Approximation by q- Bernstein Schurer-SzaszMirakyan Operators for Functions in Two Variables

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Abstract

In this paper, we define operators of summation-integral q-type of BernsteinSchurer – SzaszMirakyan operators F_n in two dimensional space. Firstly we restrict the operators in BernsteinSchurer operators and study the restrictionQ_n, then we discuss the convergence for the operators and then we prove a Voronovskaya- type asymptotic formula for this operators.

Key words: q-Bernstein-Schurer operators, q-SzaszMirakjan operators, linear positive operators, Korovkin theorem, Voronovskaja-type asymptotic formula.

1. Introduction

In 1912, Bernstein defined a sequence of linear positive operators called the Bernstein polynomials as: [1]

$$\beta_n(f;x) = \sum_{k=0}^{n} b_{n,k}(x)\, f\left(\frac{k}{n}\right),$$
$$where\ \ b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1]. \ (1.1)$$

In 1932, Voronovskaya showed that the convergence of $\beta_n(f;x)$ to $f(x)$ as $n \to \infty$ is slow but sure. [2]

In 1950, Szasz generalized the Bernstein operators to infinite interval$[0,\infty)$, which is calledSzasz-Mirakyan operators defined as:[3]

$$S_n(f(t);x) = \sum_{k=0}^{\infty} z_{n,k}(x) f\left(\frac{k}{n}\right), \quad (1,2)$$

where$z_{n,k}(x) = \frac{e^{-nx}(nx)^k}{k!}$.

In 1962, Schurer , developed the Bernstein operators from [0,1] to [0, c+1] define as:[4]

$$\beta_{n,c}(f;x) = \sum_{k=0}^{n+c} b_{n+c,k}(x)\, f\left(\frac{k}{n}\right),$$
$$x \in [0,1], \ \ c \in \aleph. (1.3)$$

We used the notation [n] instead $[n]_q$ where q is value or sequence and we need the following definitions: [5], [6]

For $n \in N^0$, $N^0 = \{0,1,2,3,\dots\}$. The q-analogue is defined as:

$$[n] = \begin{cases} \dfrac{1-q^n}{1-q} if \quad n \neq 0\ , \quad q \in \mathbb{R}^+/\{1\} \\ \quad 0 if n = 0 \quad . \end{cases}$$

For n positive integer we can write

$$[n] = 1 + q + q^2 + q^3 + \cdots + q^{n-1};$$

The q- factorial is defined as:

$$[n]! = \begin{cases} [1][2][3] \dots [n] & ; n \in N \\ 1 & ; n = 0. \end{cases}$$

The q-derivative, when $q \neq 1$, of a function $f(x)$ is define as:

$$\left(D_q f\right)(x) = \frac{f(qx) - f(x)}{(q-1)x}\ , x \neq 0.$$

The formula for the q-derivative of a product of two functions is define as:

$$D_q\big(u(x)v(x)\big) = D_q\big(u(x)\big)v(x) + u(qx)D_q\big(v(x)\big),$$

and

$$D_q^n f = \begin{cases} D_q^{n-1}(Df); & n \neq 0 \\ f\ ; & n = 0 \end{cases}$$

The q-analogue of $(t-x)^n$ is define by:
$(t-x)_q^n = (t-x)(t-qx)$

$$(t-q^2x)\dots(t-q^{n-1}x).$$

The q-Taylor's formula defined as:

$$f(t)=\sum_{k=0}^{\infty}\frac{(t-x)_q^k}{[k]!}D_q^k f(x).$$

The q-exponential function define as:

$$e_q(x)=\sum_{k=0}^{\infty}\frac{x^k}{[k]_q!}.$$

In 1987, Lupas generalized the Bernstein polynomials involving q-integers which defined as: (7)

$$\beta_n(f,q_n;x)=\sum_{k=0}^{n}b_{n,k}(q_n,x)\,f\left(\frac{[k]}{[n]}\right), \qquad (1,4)$$

where

$$b_{n,k}(q_n,x)=\begin{bmatrix}n\\k\end{bmatrix}x^k(1-x)_{q_n}^{n-k},$$

$x\in[0,1], q_n\in(0,1), q_n\to 1\ as n\to\infty.$

In 2007,Delen and Tuncaintroduced certain linear positive operators of Bernsten – Szaszin two dimension defined as: (8)

$$L_{n,m}^{\alpha_i\beta_j}(f(t,s);x,y)=\sum_{k=0}^{\infty}\sum_{p=0}^{m}z_{n,k}(x)\,b_{m,p}(y)\times f\left(\frac{k+\alpha_1}{n+\beta_1},\frac{p+\alpha_1}{m+\beta_1}\right). \qquad (1.5)$$

In 2010, Mahamudovintroduced another type of q-Szasz-Mirakyan operators defined as: (9)

$$A_n(f,q;x)=\sum_{k=0}^{\infty}z_{n,k}(q;x)f\left(\frac{[k]_q}{[n]_q}\right), \quad (1.6)$$

where

$$z_{n,k}(q;x)=\frac{1}{q^{k(k-1)/2}}\frac{{[n]_q}^k x^k}{[k]_q!}e_q\left(-[n]_q\frac{x}{q^k}\right)$$

and$q>1$.

