling procedure which gives unequal chances to the units in the original. The usefulness and applications of weighted distributions of biased samples in various areas including medicine, ecology, reliability, and branching processes can be seen in Patil and Rao[3], Oluyede[4]. Within the context of cell kinetics and the early detection of disease, Zelen[5] introduced weighted distributions to represent what he broadly perceived at length-biased sampling (introduced earlier in (Cox, D.R. et al.))[6]. For additional and important results onweighted distributions, see Rao[7], Patil and Ord[8], Zelen and Feinleib[9], Application examples for weighted distribution see(El-Shaarawi et al)[10], and there are many researches for weighted distribution as, Hewa(Nanda, A. K. 2001)[11] introduced a new class of weighted generalized gamma distribution and related distribution, theoretical properties of the generalized gamma model, Jing [12] introduced the weighted inverse Weibull distribution and beta-inverse Weibull distribution, theoretical properties of them, Castillo and Perez-Casany[13] introduced new exponential families, that come from the concept of weighted distribution, that include and generalize the poisson distribution, Shaban and Boudrissa[14](Priyadarshani, H. A.)[15] have shown that the length-biased version of the Weibull distribution known as Weibull Length-biased (WLB) distributin is unimodal throughout examining its shape, with other properties, Das and Roy[16] discussed the length-baised Weighted Generalized Rayleigh distribution with its properties, also they are develop the length-baised from of the weighted Weibull distribution[17], introduced the concept of size-biased sampling and weighted distributions by identifying some of the situations where the underlying models retain their form. In this paper, we present the Double Weighted Inverse Weibull DWIW using different weight functions. In particular, we derive the pdf, cdfderive the pdf, cdf , reliability, hazard and reverse hazard functions.

A mathematical definition of the weighted distribution is as follows. Let $(\Omega; \Upsilon, P)$ be aprobability space, $X: \Omega \rightarrow H$ be a random variable (rv) where $H = (a, b)$be an intervalon real line with $a > 0$ and $(b > a)$can be finite or infinite. When the distribution function (df) $F(x)$of $X$is absolutely continuous with probability density function (pdf) $f(x)$and$w(x)$be a non-negative weight function satisfying $\mu_w = E(w(X)) < \infty$, then the (rv)$X_w$having pdf

$$f_w(x) = \frac{w(x)f(x)}{\mu_w}\ , \quad a < x < b \quad (1.1)$$

is said to have weighted distribution, corresponding to the distribution of X. Thedefinition in the discrete case is analogous.

One of the basic problems when one use weighted distributions as a tool in the selectionof suitable models for observed data is the choice of the weight function that fits the data.Depending upon the choice of weight function $w(x)$, we have different weighted models.For example, when the weight function depends on the lengths of units of interest(i.e. $w(x) = x$), the resulting distribution is called length-biased. In this case, the pdf of alength-biased (rv)$X_L$is defined as

$$f_L(x) = \frac{xf(x)}{\mu}\ , \quad a < x < b \quad (1.2)$$

Where $\mu = E(X) < \infty$. More generally, when the sampling mechanism selects units with probability proportional to some measure of the unit size, i.e., when $w(x) = x^c\ ;\ c > 0$, then the resulting distribution is called size-biased. This type of sampling is a generalization of length-biased samplingand majority of the literature is centered on this weight function. Denoting$\mu_c = E(x^c) < \infty$, distribution of the size-biased (rv)$X_s$of order$c$is specified by the pdf

$$f_c(x) = \frac{x^c f(x)}{\mu_c}\ , \quad a < x < b \quad (1.3)$$

Clearly, when $c = 1$, (1.3) reduces to the pdf of a length-biased (rv).

1.1 Definition The Double weighted distribution is given by:-

$$f_w(x; c) = \frac{w(x)f(x)F(cx)}{W}\ , \quad x \geq 0, c > 0 \qquad (1.1)$$

Where

$$W = \int_0^\infty w(x)f(x)F(cx)dx$$

where

1) $w(x) = x$,
2) $w(x) = F(cx)$, $F(cx)$ depend on the original distribution $f(x)$.

1.2 Double weighted Inverse Weibull distribution

Consider the first weight function $w_1(x) = x$ and inverse weibull distribution function of cX given by :-

$$f(cx;\alpha,\beta) = \beta(c\alpha)^{-\beta}x^{-\beta-1}e^{-(\alpha x)^{-\beta}}, \quad cx \geq 0, \; c,\alpha,\beta > 0$$

So

$$F(cx;\alpha,\beta) = e^{-(\alpha cx)^{-\beta}}, \quad \alpha, c, \beta > 0$$

And

$$W = \int_0^\infty w_1(x) f(x) F(cx) dx$$

$$= \int_0^\infty x\beta\alpha^{-\beta}x^{-\beta-1}e^{-(\alpha x)^{-\beta}}e^{-(\alpha cx)^{-\beta}}dx$$

$$= \beta\alpha^{-\beta}\int_0^\infty x^{-\beta}e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}}dx$$

