Variable Order Two Steps Runge – Kutta Method for Solving Stochastic Ordinary Differential Equation

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Abstract

In this paper, the variable order method in connection with two steps Runge – Kutta method for solving stochastic ordinary differential equations have been proposed in order to improve the accuracy of the obtained results by increasing the order of convergence of the numerical schemes. The proposed approach has been introduced for Stratonovich type stochastic differential equation.

Keywords: Stochastic Ordinary Differential Equations, Stochastic Runge – Kutta Methods, Variable Order Method.

1. Introduction:

The stochastic ordinary differential equations (SODE's for short) are differential equations in which one or more of its terms are stochastic processes, and therefore will give solutions which are itself stochastic processes, **(1)**. Also, they are used in a wide range of applications, such as environmental modeling, engineering and biological modeling, etc., **(2), (3)**.

It is remarkable that the stochastic differential equation in Stratonovich case that will be considered has the form:

$$\left.\begin{aligned} dy_t &= f(y_t)\,dt + g(y_t) \circ dW_t, \\ y_{t_0} &= y_0 \end{aligned}\right\} \quad \ldots (1)$$

where $t \in [t_0,T]$, $y_t \in \boldsymbol{R}^m$ and W_t is the Wiener process whose increment $\Delta W(t) = W_{t+\Delta t} - W_t$ is a Gaussian random variable with mean 0 and variance Δt.

In this paper the attentions was paid toward the constructing of variable order method that is based on the global error expansions for explicit two steps stochastic Runge – Kutta method.

In addition, the variable order methods provide a class of higher order strong approximation methods which are efficient in many cases; there are also important practical situations in the variable order methods providing general and efficient class of algorithms for the higher order strong approximation of SODE's. However, further investigations are still required to develop variable order methods for SODE's that have some performances comparable to those already known methods for solving ordinary differential equations. With this aim, in this paper we will establish global error expansions for higher-order strong Taylor scheme, and we shall then use these expansions to construct variable order methods based on these higher-order schemes. This will allow range of SODE's to be handled numerically.

2. Preliminaries:

In this section, some fundamental and necessary concepts relative to SDE's are given:

***Definition (2.1):* (4)**

A stochastic process W_t, $t \in [0, \infty)$ is said to be a Brownian motion or Wiener process if:

1. $P(W_0 = 0) = 1$, where P refers to the probability.

2. For 0 < t_0 < t_1 < … < t_n; the increments $W_{t_1}-W_{t_2}, W_{t_2}-W_{t_1},\ldots, W_{t_n}-W_{t_{n-1}}$ are independent.
3. For an arbitrary t and h > 0, $W_{t+h}-W_t$ has a Gaussian distribution with mean 0 and variance *h*.

2.1 Stochastic Differential Equations and their Models: (2), (4), (5)

Consider the SDE:

$$\left.\begin{aligned} dy_t &= f(t,y_t)\,dt + g(t,y_t)\,dW_t; \\ y_{t_0} &= y_0 \end{aligned}\right\} \quad \ldots (2)$$

where $f: I\times \boldsymbol{R} \longrightarrow \boldsymbol{R}$, $g: I\times \boldsymbol{R} \longrightarrow \boldsymbol{R}$ be a Borel-measurable functions, we call (f) the ***drift function*** and (g) the ***diffusion function***. Then a solution y_t of the equation (2) must also satisfy equation (2) when it is written as a stochastic integral equation of the form:

$$\left.\begin{aligned} y_t = y_{t_0} &+ \int_{t_0}^{t} f(s,y_s)\,ds \\ &+ \int_{t_0}^{t} g(s,y_s)\,dW_s \end{aligned}\right\} \quad \ldots (3)$$