In 2011, Muraru introduce a generalization of the Bernstein-Schurer operators based on q-integers define as: (10)

$$\beta_{n,c}(f,q;x)=\sum_{k=0}^{n+c}\begin{bmatrix}n+c\\k\end{bmatrix}x^k(1-x)_q^{n+c-k}f\left(\frac{[k]}{[n]}\right),x\in[0,1]. \qquad (1.7)$$

In 2012, Ghadhban introduces a new modification define as: (11)

$$B_n(f(t),q_n;x)=[n]_{q_n}\sum_{k=0}^{\infty}z_{n,k}(q_n;x)\times\int_{[k]_{q_n}/[n]_{q_n}}^{[k+1]_{q_n}/[n]_{q_n}}f(t)\,d_{q_n}t, \qquad (1.8)$$

$q_n\in(0,1) and q_n\to 1\ as n\to\infty.$

In this paper, we define and study the following operators :

For$\alpha>0$, the space of all continuous real-valued functions on the are $[0,c+1]\times[0,\infty)$ such that $|f(t,s)|\le Ae_q^{\alpha(t+s)}$ for some constant $A>0$.

Suppose that $q_n,\breve{q}_n\in(0,1)$; $q_n,\breve{q}_n\to 1\ as\ n\to\infty$, , and $f\in C_\alpha([0,c+1]\times[0,\infty))$ $x\in[0,1], y\in[0,\infty), c\in\aleph$ and for$q_n^n=a,\ a<1$and $(\breve{q}_n-1)=o([n]^{-2})$we define:

$$F_n\big(f(t,s),q_n,\breve{q}_n,;(x,y)\big)=[n+1][n]\times\sum_{k=0}^{n+c}\sum_{p=0}^{\infty}b_{n+c,k}(q_n;x)\,z_{n,p}(\breve{q}_n;y)\times\int_{\frac{[k]_{q_n}}{[n+1]_{q_n}}}^{\frac{[k+1]_{q_n}}{[n+1]_{q_n}}}\int_{\frac{[p]_{\breve{q}_n}}{[n]_{\breve{q}_n}}}^{\frac{[p+1]_{\breve{q}_n}}{[n]_{\breve{q}_n}}}f(t,s)d_{\breve{q}_n}s\,d_{q_n}t. \quad (1.9)$$

We prove the convergence of F_n by applying the Korovkin theorem, and establish Voronovskaja-type asymptotic formula for this operator.

2. Preliminaries and results.

Here we will give some of the lemmas and theorems, which we used in our work.

Lemma (2.1):[11]

For the function $z_{n,k}(q_n;x)$ we have:

$$(i)\sum_{k=0}^{\infty} z_{n,k}(q_n;x) = 1;$$

$$(ii)\sum_{k=0}^{\infty}[k]z_{n,k}(q_n;x) = [n]x;$$

$$(iii)\sum_{k=0}^{\infty}[k]^2 z_{n,k}(q_n;x) = [n]^2x^2 + [n]x;$$

$$(iv)\sum_{k=0}^{\infty}[k]^3 z_{n,k}(q_n;x) = [n]^3x^3 + (2+q_n)[n]^2x^2 + [n]x.$$

$$(v)\ x\,D_{q_n} z_{n,k}(q_n;x) = ([k]-[n]x)z_{n,k},$$

(vi) Suppose that

$$\varphi_{n,m}(q_n;x) = \sum_{k=0}^{\infty}[k]^m z_{n,k}(q_n;x),$$

then

$$\varphi_{n,m+1}(q_n;x) = xD_{q_n}\varphi_{n,m}(q_n;x) + [n]x\varphi_{n,m}(q_n;x).$$

Theorem(2.1)(*Korovkin theorem*):[12]