Now let $y = (c^{-\beta}+1)(\alpha x)^{-\beta} \Rightarrow x = \frac{y^{-\frac{1}{\beta}}}{(c^{-\beta}+1)^{-\frac{1}{\beta}}\alpha}$

$$\Rightarrow dx = \frac{y^{-\frac{1}{\beta}-1}}{(c^{-\beta}+1)^{-\frac{1}{\beta}}\beta\alpha}dy \Rightarrow W = \frac{\Gamma(1-\frac{1}{\beta})}{\alpha(c^{-\beta}+1)^{1-\frac{1}{\beta}}}$$

Then the probability density function of the Double Weighted Inverse Weibull distribution DWIWD is given by :-

$$f_{w_1}(x;\alpha,\beta,c) = \frac{\beta\alpha^{1-\beta}(c^{-\beta}+1)^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})}x^{-\beta}e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}} \quad (1.2)$$

For $x \geq 0$, $c,\alpha > 0$, $\beta > 1$

Now let $w_2(x,\theta) = x^\theta$, $\theta \in \Re$, (where $\Re$ is the real numbers set).

The probability density function of DWIWD is:-

$$f_{w_2}(x;\alpha,\beta,c,\theta) = \frac{\beta\alpha^{\theta-\beta}(c^{-\beta}+1)^{1-\frac{\theta}{\beta}}}{\Gamma(1-\frac{\theta}{\beta})}x^{\theta-(\beta+1)}e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}}, \quad \theta < \beta \quad (1.3)$$

Note that if $\theta = 1$ then the distribution becomes as $f_{w_1}(x;\alpha,\beta,c)$.

The cumulative function of DWIWD is given by:-

$$F_{w_1}(x;\alpha,\beta,c) = \frac{\beta\alpha^{-\beta+1}(c^{-\beta}+1)^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})}\int_0^x t^{-\beta}e^{-(c^{-\beta}+1)(\alpha t)^{-\beta}}dt$$

$$= \frac{\alpha^{-\beta+1}(c^{-\beta}+1)^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})} \times \frac{1}{\alpha^{-\beta+1}(c^{-\beta}+1)^{1-\frac{1}{\beta}}}\int_{(c^{-\beta}+1)(\alpha x)^{-\beta}}^{\infty} y^{-\frac{1}{\beta}}e^{-y}dy$$

$$= \frac{1}{\Gamma(1-\frac{1}{\beta})}\int_{(c^{-\beta}+1)(\alpha x)^{-\beta}}^{\infty} y^{-\frac{1}{\beta}}e^{-y}dy$$

$$= 1 - \frac{1}{\Gamma(1-\frac{1}{\beta})}\int_0^{(c^{-\beta}+1)(\alpha x)^{-\beta}} y^{-\frac{1}{\beta}}e^{-y}dy$$

$= 1 - \frac{\gamma(1-\frac{1}{\beta}\ ,\ (c^{-\beta}+1)(\alpha x)^{-\beta})}{\Gamma(1-\frac{1}{\beta})}$(1.4)

Where

$$\gamma\left(1-\frac{1}{\beta}\ ,\ \left(c^{-\beta}+1\right)(\alpha x)^{-\beta}\right) = \int_0^{(c^{-\beta}+1)(\alpha x)^{-\beta}} y^{-\frac{1}{\beta}} e^{-y} dy$$

And

$F_{w_2}(x;\alpha,\beta,c,\theta) = 1 - \frac{\gamma(1-\frac{\theta}{\beta}\ ,\ (c^{-\beta}+1)(\alpha x)^{-\beta})}{\Gamma(1-\frac{\theta}{\beta})}$(1.5)

1.3The shape

The shapes of the density functions given in (1.2)& (1.3) can be clarified by studying those functions defined over the positive real line [0, ∞] and the behavior of its derivative as follows:

1.2.1 Limit and Mode of the function

Note that the limits of the density functions given in (1.2) & (1.3) are as follow:-

$\lim_{x\to 0} f_{w_1}(x;\alpha,\beta,c) = 0$(1.6)

That is

$$\frac{\beta\alpha^{1-\beta}\left(c^{-\beta}+1\right)^{1-\frac{1}{\beta}} \lim_{x\to 0} x^{-\beta} e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}}{\Gamma(1-\frac{1}{\beta})} = 0$$

Also

$\lim_{x\to 0} f_{w_2}(x;\alpha,\beta,c,\theta) = 0$(1.7)

That is

$$\frac{\beta\alpha^{\theta-\beta}\left(c^{-\beta}+1\right)^{1-\frac{\theta}{\beta}} \lim_{x\to 0} x^{\theta-(\beta+1)} e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}}{\Gamma\left(1-\frac{\theta}{\beta}\right)} = 0$$

Since $\lim_{x\to 0} e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}} = 0$

Now

$\lim_{x\to\infty} f_{w_1}(x;\alpha,\beta,c) = 0$(1.8)Where

$$\frac{\beta\alpha^{1-\beta}\left(c^{-\beta}+1\right)^{1-\frac{1}{\beta}} \lim_{x\to\infty} x^{-\beta} e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)} = 0$$