<u>Remarks (2.1):</u>

1. The second integral given in equation (3) cannot be defined in usual meaning, where W_s is the Wiener. The variance of the Wiener process satisfies $\mathrm{Var}(W_t)=t$, which is increases as the time increases even thought the mean stays at 0.
2. There are two types of SODE's according to the calculations of the second integral, the first type is known as Itô SDE and is given by equation (2) and the stochastic integral $\int_{t_0}^{t} g(s,y_s)\,dW_t$ that appears in equation (3) refers to the Itô stochastic integral while the second type of SODE's is known as Stratonovich that is given by:

$$\left.\begin{aligned} dy_t &= f(t,y_t)\,dt + g(t,y_t)\circ dW_t; \\ y_{t_0} &= y_0 \end{aligned}\right\} \quad \ldots (4)$$

Also, equation (4) can be written as Stratonovich stochastic integral equation by:

$$\left.\begin{aligned} y_t = y_{t_0} &+ \int_{t_0}^{t} f(s,y_s)\,ds \\ &+ \int_{t_0}^{t} g(s,y_s)\circ dW_s \end{aligned}\right\} \quad \ldots (5)$$

and the stochastic integral $\int_{t_0}^{t} g(s,y_s)\circ dW_t$ that appears in equation (5) refers to the stochastic Stratonovich integral, **(6)**.

3. We have a simple relationship between the solution of an Itô SODE and the Stratonovich SODE's that are given in equations (2) and (4). Let y_t be the solution of one-dimensional Itô SODE that given in equation (2), then y_t is also a solution of the Stratonovich SODE, such that when, **(6)**:

$$dy_t = \underline{f}(t,y_t)\,dt + g(t,y_t)\circ dW_t;\ y_{t_0} = y_0$$

then:

$$y_t = y_0 + \int_{t_0}^{t} \underline{f}(s,y_s)\,ds + \int_{t_0}^{t} g(s,y_s)\circ dW_s$$

where:

$$\underline{f}(t,y_t) = f(t,y_t) - \frac{1}{2}\frac{\partial g}{\partial y}(t,y_t)g(t,y_t)$$

4. The formulation of two steps stochastic Runge - Kutta methods take the form, **(6)**:

$$\left.\begin{aligned} Y_1 &= y_n + h\,[a_{11}f(Y_1)+a_{12}f(Y_2)] \\ &\quad + J_1[b_{11}g(Y_1)+b_{12}g(Y_2)] \\ Y_2 &= y_n + h[a_{21}f(Y_1)+a_{22}f(Y_2)] \\ &\quad + J_1[b_{21}g(Y_1)+b_{22}g(Y_2)] \\ y_{n+1} &= y_n + h[\alpha_1 f(Y_1)+\alpha_2 f(Y_2)] \\ &\quad + J_1[\gamma_1 g(Y_1)+\gamma_2 g(Y_2)] \end{aligned}\right\} \quad \ldots (6)$$

Where :

$$A=(a_{ij})=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ \alpha^T=\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix}$$

$$B=(b_{ij})=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},\ \gamma^T=\begin{bmatrix}\gamma_1 & \gamma_2\end{bmatrix},$$

and a_{ij}, b_{ij}, α_j and γ_j are constants with $i,j=1,2$, $h=t_{n+1}-t_n$, $J_1=\Delta W_n$

$= W_{t_{n+1}} - W_{t_n}$ for all $n = 0,1,2,...$.

5. In this paper, we choose an explicit stochastic Runge - Kutta methods (R2 and PL) models such that in R2 model, the stochastic Runge – Kutta method (SRKM for short) take the form, **(6)**:

$$\left.\begin{aligned} &Y_1 = y_n \\ &Y_2 = y_n + \frac{2}{3} h\, f(Y_1) + \frac{2}{3} J_1 g(Y_1) \\ &y_{n+1} = y_n + h[\frac{1}{4} f(Y_1) + \frac{3}{4} f(Y_2)] \\ &\qquad + J_1[\frac{1}{4} g(Y_1) + \frac{3}{4} g(Y_2)] \end{aligned}\right\} \quad ... (7)$$

and PL model of SRKM take the form, **(6)**:

$$\left.\begin{aligned} &Y_1 = y_n \\ &Y_2 = y_n + h\, f(Y_1) + J_1 g(Y_1) \\ &y_{n+1} = y_n + h\, f(Y_1) \\ &\qquad + J_1[\frac{1}{2} g(Y_1) + \frac{1}{2} g(Y_2)] \end{aligned}\right\} \quad ... (8)$$