If $M_{n,m}(f(t,s);x,y)$ be a sequence of linear positive operators of 2 dimensional space with the norm $\|.\|$ and the four conditions are hold:

$$(i)\lim_{n,m\to\infty}\|M_{n,m}(1;(x,y))-1\| = 0;$$

$$(ii)\lim_{n,m\to\infty}\|M_{n,m}(t;(x,y))-x\| = 0;$$

$$(iii)\lim_{n,m\to\infty}\|M_{n,m}(s;(x,y))-y\| = 0;$$

$$(iv)\lim_{n,m\to\infty}\|M_{n,m}(t^2+s^2;(x,y)) - (x^2+y^2)\| = 0.$$

Then

$$\|M_{n,m}(f(t,s);(x,y)) - f(x,y)\| = 0 \ \text{as}\ n,m\to\infty.$$

For $m\in N^0$, the m-th order q-moments $(T_{n,m})_2(q_n;x)$ for the operators(1.8) are defineas:[11]

$$(T_{n,m})_2(q_n;x) = B_n((t-x)_{q_n}^m, q_n;x)$$

$$= [n]\sum_{k=0}^{\infty} S_{n,k}(q_n;x) \times \int_{[k]/[n]}^{[k+1]/[n]} (t-x)_{q_n}^m\, d_{q_n}t.$$

Lemma (2.2):[11]

For the function $(T_{n,m})_2(q_n;x)$ we have:

$$(i)(T_{n,0})_2(q_n;x) = (q_n-1)[n]x+1;$$

$$(ii)(T_{n,1})_2(q_n;x) = \frac{q_n^2+q_n-2}{[2]}x + \frac{1}{[2][n]};$$

$$(iii)(T_{n,2})_2(q_n;x) = \frac{2q_n^2-q_n-1}{[3]}x^2 + \frac{q_n^3+2q_n^2+2q_n-2}{[3][n]}x + \frac{1}{[3][n]^2};$$

$$(iv)(T_{n,3})_2(q_n;x) = \frac{-2q_n^5+q_n^4+2q_n-1}{[4]}x^3 + \frac{q_n^4+4q_n^3+q_n^2-4q_n-2}{[4][n]}x^2 + \frac{q_n^4+3q_n^3+5q_n^2+3q_n-2}{[4][n]^2}x + \frac{1}{[4][n]^3}.$$

3. Main results

We need to know some properties of the restriction operators

$$Q_n(f(t),q_n;x) \equiv F_n(f(t,s),q_n,\check{q}_n;(x,y))|_{s=0}$$

in the study of the operators F_n. We do not find a study give as this object, so, we firstly study the operators Q_n as follows:

$$Q_n(f(t),q_n;x) = [n+1]_{q_n} \times \sum_{k=0}^{n+c} b_{n+c,k}(q_n;x) \times \int_{[k]_{q_n}/[n+1]_{q_n}}^{[k+1]_{q_n}/[n+1]_{q_n}} f(t)\, d_{q_n}t. \quad (3.1)$$

Lemma(3.1):

For the weight functions $b_{n,k}(q_n;x)$, we have:

(1) $\sum_{k=0}^{n+c} b_{n+c,k}(q_n;x) = 1$;

(2) $\sum_{k=0}^{n+c} b_{n+c,k}(q_n;x)\,[k] = [n+c]x$;

(3) $\sum_{k=0}^{n+c} b_{n+c,k}(q_n;x)\,[k]^2 = [n+c]x + q_n[n+c][n+c-1]x^2$;

(4) $\sum_{k=0}^{n+c} b_{n+c,k}(q_n;x)\,[k]^3 = [n+c]x + (2+q)q[n+c] \times[n+c-1]x^2 + q^3[n+c][n+c-1] \times[n+c-2]x^3$;

(5) $\sum_{k=0}^{n+c} b_{n+c,k}(q_n;x)\,[k]^4 = [n+c]x + (3q_n + 3{q_n}^2 + {q_n}^3) \times[n+c][n+c-1]x^2 + (3q^3 + 2q^4 + q^5)[n+c] \times[n+c-1][n+c-2]x^3 + q^6[n+c][n+c-1][n+c-2] \times[n+c-3]x^4$;

(6) $\sum_{k=0}^{n+c} b_{n+c,k}(q_n;x)\,[k]^5 = [n+c]x + \left(4q_n + 6{q_n}^2 + 4{q_n}^3 + {q_n}^4\right) \times[n+c][n+c-1]x^2 + \left(6{q_n}^3 + 8{q_n}^4 + 7{q_n}^5 + 3{q_n}^6 + {q_n}^7\right) \times[n+c][n+c-1][n+c-2]x^3 + (4{q_n}^6 + 3{q_n}^7 + 2{q_n}^8 + {q_n}^9) \times[n+c][n+c-1][n+c-2] \times[n+c-3]x^4 + {q_n}^{10}[n+c][n+c-1] \times[n+c-2][n+c-3] \times[n+c-4]x^5$.