$\lim_{x\to\infty} f_{w_2}(x;\alpha,\beta,c,\theta) = 0$ (1.9)

Where

$$\lim_{x\to\infty} \frac{\beta\alpha^{\theta-\beta}\left(c^{-\beta}+1\right)^{1-\frac{\theta}{\beta}}\ x^{\theta-(\beta+1)} e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}}{\Gamma(1-\frac{\theta}{\beta})} = 0$$

Since $\lim_{x\to\infty} x^{-\beta} = \lim_{x\to\infty} \frac{1}{x^{\beta}} = 0$ and $\lim_{x\to\infty} e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}} = e^{0} = 1$

And the **modes** of the functions $f_{w_1}(x;\alpha,\beta,c)$ and $f_{w_2}(x;\alpha,\beta,c,\theta)$ are given by:

Case 1 Consider the Double Weighted Inverse Weibull distribution given by equation (2.2) .

Note that

$$\text{Log} f_{w_1}(x;\alpha,\beta,c) = \log\left(\frac{\beta\alpha^{1-\beta}\left(c^{-\beta}+1\right)^{1-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}\right) - \beta\log x - \left(c^{-\beta}+1\right)(\alpha x)^{-\beta} \quad (1.10)$$

Differentiating equation (2.10) with respect to x, we obtain

$$\frac{\partial}{\partial x}\text{Log}f_{w_1}(x;\alpha,\beta,c) =$$

$$-\frac{\beta}{x}+\beta\left(c^{-\beta}+1\right)\alpha^{-\beta}x^{-\beta-1}$$

Now , $\frac{\partial}{\partial x}\text{Log}f_{w_1}(x;\alpha,\beta,c) = 0$

Implies$-\frac{\beta}{x}+\beta\left(c^{-\beta}+1\right)\alpha^{-\beta}x^{-\beta-1} = 0$(1.11)

So that

$$\left(c^{-\beta}+1\right)\alpha^{-\beta}x^{-\beta} = 1$$

$$\Rightarrow x^{-\beta} = \frac{1}{\left(c^{-\beta}+1\right)\alpha^{-\beta}}$$

$$x_0 = \left(c^{-\beta}+1\right)^{\frac{1}{\beta}}\alpha^{-1}\ , where\alpha \neq 0$$

Then x_0 is the mode of Double Weighted Inverse Weibulldistribution, the derivative equal to zero (equation (1.11)) at x_0.Thenthe function $f_{w_1}(x;\alpha,\beta,c)$ increases it takes its maximum at x_0 then it decreases again. To verify, the second derivative of $f_{w_1}(x;\alpha,\beta,c)$ with respect to x is derived which is equal to:

$$\frac{\partial^2\text{Log}f_{w_1}(x;\alpha,\beta,c)}{\partial x^2} = \frac{\beta}{x^2} - \beta(\beta+1)\left(c^{-\beta}+1\right)\alpha^{-\beta}x^{-\beta-2} \qquad (1.12)$$

The quantity is negative for all value of x. The following Figures illustrates some of the possible shapes of the density $f_{w_1}(x;\alpha,\beta,c)$ for specified values of α,β and , c.

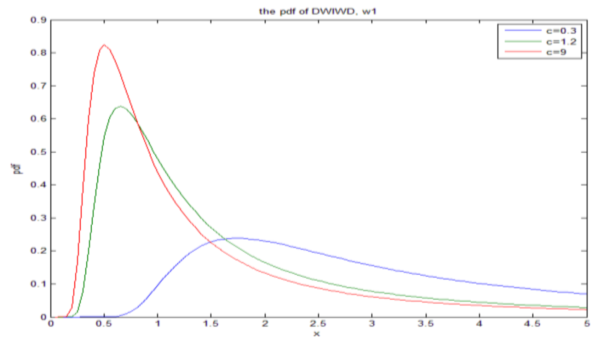


Figure1-1pdf of DWIWD using $w_1(x) = x$ with fixed , β , and c take the values (0.3,1.2,9)

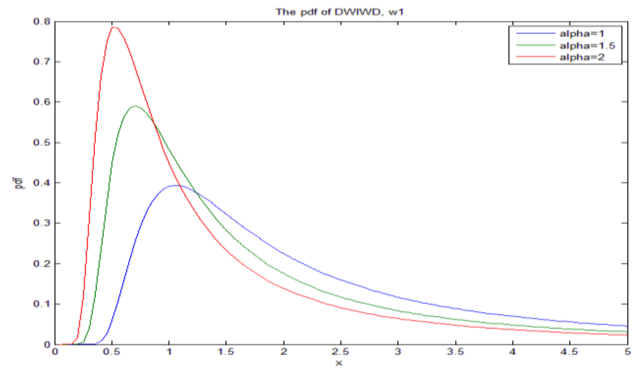


Figure 1-2 : pdf of DWIWD using $w_1(x) = x$ with fixed c,β and α take the values (1,1.5,2)