2.2 Strong convergence criterion: (7), (8)

Consider the sample path of the Wiener process, W_t that is given (and hence known), therefore, there is no randomness in the SDE and hence no randomness in X_T, The increments in the given Wiener process are then used to obtain the numerical solution y(h). The expectation of the absolute error is defined as:

$$\varepsilon = E(|X_T - y(h)|)$$

Here, the Euclidean norm is used, X_T is the Itô process at time T, while y(h) is the numerical solution obtained by approximately integrating the stochastic differential equation in a sequence of time steps.

The numerical scheme is consistent if the numerical solution y(h) converge to X_T as Δt tends to zero. Therefore, a discrete time numerical solution y(h) with maximum time step size δ converges strongly to X_T at time T if :

$$\lim_{\delta \to 0} E(|X_T - y(h)|) = 0$$

A discrete time approximation y(h) converge strongly with order $p > 0$ at time t if there exists a positive constant C, which does not depend on δ, and $\delta_0 > 0$, such that:

$$\varepsilon(\delta) = E(|X_T - y(h)|) \le C(\Delta t)^p$$

for each $\delta \in (0, \delta_0)$, where . $(0, \delta_0)$ is the interval of stability of the method.

3. Variable Order Methods for Solving SODE's

Variable order method is an accurate method that may be used to improve the accuracy of the obtained results. Therefore in this section, this method will be used in connection with SRKM's for solving Stratonovich SODE's and derive a new approach for solving Stratonovich SODE's with more accurate results. This method will be referred to as the variable order method for solving Stratonovich SODE's.

In this investigation, the weak numerical solution E(y(T)) will be studied. The weak error is defined as E(y(T) − y(h)) and the primary goal of this investigation is to derive the variable order method which has an error expansion of the form:

$$E(y(T) - y(h)) = a_1 h + a_2 h^2 + \qquad \ldots(9)$$

where a_1, a_2, … are some constants independent of the step size discretization h

Now, to successively eliminate the terms in the error expansion, thereby producing solutions using methods of higher and higher order. If a_1 in equation (9) is not zero, then the scheme of evaluating E(y(T)) is only of order h. To obtain the scheme of evaluating E(y(T)) of order h^2, we proceed as follows:

Find the error expansion using two different step sizes h_0 and h_1, such that $h_1 < h_0$, as follows:

$$\left.\begin{aligned} &E(y(T) - y(h_0)) = a_1 h_0 + a_2 h_0^2 \\ &\qquad + a_3 h_0^3 + ... \end{aligned}\right\} \quad \ldots (10)$$

$$E(y(T)-y(h_1)) = a_1h_1 + a_2h_1^2 + a_3h_1^3 + \ldots \quad \ldots (11)$$

and upon subtracting h_0-times of equation (11) from h_1-times of equation (10) and solving for E(y(T)), one may get:

$$E\big(y(T)\big) = \frac{h_1E(y(h_0)) - h_0E(y(h_1))}{h_1 - h_0} - a_2h_0h_1 - a_3h_0h_1\big(h_0 + h_1\big) - a_4(h_0^2 + h_0h_1 + h_1^2) - \ldots$$

$$= E\big(y(h_1)\big) + \frac{E(y(h_1)) - E(y(h_0))}{\frac{h_0}{h_1} - 1} - a_2h_0h_1 - a_3h_0h_1\big(h_0 + h_1\big) - a_4(h_0^2 + h_0h_1 + h_1^2) - \ldots$$