Proof:

We can get (1), (2) by using the direct computation,

(3) $\sum_{k=0}^{n+c} b_{n+c,k}(q_n;x)\,[k]^2$

$$= \sum_{k=1}^{n+c} \frac{[n+c]!}{[k]!\,[n+c]!} x^k (1-x)_q^{n-k}\,[k]^2$$

$$= \sum_{k=1}^{n+c} \frac{[n+c]!}{[k-1]!\,[n+c]!} x^k (1-x)_q^{n-k} \times(1+q[k-1])$$

$$= \sum_{k=1}^{n+c} \frac{[n+c][n+c-1]!}{[k-1]!\,[n+c]!} x^k (1-x)_q^{n-k} + \sum_{k=2}^{n+c} \frac{[n+c][n+c-1][n+c-2]!}{[k-1]!\,[n+c]!}$$

$$= [n+c]x \sum_{k=0}^{n+c-1} b_{n+c-1,k}(q_n;x) + q[n+c][n+c-1]x^2 \times \sum_{k=0}^{n+c-2} b_{n+c-2,k}(q_n;x)$$

$$= [n+c]x + q[n+c][n+c-1]x^2;$$

By the same way we get (4), (5) and (6).

Lemma (3.2):

For $x \in [0, c+1]$the following conditions are holds

(1) $Q_n(1,q_n;x) = (q_n - 1)[n+c]x + 1 \to 1 \;\; as \;\; n \to \infty$;

(2) $$Q_n(t,q_n;x) = \frac{1}{[2][n+1]} + \frac{q_n^2 + 2q_n - 1}{[2][n+1]}[n+c]x + \frac{(q_n^3 - q_n)}{[2][n+1]}[n+c][n+c-1]x^2 \to x \;\; as \;\; n \to \infty \,;$$

(3) $$Q_n(t^2,q_n;x) = \frac{1}{[3][n+1]^2} + \frac{q_n^3 + 3q_n^2 + 3q_n - 1}{[3][n+1]^2}[n+c]x + \frac{q_n^5 + 2q_n^4 + 3q_n^3 - {q_n}^2 - 2q_n}{[3][n+1]^2} \times[n+c][n+c-1]x^2$$

$$+\frac{q_n^3\left({q^3}_n-1\right)}{[3][n+1]^2}[n+c][n+c-1]$$

$$\times[n+c-2]x^3 \to \ x^2 \ as \ n \to \ \infty;$$

$$(4) Q_n(t^3,q_n;x) = \frac{1}{[4][n+1]^3}$$

$$+\frac{q_n^4+4q_n^3+6q_n^2+4q_n-1}{[4][n+1]^3}[n+c]x$$

$$+\frac{q_n^7+3q_n^6+7q_n^5+8q_n^4+3q_n^3+3q_n^2-3q_n}{[4][n+1]^3}$$

$$\times[n+c][n+c-1]x^2$$

$$+\frac{q_n^9+2q_n^8+3q_n^7+4q_n^6-q_n^5-2q_n^4-3q_n^3}{[4][n+1]^3}$$

$$\times[n+c][n+c-1][n+c-2]x^3$$

$$+\frac{q_n^6(q_n-1)}{[4][n+1]^3}[n+c][n+c-1]$$

$$\times[n+c-2][n+c-3]x^4;$$

$$(5) Q_n(t^4,q_n;x) = \frac{1}{[5][n+1]^4}$$

$$+\frac{q_n^5+5q_n^4+10q_n^3+10q_n^2+5q_n-1}{[5][n+1]^4}$$

$$\times[n+c]x$$

$$+\frac{\begin{Bmatrix} q_n^9+4q_n^8+11q_n^7+19q_n^6+25q_n^5 \\ +19q_n^4+9q_n^3-6q_n^2-4q_n \end{Bmatrix}}{[5][n+1]^4}$$

$$\times[n+c][n+c-1]x^2$$

$$+\frac{\begin{Bmatrix} q_n^{12}+3q_n^{11}+7q_n^{10}+5q_n^9+16q_n^8+14q_n^7 \\ +15q_n^6-7q_n^5-8q_n^4-6q_n^3 \end{Bmatrix}}{[5][n+1]^4}$$

$$\times[n+c][n+c-1][n+c-2]x^3$$

$$+\frac{\begin{Bmatrix} q_n^{14}+2q_n^{13}+3q_n^{12}+4q_n^{11}+5q_n^{10} \\ -q_n^9-2q_n^8-3q_n^7-4q_n^6 \end{Bmatrix}}{[5][n+1]^4}$$

$$\times[n+c][n+c-1][n+c-2]$$

$$\times[n+c-3]x^4$$

$$+\frac{q_n^{10}(q_n^5-1)}{[5][n+1]^4}[n+c][n+c-1]$$

$$\times[n+c-2][n+c-3][n+c-4]x^5.$$

Proof:

Using Lemma (3.1), our consequences hold immediately.