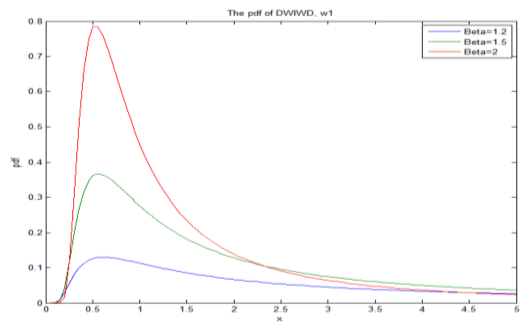


Figure1-3: pdf of DWIWD using $w_1(x) = x$ with fixed c,α and β take the values (1.2,1.5,2)

From the figures above, we note that the all parameters (c,α and β) parameters behaves as a scale parameters as in the original distribution (IWD).

Case 2 Consider the Double Weighted Inverse Weibull distribution given by equation (2.3) .

Note that

$$\log f_{w_2}(x;\alpha,\beta,c,\theta) = \log\left(\frac{\beta\alpha^{\theta-\beta}\left(c^{-\beta}+1\right)^{1-\frac{\theta}{\beta}}}{\Gamma(1-\frac{\theta}{\beta})}\right) + \left(\theta-(\beta+1)\right)\log x - \left(c^{-\beta}+1\right)(\alpha x)^{-\beta} \quad (1.13)$$

Differentiating equation (2.13) with respect to x, we obtain

$$\frac{\partial}{\partial x}\mathrm{Log}f_{w_2}(x;\alpha,\beta,c) = \frac{\theta-(\beta+1)}{x} + \beta\left(c^{-\beta}+1\right)\alpha^{-\beta}x^{-\beta-1}$$

Now , $\frac{\partial}{\partial x}\mathrm{Log}f_{w_2}(x;\alpha,\beta,c) = 0$

Implies $\frac{\theta-(\beta+1)}{x} + \beta\left(c^{-\beta}+1\right)\alpha^{-\beta}x^{-\beta-1} = 0 \quad (1.14)$

So that

$$x^{-\beta} = \frac{\beta-\theta+1}{\beta(c^{-\beta}+1)\alpha^{-\beta}}$$

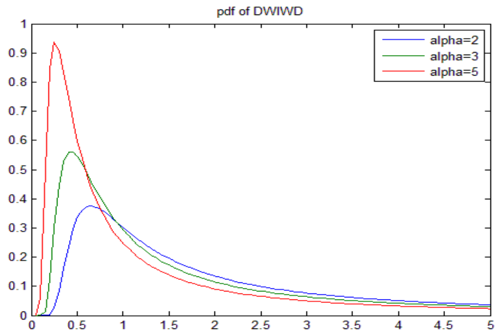
$$\Longrightarrow x_0 = \frac{1}{\alpha}\left(\frac{\beta\left(c^{-\beta}+1\right)}{\beta-\theta+1}\right)^{\frac{1}{\beta}}$$

Where $\alpha \neq 0$ and $\beta-\theta+1 \neq 0$

As in case1, x_0 represent the mode of DWIWD where $w_2(x) = x^\theta$and the derivative in equation (2.14) equal to zero at x_0.Thenthe function $f_{w_2}(x;\alpha,\beta,c,\theta)$increases it takes its maximum at x_0 then it decreases again.To verify, the second derivative of $f_{w_2}(x;\alpha,\beta,c,\theta)$ with respect to x is derived which is equal to:

$$\frac{\partial^2 \log\ f_{w_2}(x;\alpha,\beta,c,\theta)}{\partial x^2} = \frac{(\beta+1)-\theta}{x^2} - \beta(\beta+1)\left(c^{-\beta}+1\right)\alpha^{-\beta}x^{-\beta-2} \quad (1.15)$$

The quantity is negative for all value of x.The following Figures illustrates some of the possible shapes of the density $f_{w_2}(x;\alpha,\beta,c,\theta)$ for specified values of α,β,c and θ.



re1-4: pdf of DWIWD using $w_2(x) = x^\theta$ with fixed $c = 2, \beta = 2,\ \theta = 1.5$, and α take the values (2,3,5)

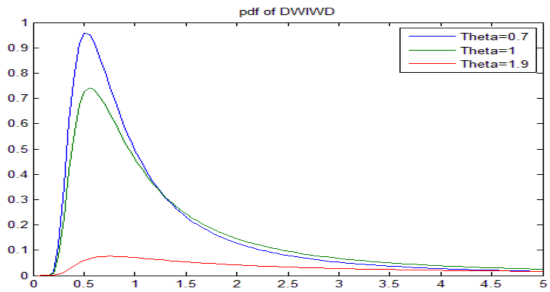


Figure1-5: pdf of DWIWD using $w_2(x) = x^\theta$ with fixed $c = 2, \beta = 2$, $\alpha = 2$ and θtake the values (0.7,1,1.9)