Thus, letting:

$$E_1\big(y(h_0)\big) = E\big(y(h_1)\big) + \frac{E(y(h_1))) - E(y(h_0))}{\frac{h_0}{h_1} - 1}$$

which is an $O(h_0^2)$ approximation to E(y(T)). Since $h_1 < h_0$ and for any two step sizes h_j and h_{j+1} may be used in the above elimination process, one may see that in general:

$$E_1(y(h_j)) = E(y(h_{j+1})) + \frac{E(y(h_{j+1})) - E(y(h_j))}{\frac{h_j}{h_{j+1}} - 1}$$

$, \ j = 0, 1, 2, \ldots$

which is also an $O(h_j^2)$ approximation to E(y(T)). Now, we have:

$$E\big(y(T)\big) = E_1\big(y(h_0)\big) - a_2h_0h_1 - a_3h_0h_1\big(h_0 + h_1\big) - a_4h_0h_1(h_0^2 + h_0h_1 + h_1^2) - \ldots$$

and

$$E\big(y(T)\big) = E_1\big(y(h_1)\big) - a_2h_1h_2 - a_3h_1h_2\big(h_1 + h_2\big) - a_4h_1h_2(h_1^2 + h_1h_2 + h_2^2) - \ldots$$

and upon eliminating the terms involving a_2, we obtain:

$$E\big(y(T)\big) = E_2\big(y(h_0)\big) + a_3h_0h_1h_2 + a_4h_0h_1h_2\big(h_0 + h_1 + h_2\big) + \ldots$$

where:

$$E_2(y(h_0)) = E_1(y(h_1)) + \frac{E_1(y(h_1)) - E_1(y(h_0))}{\frac{h_0}{h_2} - 1}$$

which is an $O(h_0^3)$ approximation to E(y(T)). More generally:

$$E_2(y(h_j)) = E_1(y(h_{j+1})) + \frac{E_1(y(h_{j+1})) - E_1(y(h_j))}{\frac{h_j}{h_{j+1}} - 1}$$

$, \ j = 0, 1, 2, \ldots$

which is also an $O(h_j^3)$ approximation to E(y(T)).

Similarly, continuing in this process, the following recursively sequence may be derived:

$$E_0(y(h_j)) = E(y(h_j)) \quad \ldots (12)$$

$$E_n(y(h_j)) = E_{n-1}(y(h_{j+1})) + \frac{E_{n-1}(y(h_{j+1})) - E_{n-1}(y(h_j))}{\frac{h_j}{h_{j+n}} - 1} \quad \ldots (13)$$

for all n = 1, 2, …; j = 0, 1, 2, …

On the basis of the results for $E(y(h_j))$ and $E_n(y(h_j))$, it seems that $E_n(y(h_j))$ provides an $O(h_j^{n+1})$ approximation to E(y(T)). This may be verified directly by following the evolution of the general term $a_n h^n$ in the error expansion, but is perhaps obtained more easily by the following an alternative approach obtained from equations (12) and (13), which is given in the following table:

Level	$O(h_j)$	$O(h_j^2)$	$O(h_j^3)$	$O(h_j^4)$	
0	$E_0(y(h_0))$				
1	$E_0 y(h_1))$	$E_1(y(h_0))$			
2	$E_0(y(h_2))$	$E_1(y(h_1))$	$E_2(y(h_0))$		
3	$E_0(y(h_3))$	$E_1(y(h_2))$	$E_2(y(h_2))$	$E_3(y(h_0))$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The following algorithm may be used for presenting computer programs:

<u>Algorithm:</u>

1. Input y_0 (initial condition) , h_0 , h_1 …;

   $h_j := \frac{h_j}{2^j}$, j:=0,1,2,…,(step sizes).
2. Find the numerical solution y_j with $h_j := \frac{h_j}{2^j}$, by using explicit SRKM's which are given in equations (7) and (8).
3. Evaluate: $E(y(h_j))$, j=0,1, 2, …
4. Find:
   a. $E_0(y(h_j)) = E(y(h_j))$

   b. $$E_n(y(h_j)) = E_{n-1}(y(h_{j+1})) + \frac{E_{n-1}(y(h_{j+1})) - E_{n-1}(y(h_j))}{\frac{h_j}{h_{j+n}} - 1};$$

   for $n := 1, 2, \ldots,\ j = 0, 1,\ 2,\ \ldots$

4. Numerical Simmulation:

As an illustration and for comparison purpose, we consider in this section, two illustrative examples, which are compared with the exact solution, but, first, it is remarkable that the argument of the considered examples is $t \in [0, 1]$ and the step sizes used for discretizing this interval are h, $\frac{h}{2}$, $\frac{h}{4}$, … with h = 0.1. Also, the obtained results for the given examples are represented at average of 10000 simulated solution by using N(0, h) random number generations for the Wiener process W_t.