For $m \in N^0$, we define the m-th order q-moments $\left(U_{n,m}\right)_2(q_n;x)$ for the operators(3.1)as:

$$\left(U_{n,m}\right)_2(q_n;x) = [n+1] \times\sum_{k=0}^{n+c} b_{n+c,k}(q_n;x) \times \int_{[k]/[n+1]}^{[k+1]/[n+1]} (t-x)_{q_n}^m \, d_{q_n}t. \quad (3.2)$$

<u>Lemma (3.3)</u>:

For the function $\left(U_{n,m}\right)_2(q_n;x)$, we have:

$$(1) \left(U_{n,0}\right)_2(q_n;x) = (q_n-1)[n+c]x+1;$$

$$(2) \left(U_{n,1}\right)_2(q_n;x) = \frac{1}{[2][n+1]} + \left(\frac{(q_n^2+2q_n-1)}{[2][n+1]}[n+c]-1\right)x + \left(\frac{(q_n^3-q_n)}{[2][n+1]}[n+c][n+c-1] - (q_n-1)[n+c]\right)x^2;$$

$$(3) \left(U_{n,2}\right)_2(q_n;x) = \frac{1}{[3][n+1]^2} + \frac{(q_n^3+2q_n^2+3q_n-1)}{[3][n+1]^2}[n+c]x - \frac{(1+q_n)}{[2][n+1]}x + \frac{(q_n^5+2q_n^4+3q_n^3-q_n^2-2q_n)}{[3][n+1]^2}$$

$$\times[n+c][n+c-1]$$
$$-\frac{(1+q_n)(q_n^2+2q_n-1)}{[2][n+1]}$$
$$\times([n+c]+q)x^2$$
$$+\frac{(q_n^6-q_n^3)}{[3][n+1]^2}[n+c][n+c-1]$$
$$\times[n+c-2]x^3$$
$$-\frac{(q_n^3-q_n)}{[2][n+1]}[n+c][n+c-1]x^3$$
$$+q_n(q_n-1)[n+c]x^3;$$

$$(4)\left(U_{n,3}\right)_2(q_n;x)=\frac{1}{[4][n+1]^3}$$
$$\frac{(q_n^4+4q_n^3+6q_n^2+4q_n-1)[n+c]}{[4][n+1]^3}x$$
$$-\frac{(1+q_n+q_n^2)}{[3][n+1]^2}x$$
$$+\frac{(q_n^7+3q_n^6+7q_n^5+8q_n^4+3q_n^3+3q_n^2-3q_n)}{[4][n+1]^3}$$
$$\times[n+c][n+c-1]x^2$$
$$-\frac{(q_n^5+4q_n^4+7q_n^3+5q_n^2+2q_n-1)}{[3][n+1]^2}$$
$$\times[n+c]x^2$$
$$+\frac{(q_n+q_n^2+q_n^3)}{[2][n+1]}x^2$$
$$+\frac{(q_n^9+2q_n^8+3q_n^7+4q_n^6-q_n^5-2q_n^4-3q_n^3)}{[4][n+1]^3}$$
$$\times[n+c][n+c-1][n+c-2]x^3$$
$$-\frac{\left(q_n^7+3q_{n]}^6+6q_n^5+4q_n^4-3{q_n}^2-2q_n\right)}{[3][n+1]^2}$$
$$\times[n+c][n+c-1]x^3$$
$$+\frac{(q_n^5+3q_n^4+2q_n^3+q_n^2-q_n)}{[2][n+1]}$$
$$\times[n+c]x^3-q_n^3x^3$$
$$+\frac{(q_n^7-q_n^6)}{[4][n+1]^3}[n+c][n+c-1]$$
$$\times[n+c-2][n+c-3]x^4$$
$$-\frac{(1+q_n+q_n^2)(q_n^6-q_n^3)}{[3][n+1]^2}[n+c]$$
$$\times[n+c-1][n+c-2]x^4$$
$$+\frac{(q_n+q_n^2+q_n^3)(q_n^3-q_n)}{[2][n+1]}[n+c]$$
$$\times[n+c-1]x^4$$
$$-q_n^3\big((q_n-1)[n+c]\big)x^4;$$