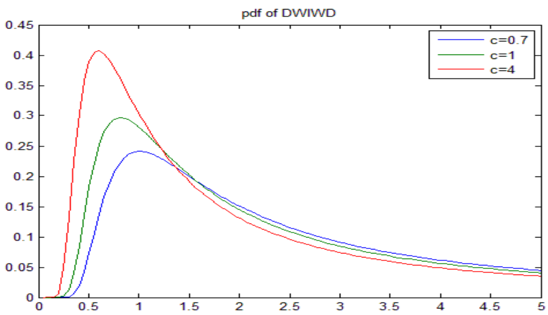


Figure1-6: pdf of DWIWD using $w_2(x) = x^{\theta}$ with fixed $\beta = 2,\ \alpha = 2,\ \theta = 1.5$ and $c$take the values (0.7,1,4)

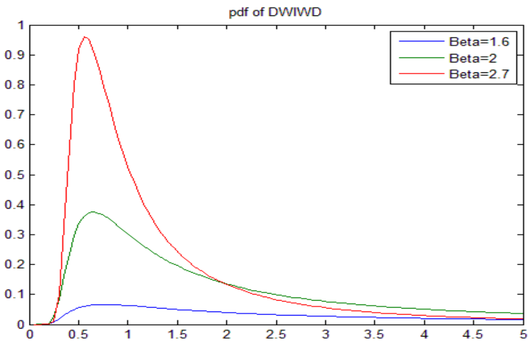


Figure1-7: pdf of DWIWD using $w_2(x) = x^{\theta}$ with fixed $c = 2,\ \alpha = 2,\ \theta = 1.5$ and βtake the values (1.6,2,2.7)

From the **figures 1-(4, 5, 6,7)** we note that all parameters behaves as a scale parameters as in the original distribution (IWD).

1.4 Reliability function

1-The reliability function of DWIWD with weighted function$w_1(x)$is

given by:-

$$R_{fw_1}(x;\alpha,\beta,c) = 1 - F_{w_1}(x;\alpha,\beta,c)$$

$$= 1 - 1 + \frac{\gamma(1-\frac{1}{\beta}\ ,\ \left(c^{-\beta}+1\right)(\alpha x)^{-\beta})}{\Gamma(1-\frac{1}{\beta})}$$

$$= \frac{\gamma\left(1-\frac{1}{\beta}\ ,\ \left(c^{-\beta}+1\right)(\alpha x)^{-\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}$$

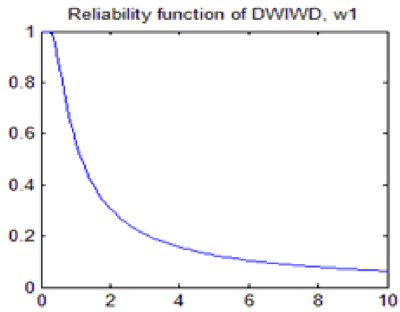
$,\ \beta > 1(1.16)$

2- The reliability function of DWIWD with weighted function $w_2(x;\theta)$ is given by:-

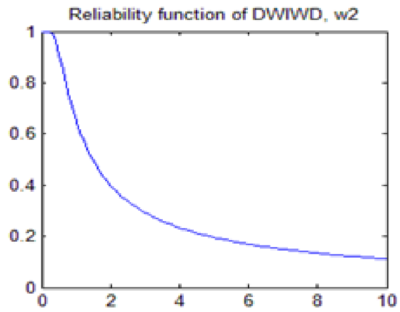
$$R_{fw_2}(x;\alpha,\beta,c,\theta) = 1 - F_{w_2}(x;\alpha,\beta,c,\theta)$$

$$= 1 - 1 + \frac{\gamma(1-\frac{\theta}{\beta}\ ,\ \left(c^{-\beta}+1\right)(\alpha x)^{-\beta})}{\Gamma(1-\frac{\theta}{\beta})}$$

$$\therefore R_{fw_2}(x;\alpha,\beta,c,\theta) = \frac{\gamma(1-\frac{\theta}{\beta}\ ,\ \left(c^{-\beta}+1\right)(\alpha x)^{-\beta})}{\Gamma(1-\frac{\theta}{\beta})}\ ,\ \beta > \theta(1.17)$$



-1-



-2-

From **Figure 2.12,** 1 and 2 we can see that the curves are decreasing, with most rapid decrease initially. Also from 3 and 4 the curves are rapid decrease.