<u>Example (1):</u>

Consider to the nonlinear Stratonovich SODE, **(3), (9**)

$$dy_t = (y^2 - 1)\,dt + (0.1 - 0.1y^2) \circ dW_t,$$

with the initial condition $y_0 = 0$, and for comparison purpose, the exact solution is given by:

$$y_t = \frac{(1 + y_0)\exp(-2t + 0.2W_t) + y_0 - 1}{(1 + y_0)\exp(-2t + 0.2W_t) - y_0 + 1}$$

and using two steps SRKM's with (R2 and PL) models that are given in equations (7) and (8) with step sizes $h_0 = 0.1$, $h_1 = 0.05$, $h_2 = 0.025$ and $h_3 = 0.013$ and is defined for all n=0,1,…,N.

Therefore, using equations (12) and (13), the following results given in tables (1) - (6) which represent the approximate variable order method, exact results and the absolute error, respectively for the weak solution at x=0.1:

Table (1)
The numerical results for the weak solutions using variable order method with R2-model

Level	O(h_j)	O(h_j^2)	O(h_j^3)	O(h_j^4)	
0	-0.09963055				
1	-0.09963032	-0.09963009			
2	-0.09972327	-0.09981622	-0.10000235		
3	-0.09966088	-0.09959848	-0.09938075	-0.09875915	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Table (2)
The exact results solutions by using variable order method

Level	O(h_j)	O(h_j^2)	O(h_j^3)	O(h_j^4)	
0	-0.09963202				
1	-0.09963085	-0.09962968			
2	-0.09972341	-0.09981598	-0.10000228		
3	-0.09966091	-0.09959838	-0.09938078	-0.09875928	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Table (3)
The absolute error between the numerical results for the weak solutions with R2-model and the exact results for the solutions using variable order method

Level	O(h_j)	O(h_j^2)	O(h_j^3)	O(h_j^4)	
0	1.4618×10^{-6}				
1	5.26297×10^{-7}	4.0921×10^{-7}			
2	1.42624×10^{-7}	2.41048×10^{-7}	7.2887×10^{-8}		
3	3.60967×10^{-8}	1.06527×10^{-7}	2.79934×10^{-8}	1.28874×10^{-7}	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Table (4)
The numerical results for the weak solutions using variable order method with PL-model

Level	O(h_j)	O(h_j^2)	O(h_j^3)	O(h_j^4)	
0	-0.10006628				
1	-0.0999485	-0.09983072			
2	-0.099768	-0.09958751	-0.09934429		
3	-0.09969615	-0.09962429	-0.09966108	-0.09997787	⋯

⋮	⋮	⋮	⋮	⋮	⋱

Table (5)
The exact results solutions by using the variable order method

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	-0.09973404				
1	-0.09974248	-0.09975092			
2	-0.09965491	-0.09956734	-0.09938376		
3	-0.09963703	-0.09967827	-0.09978919	-0.10019463	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Table (6)
The absolute error between the numerical results for the weak solutions with PL-model and the exact results for the solutions using variable order method

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	3.32244×10^{-4}				
1	2.06022×10^{-4}	7.98008×10^{-5}			
2	1.13094×10^{-4}	2.0166×10^{-5}	3.94688×10^{-5}		
3	5.91213×10^{-5}	5.3973×10^{-5}	1.28112×10^{-4}	2.16755×10^{-4}	⋯
⋮	⋮	⋮	⋮	⋮	⋱