$$(5)\left(U_{n,4}\right)_2(q_n;x)=\frac{1}{[5][n+1]^4}$$
$$+\frac{(q_n^5+5q_n^4+10q_n^3+10q_n^2+4q_n-1)}{[4][n+1]^3}$$
$$\times[n+c]x-\frac{(1+q_n+q_n^2+q_n^3)}{[4][n]^3}x$$
$$+\frac{\begin{pmatrix}q_n^9+4q_n^8+11q_n^7+19q_n^6+25q_n^5\\+19q_n^4+9q_n^3-6q_n^2-4q_n\end{pmatrix}}{[5][n+1]^4}$$
$$\times[n+c][n+c-1]x^2$$
$$-\frac{(1+q_n+q_n^2+q_n^3)(q_n^4+4q_n^3+6q_n^2+4q_n-1)}{[4][n+1]^3}$$
$$\times[n+c]x^2$$
$$+\frac{(q_n+q_n^2+2q_n^3+q_n^4+q_n^5)}{[3][n+1]^2}x^2$$
$$+\frac{\begin{Bmatrix}q_n^{12}+3q_n^{11}+7q_n^{10}+5q_n^9+16q_n^8\\+14q_n^7+15q_n^6-7q_n^5-8q_n^4-6q_n^3\end{Bmatrix}}{[5][n+1]^4}$$
$$\times[n+c][n+c-1][n+c-2]x^3$$
$$-\frac{\begin{Bmatrix}(1+q_n+q_n^2+q_n^3)\\\times(q_n^6+3q_n^5+7q_n^4+8q_n^3+5q_n^2-3q_n-3)\end{Bmatrix}}{[4][n]^3}$$
$$\times[n+c][n+c-1]x^3$$

$$+\frac{\left\{\begin{matrix}(q_n+q_n^2+2q_n^3+q_n^4+q_n^5)\\ \times(q_n^3+3q_n^2+3q_n-1)[n+c]\end{matrix}\right\}}{[3][n+1]^2}x^3$$
$$-\frac{(q_n^3+q_n^4+q_n^5+q_n^6)}{[2][n]}x^3$$
$$+\frac{\left\{\begin{matrix}q_n^{14}+2q_n^{13}+3q_n^{12}+4q_n^{11}+5q_n^{10}\\ -q_n^9-2q_n^8-3q_n^7-4q_n^6\end{matrix}\right\}}{[5][n+1]^4}$$
$$\times[n+c][n+c-1][n+c-2]$$
$$\times[n+c-3]x^4$$
$$-\frac{\begin{matrix}(1+q_n+q_n^2+q_n^3)\\ (q_n^9+2q_n^8+3q_n^7+4q_n^6-q_n^5-2q_n^4-3q_n^3)\end{matrix}}{[4][n+1]^3}$$
$$\times[n+c][n+c-1][n+c-2]x^4$$
$$+\frac{\begin{matrix}(q_n+q_n^2+2q_n^3+q_n^4+q_n^5)\\ \times(q_n^5+2q_n^4+3q_n^3-q_n{}^2-2q_n)\end{matrix}}{[3][n+1]^2}$$
$$\times[n+c][n+c-1]x^4$$
$$-\frac{(q_n^3+q_n^4+q_n^5+q_n^6)(q_n^2+2q_n-1)}{[2][n+1]}$$
$$\times([n+c]+q_n^6)x^4$$
$$+\frac{q_n^{10}(q_n^5-1)}{[5][n+1]^4}[n+c][n+c-1]$$
$$\times[n+c-2][n+c-3][n+c-4]x^5$$
$$-\frac{q_n^6(q_n-1)(1+q_n+q_n^2+q_n^3)}{[4][n+1]^3}$$
$$\times[n+c][n+c-1][n+c-2]$$
$$\times[n+c-3]x^5$$
$$+\frac{(q_n+q_n^2+2q_n^3+q_n^4+q_n^5)q_n^3(q_n-1)}{[3][n+1]^2}$$
$$\times[n+c][n+c-1][n+c-2]x^5$$
$$-\frac{(q_n^3+q_n^4+q_n^5+q_n^6)(q_n^3-q_n)}{[2][n+1]}$$
$$\times[n+c][n+c-1]x^5$$
$$+q_n^6(q_n-1)[n+c]x^5.$$

Proof:

Byeasy evaluation the consequence (1) can be follows:

(2) $\left(U_{n,1}\right)_2(q_n;x)=Q_n\left((t-x)^1_{q_n},q_n;x\right)$

$$=Q_n(t,q_n;x)-xQ_n(1,q_n;x)$$
$$=\frac{1}{[2][n+1]}+\frac{q_n^2+2q_n-1}{[2][n+1]}[n+c]x$$
$$+\frac{(q_n^3-q_n)}{[2][n+1]}[n+c][n+c-1]x^2$$
$$-x((q_n-1)[n+c]x+1)$$
$$=\frac{1}{[2][n+1]}$$
$$+\left(\frac{(q_n^2+2q_n-1)}{[2][n+1]}[n+c]-1\right)x$$
$$+\frac{(q_n^3-q_n)}{[2][n+1]}[n+c][n+c-1]x^2$$
$$-(q_n-1)[n+c]x^2;$$

By the same way, we can evaluate (3), (4) and (5).

The operators$\boldsymbol{F_n\big(f(t,s),q_n,\breve{q}_n;(x,y)\big)}$:

The first result shows that the operators $F_n(f,q_n,\breve{q}_n;(x,y))$converges to the function $f(x,y)$ as$n\to\infty$, (see the equation (1.9)).