1.5 Hazard function

1- Hazard function for DWIWD using $w_1(x)$is given by:-

$$h_{fw_1}(x;\alpha,\beta,c) = \frac{f_{w_1}(x;\alpha,\beta,c)}{R_{fw_1}(x;\alpha,\beta,c)}$$

$$= \frac{\frac{\beta\alpha^{1-\beta}(c^{-\beta}+1)^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})} x^{-\beta} e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}}}{\frac{\gamma(1-\frac{1}{\beta}\,,\ (c^{-\beta}+1)(\alpha x)^{-\beta})}{\Gamma(1-\frac{1}{\beta})}}$$

$$= \frac{\beta\alpha^{1-\beta}(c^{-\beta}+1)^{1-\frac{1}{\beta}}}{\gamma(1-\frac{1}{\beta}\,,\ (c^{-\beta}+1)(\alpha x)^{-\beta})} x^{-\beta} e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}} \quad (1.18)$$

2- Hazard function for DWIWD using $w_2(x;\theta)$ is given by:-

$$h_{fw_2}(x;\alpha,\beta,c,\theta) = \frac{\frac{\beta\alpha^{\theta-\beta}(c^{-\beta}+1)^{1-\frac{\theta}{\beta}}}{\Gamma(1-\frac{\theta}{\beta})} x^{\theta-(\beta+1)} e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}}}{\frac{\gamma(1-\frac{\theta}{\beta}\,,\ (c^{-\beta}+1)(\alpha x)^{-\beta})}{\Gamma(1-\frac{\theta}{\beta})}}$$

$$= \frac{\beta\alpha^{\theta-\beta}(c^{-\beta}+1)^{1-\frac{\theta}{\beta}}}{\gamma(1-\frac{\theta}{\beta}\,,\ (c^{-\beta}+1)(\alpha x)^{-\beta})} x^{\theta-(\beta+1)} e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}} \quad (1.19)$$

We study the behavior of the hazard function of the DWIW distribution via the following lemma, due to Glaser[18].

Lemma (2-1):

Let X be a continuous random variable with twice differentiable density function. Define the quantity $\eta(x) = -\frac{f(x)'}{f(x)}$, where $f(x)'$ denote the first derivative of the density function with respect to x. suppose that the first derivative of $\eta(x)$-named $\eta'(x)$-exists. Glaser[18] gave the following results.

1- If $\eta'(x) < 0$, for all $x > 0$, then the hazard function is monotonically decreasing (DHF).

2- If $\eta'(x) > 0$ for all > 0, then the hazard function is monotonically increasing (IHF).

3- If there exists x_0 such that $\eta'(x) > 0$ for all $0 < x < x_0$; $\eta'(x_0) = 0$ and $\eta'(x) < 0$ for all $> x_0$. In addition to that $\lim_{x\to 0} f(x) = 0$;then the hazard function is upside down bathtub shaped (UBT).

4- If there exists x_0 , such that $\eta'(x) < 0$ for all $0 < x < x_0$; $\eta'(x_0) = 0$ and $\eta'(x) > 0$ for all $x > x_0$ .Adding to that $\lim_{x\to 0} f(x) = \infty$. it consequences that the hazard function is bathtub shaped (BT)

For DWIWD we begin by computing the quantity $\eta(x)$; by first taking the derivative of the density function given in (1.2) with respect to x which is given by

$$f'_{w_1}(x) = \frac{\partial f_{w_1}(x;\alpha,\beta,c)}{\partial x}$$

$$= \frac{\beta\alpha^{1-\beta}(c^{-\beta}+1)^{1-\frac{1}{\beta}}}{\Gamma(1-\frac{1}{\beta})}\left[\beta x^{-\beta-1} e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}}\right] \times\left[(c^{-\beta}+1)(\alpha x)^{-\beta}-1\right]$$

$$= \frac{\beta f_{w_1}(x)}{x}\left[(c^{-\beta}+1)(\alpha x)^{-\beta}-1\right] \quad (1.20)$$

Dividing both sides of the equation (1.20)by the measure $-f_{w_1}(x)$ we obtain :-

$\eta_{fw_1}(x) = \frac{\beta}{x}\left[1-(c^{-\beta}+1)(\alpha x)^{-\beta}\right]$, taking its derivative with respect to x yields:-

$$\eta'_{fw_1}(x) = \frac{\beta}{x^2}\left[(c^{-\beta}+1)(\beta+1)(\alpha x)^{-\beta}-1\right] \quad (1.21)$$

Since $\alpha, c > 0$, $\beta > 1$ and $x > 0$, we have $\eta'_{fw_1}(x) > 0$ if $(c^{-\beta}+1)(\beta+1)(\alpha x)^{-\beta}-1 > 0$

Theorem 2-2: Let $\beta > 1$, $\alpha, c > 0$ and $x_0 = \frac{[(c^{-\beta}+1)(\beta+1)]^{\frac{1}{\beta}}}{\alpha}$. Then $\eta'_{f_{w1}}(x) = 0$ if $x = x_0$,where $x < x_0$,$\eta'_{f_{w1}}(x) > 0$ and $\eta'_{f_{w1}}(x) < 0$ if $x > x_0$.

Note: The results in the theorem follows from the fact that $\eta'_{f_{w1}}(x) = 0$ implies

$$x = \frac{[(c^{-\beta}+1)(\beta+1)]^{\frac{1}{\beta}}}{\alpha}$$

Lemma 2-3: For $\beta > 1$, $\alpha, c > 0$, the hazard function of the DWIWD in case $w_1(x)$ is an upside down function.