<u>Example (2):</u>

Consider to the linear Stratonovich SODE given by, **(10)**:

$$d\,y_t = \frac{7}{8} y_t\,dt + \frac{1}{2} y_t \circ dW_t,$$

with the initial condition $y_0 = 0.5$, and for comparison purpose, the exact solution is given by:

$$y_t = y_0 \exp\left(\frac{7}{8} t + W_t\right)$$

and using two steps SRKM's with (R2 and PL) models that are given in equations (7) and (8) with step sizes $h_0 = 0.1$, $h_1 = 0.05$, $h_2 = 0.025$ and $h_3 = 0.013$ and is defined for all n=0,1,…,N.

Therefore, using equations (12) and (13), the following results given in tables (7) - (12) which represent the approximate variable order method, exact results and the absolute error, respectively for the weak solution at x=0.1:

Table (7)
The numerical results for the weak solutions using variable order method with R2-model

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	0.54607854				
1	0.54569265	0.54530676			
2	0.54569103	0.54568941	0.54607206		

3	0.54568609	0.54568114	0.54567288	0.54527369	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Table (8)
The exact results solutions by using variable order method

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	0.54619186				
1	0.54572822	0.54526458			
2	0.54570117	0.54567411	0.54608365		
3	0.54568873	0.54567365	0.54567318	0.54526272	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Table (9)
The absolute error between the numerical results for the weak solutions with R2-model and the exact results for the solutions using variable order method

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	1.13325×10^{-4}				
1	3.55705×10^{-5}	4.21837×10^{-5}			
2	1.01355×10^{-5}	1.52995×10^{-5}	1.15848×10^{-5}		
3	2.63722×10^{-6}	7.49827×10^{-6}	3.02935×10^{-7}	1.09789×10^{-5}	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Table (10)
The numerical results for the weak solutions using variable order method with PL-model

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	0.54417286				
1	0.54491478	0.5456567			
2	0.54529575	0.54567672	0.54569675		
3	0.54553026	0.54576477	0.54585281	0.54600887	⋯
⋮	⋮	⋮	⋮	⋮	⋱

Table (11)
The exact results solutions by using variable order method

Level	**O(h_j)**	**O(h_j^2)**	**O(h_j^3)**	**O(h_j^4)**	
0	0.54619186				
1	0.54593727	0.54568267			
2	0.54581177	0.54568627	0.54568988		
3	0.54578991	0.54550839	0.54533051	0.54497115	⋯

⋮	⋮	⋮	⋮	⋮	⋱

Table (12)
The absolute error between the numerical results for the weak solutions with PL-model and the exact results for the solutions using variable order method

Level	$O(h_j)$	$O(h_j^2)$	$O(h_j^3)$	$O(h_j^4)$	
0	2.019×10^{-3}				
1	1.02249×10^{-3}	2.59703×10^{-5}			
2	5.16019×10^{-4}	9.55111×10^{-6}	6.86804×10^{-6}		
3	2.59647×10^{-4}	2.56372×10^{-4}	5.22295×10^{-4}	1.03772×10^{-3}	…
⋮	⋮	⋮	⋮	⋮	⋱

5. Conclusion:

1. Variable order method gives very high accurate result in comparison between the approximate result and exact result.

2. Solution of non-linear SODE's using variable order method give more accurate result than the solution of linear SODE's.

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طريقة الرتبة المتغيرة لحل المعادلات التفاضلية التصادفية الاعتيادية بأستخدام طرائق رانج – كوتا ذات الخطوتين

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الخلاصة

في هذا البحث، ولغرض تحسين دقة النتائج تم استحداث طريقة الرتبة المتغيرة لحل معادلات تفاضلية تصادفية من نوع ستراتونوفيتش وبأستخدام طرائق رانج – كوتا ذات الخطوتين وذلك عن طريق زيادة رتبة تقارب الطريقة العددية .. إن الطريقة المقترحة تم دراستها لمعادلات تصادفية من نوع ستراتونوفيتش.