Lemma (3.4):

For the operators F_n the following conditions are hold:

(1) $\lim_{n\to\infty}\|F_n(1,q_n,\breve{q}_n;(x,y))-1\|_{C_\alpha}=0;$

(2) $\lim_{n\to\infty}\|F_n(t,q_n,\breve{q}_n;(x,y))-x\|_{C_\alpha}=0;$

(3) $\lim_{n\to\infty}\|F_n(s,q_n,\breve{q}_n;(x,y))-y\|_{C_\alpha}=0;$

(4) $\lim_{n\to\infty}\left\|F_n\left(t^2+s^2,q_n,\breve{q}_n;(x,y)\right)-(x^2+y^2)\right\|_{C_\alpha}=0.$

Proof:

By using Lemma (2.1) and the direct computation, we have:

(1) $\lim_{n\to\infty}\|F_n(1,q_n,\breve{q}_n;(x,y))-1\|_{C_\alpha}$

$$=\lim_{n\to\infty}\Bigg\|\ [n+1]\times\sum_{k=0}^{n+c}b_{n,k}(q_n;x)\left(\frac{1+[k](q_n-1)}{[n]}\right)\times[n]\sum_{p=0}^{\infty}z_{n,p}(\breve{q}_n;y)\times\left(\frac{1+[p](\breve{q}_n-1)}{[n]}\right)-1\ \Bigg\|_{C_\alpha}$$
$$=\lim_{n\to\infty}\|(q_n-1)[n+c]x+1\times\left((\breve{q}_n-1)[n]y+1\right)-1\|_{C_\alpha}=0$$

(2) $\lim_{n\to\infty}\left\|F_n\big(t,q_n,\breve{q}_n;(x,y)\big)-x\right\|_{C_\alpha}$

$$=\lim_{n\to\infty}\left\|\left\{\frac{1}{[2][n+1]}+\frac{q_n^2+2q_n-1}{[2][n+1]}\times[n+c]x+\frac{(q_n^3-q_n)}{[2][n+1]}[n+c]\times[n+c-1]x^2\right\}\times(1+(\breve{q}_n-1)[n]y)-x\right\|_{C_\alpha}$$

By using the same technique of (2) we can get the consequence (3) .

(4) $\lim_{n\to\infty}\|F_n(t^2+s^2,q_n,\breve{q}_n;(x,y))-(x^2+y^2)\|_{C_\alpha}$

$$=\lim_{n\to\infty}\left\|\quad[n][n+1]\sum_{k=0}^{n+c}b_{n,k}(q_n;x)\times\sum_{p=0}^{\infty}z_{n,p}(\breve{q}_n;y)\times\int_{\frac{[k]}{[n+1]}}^{\frac{[k+1]}{[n+1]}}\int_{\frac{[p]}{[n]}}^{\frac{[p+1]}{[n]}}(t^2+s^2)\,d_{\breve{q}_n}s\,d_{q_n}t-(x^2+y^2)\quad\right\|_{C_\alpha}$$

$$=\lim_{n\to\infty}\Bigg\|\quad\frac{1}{[3][n+1]^2}+\frac{q_n^3+3q_n^2+3q_n-1}{[3][n+1]^2}[n+c]x+\frac{q_n^5+2q_n^4+3q_n^3-{q_n}^2-2q_n}{[3][n+1]^2}\times[n+c][n+c-1]x^2+\frac{q_n^3\big({q^3}_n-1\big)}{[3][n+1]^2}[n+c][n+c-1]\times[n+c-2]x^3((\breve{q}_n-1)[n]y+1)$$
$$+((q_n-1)[n+c]x+1)\times\left\{\quad\frac{(\breve{q}_n^3-1)}{[3][n]^2}([n]^3y^3+(2+\breve{q}_n)[n]^2y^2+[n]y)+\frac{3\breve{q}_n^2}{[3][n]^2}([n]^2y^2+[n]y)+\frac{3\breve{q}_n}{[3][n]^2}([n]y)+\frac{1}{[3][n]^2}\quad\right\}-(x^2+y^2)\quad\Bigg\|_{C_\alpha}=0$$

Therefore , by Theorem (2.1) we get:

$$\left\|F_n\big(f(t,s),q_n,\breve{q}_n;(x,y)\big)-f(x,y)\right\|_{C_\alpha}=0\ \ as n\to\infty.\ \blacksquare$$