Proof:
Note that it is enough to prove that $\lim_{x\to 0} h_{fw_1}(x;\alpha,\beta,c) = 0$ .from (1.19) we have

$$h_{fw_1}(x;\alpha,\beta,c) = \frac{\beta\alpha^{1-\beta}(c^{-\beta}+1)^{1-\frac{1}{\beta}}}{\gamma(1-\frac{1}{\beta}\ ,\ (c^{-\beta}+1)(\alpha x)^{-\beta})} x^{-\beta} e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}}$$

Since $\lim_{x\to 0}\gamma\left(1-\frac{1}{\beta}\ ,\ (c^{-\beta}+1)(\alpha x)^{-\beta}\right) = \lim_{x\to 0}\int_0^{(c^{-\beta}+1)(\alpha x)^{-\beta}} t^{-\frac{1}{\beta}} e^{-t} dt$

$$= \int_0^{\infty} t^{-\frac{1}{\beta}} e^{-t} dt = \Gamma(1-\frac{1}{\beta})$$

And

$$\lim_{x\to 0} e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}} = e^{-\infty} = 0$$

So

$$\lim_{x\to 0} h_{fw_1}(x;\alpha,\beta,c) = 0$$

Now using the equation (1.3), the derivative of the $f_{w_2}(x;\alpha,\beta,c,\theta)$ with respect to x is

$f'_{w_2}(x;\alpha,\beta,c,\theta) = \frac{f_{w_2(x)}}{x}\left[\beta(c^{-\beta}+1)(\alpha x)^{-\beta} + (\theta+1-\beta)\right]$, (1.22)

Dividing both sides of the equation (1.22) by the measure $-f_{w_2}(x)$ we obtain :-

$\eta_{f_{w2}}(x) = \frac{1}{x}\left[\beta(c^{-\beta}+1)(\alpha x)^{-\beta} + (\theta+1-\beta)\right]$(1.23)

taking its derivative with respect to x yields:-

$\eta'_{f_{w2}}(x) = \frac{1}{x^2}\left[\beta-1-\theta-\beta(\beta+1)(c^{-\beta}+1)(\alpha x)^{-\beta}\right]$, (1.24)

Also Since $\alpha, c > 0$, $\beta > 1$ and $x > 0$, we have $\eta'_{f_{w1}}(x) > 0$ if

$$\beta-1-\theta-\beta(\beta+1)(c^{-\beta}+1)(\alpha x)^{-\beta} > 0$$

Theorem 2-4: Let $\beta > 1$ and $x_0 = \left[\frac{\beta(\beta+1)(c^{-\beta}+1)}{\alpha^{-\beta}(\beta-\theta-1)}\right]^{\frac{1}{\beta}}$. Then $\eta'_{f_{w2}}(x) = 0$ if $x = x_0$,where $x < x_0$,$\eta'_{f_{w2}}(x) > 0$ and $\eta'_{f_{w2}}(x) < 0$ if $x > x_0$.

Note: As above the results in the theorem follows from the fact that $\eta'_{f_{w2}}(x) = 0$ implies

$$x = \left[\frac{\beta(\beta+1)(c^{-\beta}+1)}{\alpha^{-\beta}(\beta-\theta-1)}\right]^{\frac{1}{\beta}}$$

Lemma 2-5: For $\beta > 1$, the hazard function of the DWIWD in case $w_2(x;c)$ is an upside down function.

Proof:
It is enough to prove that $\lim_{x\to 0} h_{fw_2}(x;\alpha,\beta,c,\theta) = 0$ .Then we have $h_{fw_2}(x;\alpha,\beta,c,\theta) = \frac{\beta\alpha^{\theta-\beta}(c^{-\beta}+1)^{1-\frac{\theta}{\beta}}}{\gamma(1-\frac{\theta}{\beta}\ ,\ (c^{-\beta}+1)(\alpha x)^{-\beta})} x^{\theta-(\beta+1)} e^{-(c^{-\beta}+1)(\alpha x)^{-\beta}}$

Now since

$$\lim_{x\to 0}\gamma\left(1-\frac{\theta}{\beta}\,,\;\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}\right)$$

$$=\lim_{x\to 0}\int_{0}^{\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}} t^{-\frac{\theta}{\beta}}e^{-t}dt$$