<u>Theorem (3.1):</u>

For $f\in C_\alpha$, and suppose that $\frac{f_{xx}(x,y)}{\partial_q^2x},\frac{f_{yy}(x,y)}{\partial_q^2y}$ and $\frac{f_{xy}(x,y)}{\partial_qx\partial_qy}$ are exist and continuous at a point $(x,y)\in([0,c+1]\times[0,\infty))$, then:

$$\lim_{n\to\infty}[n](F_n\big(f(t,s),q_n,\breve{q}_n;(x,y)\big)-f(x,y))=\frac{1}{2}f_x(x,y)+\frac{1}{2}(x,y)+\frac{1}{4}f_{xy}(x,y)+\frac{1}{2}yf_{yy}(x,y).$$

Proof:

By q-Taylor's formula [13]for $f\in C_\alpha$, about the point (x,y) we have:

$$f(t,s)=f(x,y)+f_x(x,y)(t-x)+f_y(x,y)(s-y)+\frac{1}{2}\{f_{xx}(x,y)(t-x)^2+2f_{xy}(t-x)(s-y)+f_{yy}(x,y)(s-y)^2\}+\varphi(t,s;x,y)\sqrt{(t-x)^4+(s-y)^4},$$

where$\varphi(t,s;(x,y)):=\varphi(t,s)$ is a function in the space$C_\alpha([0,c+1]\times[0,\infty))$ and $\varphi(t,s)\to(0,0)$ as $(t,s)\to(x,y)$ thus$\varphi(x,y)=(0,0)$.

$$F_n\big(f(t,s),q_n,\breve{q}_n;(x,y)\big)=f(x,y)F_n\big(1,q_n,\breve{q}_n;(x,y)\big)+f_x(x,y)F_n\Big((t-x),q_n;x)+f_y(x,y)F_n\big((s-y),\breve{q}_n;y\big)\Big)$$
$$+\frac{1}{2}f_{xx}(x,y)F_n((t-x)^2,q_n;x)+f_{xy}(x,y)F_n\big((t-x)(s-y),q_n,\breve{q}_n;(x,y)\big)+\frac{1}{2}f_{yy}(x,y)F_n((s-y)^2,\breve{q}_n;y)+F_n\Big(\varphi(t,s)\sqrt{(t-x)^4+(s-y)^4},q_n,\breve{q}_n;(x,y)\Big).$$

Using Lemmas (2.2) and (3.3) we have:

$\lim_{n\to\infty}[n](F_n\big(f(t,s),q_n,\breve{q}_n;(x,y)\big)$

$$-f(x,y))$$
$$= f_x(x,y)\left(\frac{1}{2}\right) + f_y(x,y)\left(\frac{1}{2}\right) + \frac{1}{4} f_{xy}(x,y)$$
$$+\frac{1}{2} y f_{yy}(x,y)$$
$$+\lim_{n\to\infty}[n]F_n(\varphi(t,s)$$
$$\times\sqrt{(t-x)^4+(s-y)^4}, q_n, \breve{q}_n; (x,y)).$$

To complete the proof, we must show that the term

$$[n]F_n(\varphi(t,s)$$
$$\times\sqrt{(t-x)^4+(s-y)^4}, q_n, \breve{q}_n; (x,y))$$
$$\to (0,0) \text{as} n \to \infty.$$

By using Cauchy-Schwartz inequality, we get:

$$\|F_n(\varphi(t,s)$$
$$\times\sqrt{(t-x)^4+(s-y)^4}, q_n, \breve{q}_n; (x,y))\|$$
$$\le |F_n(\varphi^2(t,s), q_n, \breve{q}_n; (x,y)|^{\frac{1}{2}}$$
$$\times|F_n((t-x)^4, q_n; x)$$
$$+ F_n((s-y)^4, \breve{q}_n; y)|^{\frac{1}{2}}$$

By the properties of $\varphi(t,s) = (0,0)\ as\ t \to x\ and\ s \to y$ we get $F_n(1, q_n, \breve{q}_n; (x,y)) = \varphi^2(x,y)$=0

Therefore,

$$\lim_{n\to\infty}[n]F_n(\varphi(t,s)$$
$$\times\sqrt{(t-x)^4+(s-y)^4}, q_n, \breve{q}_n; (x,y))$$
$$\to (0,0) \text{as} n \to \infty.$$

Hence, the proof of the Theorem is complete.

■

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التقريب باستخدام مؤثرات ***q*-BernsteinSchurer-SzaszMirakyan** لدوال بمتغيرين

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الخلاصة

في بحثنا هذا، عرفنا مؤثرات من النمط مجموع-تكامل لـ *Bernstein Schurer – SzaszMirakyan* وهي F_n في فضاء ثنائي البعد. أولا، سنقصر المؤثر في *BernsteinSchurer* وندرس هذا القصر وهو Q_n. ثم نناقش تقارب المؤثرونبرهن الصيغة المشابهه لـ *Voronovskaya* لهذا المؤثر.