$$=\Gamma\left(1-\frac{\theta}{\beta}\right)$$

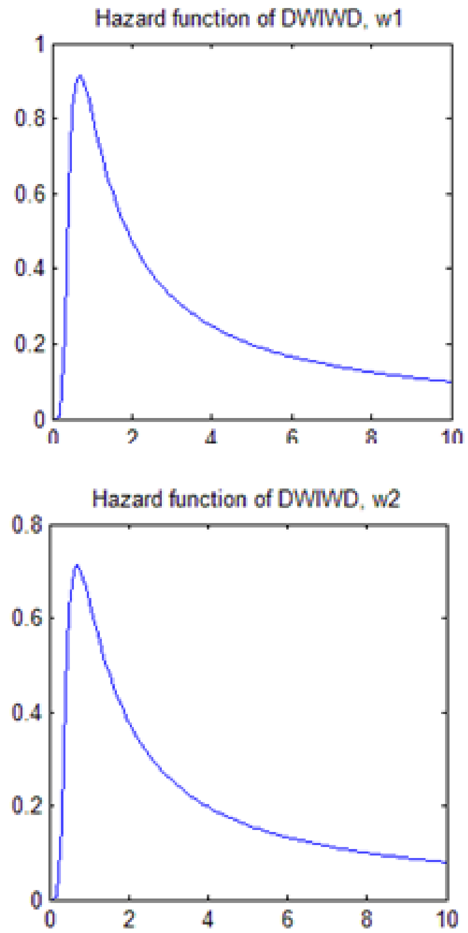
And

$$\lim_{x\to 0}e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}=0$$

Then

$$\lim_{x\to 0}h_{fw_2}(x;\alpha,\beta,c,\theta)=0$$

And the plots of the hazard function of (DWIW) is given below:-



-2-

Figure 2.13 : -1- The hazard function of DWIWDusing$w_1(x)$, -2- the hazard function of DWIWDusing$w_2(x)$.

1.6 Reverse Hazard function

1- Reverse hazard function for DWIWD with $w_1(x)$is given by:-

$$\varphi_{fw_1}(x;\alpha,\beta,c)=\frac{f_{w_1}(x;\alpha,\beta,c)}{F_{w_1}(x;\alpha,\beta,c)}=$$

$$\frac{\frac{\beta\alpha^{1-\beta}\left(c^{-\beta}+1\right)^{1-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)}x^{-\beta}e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}}{1-\frac{\gamma\left(1-\frac{1}{\beta}\,,\,\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}\right)}{\Gamma\left(1-\frac{1}{\beta}\right)}}=$$

$$\frac{\beta\alpha^{1-\beta}\left(c^{-\beta}+1\right)^{1-\frac{1}{\beta}}}{\Gamma\left(1-\frac{1}{\beta}\right)-\gamma\left(1-\frac{1}{\beta}\,,\,\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}\right)}x^{-\beta}e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}$$

(1.25)

2- Reverse Hazard function for DWIWD with $w_2(x;\theta)$is given by:-

$$\varphi_{fw_2}(x;\alpha,\beta,c,\theta)=\frac{f_{w_2}(x;\alpha,\beta,c,\theta)}{F_{w_2}(x;\alpha,\beta,c,\theta)}$$

$$=\frac{\frac{\beta\alpha^{\theta-\beta}\left(c^{-\beta}+1\right)^{1-\frac{\theta}{\beta}}}{\Gamma\left(1-\frac{\theta}{\beta}\right)}x^{\theta-(\beta+1)}e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}}{1-\frac{\gamma\left(1-\frac{\theta}{\beta}\,,\,\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}\right)}{\Gamma\left(1-\frac{\theta}{\beta}\right)}}$$

$$=\frac{\beta\alpha^{\theta-\beta}\left(c^{-\beta}+1\right)^{1-\frac{\theta}{\beta}}}{\Gamma\left(1-\frac{\theta}{\beta}\right)-\gamma\left(1-\frac{\theta}{\beta}\,,\,\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}\right)}x^{\theta-(\beta+1)}e^{-\left(c^{-\beta}+1\right)(\alpha x)^{-\beta}}$$

(1.26)

The Figure below shows the inverted hazard function of DWIWD.

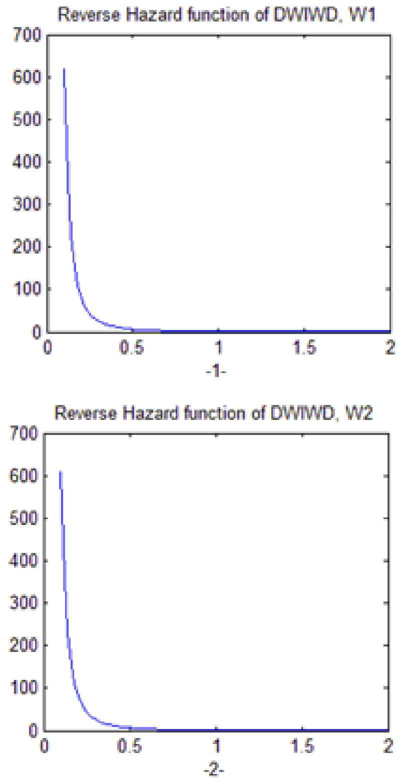


Figure 2.14 :-1-The reverse hazard function of DWIWD using $w_1(x)$, -2- the reverse hazard function of DWIWD using $w_2(x)$. Note that the parameter $\theta = 1.2$ and the all others parameters takes the value 2.

Conclusions

We can derive the Double Weighted Inverse Weibull DWIW using different weight functions. In particular, we derive the pdf, cdfderive the pdf, cdf , reliability, hazard and reverse hazard functions.

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