A Reliable Algorithm of Homotopy Analysis Method for Solving Fuzzy Integral Equations of Fractional Order

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Abstract

In this paper, we based on the homotopy analysis method (HAM), a powerful algorithm is developed for the solution of linear and nonlinear fuzzy integral equations of fractional order. The proposed algorithm presents the procedure of constructing the set of base functions and gives the high-order deformation equation in a simple form. This method different from all other analytic methods, it provides us with a simple way to adjust and control the convergence region of solution series by introducing an auxiliary parameter h. The fractional derivative is described in the Caputo sense and the fractional integration is described in the Riemann-Liouville formula. The analysis is accompanied by some numerical examples to show the accuracy and validity of this approach.

Keywords: **Homotopy Analysis Method, Fractional Calculus, Fuzzy Set Theory.**

1-Introduction:

The concept of integration of fuzzy functions was first introduced by Dubois and Prade(1). Alternative approaches were later suggested by Goetschel andVoxman(2), Kaleva(3), Matloka(4), Nanda(5)and others. While GoetschelandVoxman(2)and later Matloka(4)preferred a Riemann integral typeapproach, Kaleva(3)chose to define the integral of fuzzy function, using the Lebesgue type concept for integration.The topics of fuzzy differential equations (FDE) and fuzzy integral equations (FIE) which attracted growing interest for some time, in particular in relation to fuzzy control, have been rapidly developed in recentyears. The first step which included applicable definitions of the fuzzy derivative and the fuzzy integral was followed by introducing FDE and FIE and establishing sufficient conditions for the existence of unique solutions to these equations. Finally, numerical algorithms for calculating approximates to these solutions were designed. Prior to discussing fuzzy differential and integral equations and their associated numericalalgorithms, it is necessary to present an appropriate brief introduction to preliminary topics such as fuzzy numbers and fuzzy calculus.

The concept of fuzzy sets which was originally introduced by Zadeh(6) led to the definition of the fuzzy number and its implementation in fuzzy control and approximate reasoning problems. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka(7, 8), Nahmias(9), Dubois andPrade(10) and Ralescu(11), all of which observed the fuzzy number as a collection of α-levels, $0 \le \alpha \le 1$. One of the first applications of fuzzy integration was given by Wu and Ma(12) who investigated the Fuzzy Fredholm integral equation of the second kind (FF-2). This work which established the existence of a unique solution to (FF-2) was followed by other work on FIE given by J.Y. Park, et al(13)where a fuzzy integral equation replaced an original fuzzy differential equation.In this paper we shall treat the approximate solution of the fuzzy integral equations of fractional order of the form

$$\tilde{y}(t) = \tilde{f}(t) + I^{q} K\left[\tilde{y}(t)\right] \quad (1)$$

where $\tilde{f}(t)$ is assumed to be fuzzy function and I^q is a fractional integration in the Riemann-Liouville sense. Fractional calculus has found diverse applications in various scientific and technological fields see Hilfer R.(14),et al such as thermal engineering, acoustic, electromagnetism, control, robotics, viscoelasticity, diffusion, signal processing and many other physical processes.

Odibat and et.al,(15) extended the application of the homotopy analysis method proposed by Lioa(16) to solve nonlinear differential equations of fractional order. They also show that the procedure of constructing the set of base functions and the high order deformation equation in a simple form. Differential equations involving derivatives of non-integer order have shown to be adequate models for various physical phenomena in areas like rheology, damping laws, diffusion processes, etc. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives by Maindari F.(17), and the fluid-dynamic traffic model with fractional derivatives by G. Adomian(18,19) can eliminate the deficiency arising from the assumption of continuum traffic flow. Based on experimental data fractional partial differential equations for seepage flow in porous media are suggested by G. Adomian[18,19], and differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena by N.T. Shawagfeh[20]. Most fractional differential equations do not have exact analytic solutions, so approximation and numerical techniques must be used.

There are several definitions of a fractional derivative of order $\alpha>0$ given by N. Shawagfeh and D. Kaya[21], two most commonly used definitions are Riemann-Liouville and Caputo. Each definitions uses Riemann-Liouville fractional integration and derivatives of whole orders. Throughout this paper we will exhibit the fuzzy integral equation of fractional order of the form Eq.(1) using a reliable algorithm of HAM.

The homotopy analysis method (HAM) was first proposed in his Ph.D. thesis by Liao S.J.[22]. A systematic and clear exposition on HAM is given by S.J. Liao[16]. In recent years, this method has been successfully employed to solve many types of non-linear, homogenous or non-homogenous equations and system of equations as well as problems in science and engineering. The HAM is based on homotopy, a fundamental concept in topology and differential geometry. Briefly speaking, by means of the HAM, one construct a continuous mapping of an initial guess approximation to the exact solution of considered equations by Jafari H. andSeifi S.[23]. An auxiliary linear operator is chosen to construct such kind of continuous mapping and an auxiliary parameter is used to ensure the convergence of solution series. The method enjoys great freedom in choosing initial approximations and auxiliary linear operators. By means of this kind of freedom, a complicated non-linear problem can be transferred into an infinite number of simpler, linear sub- problems.

2- Basic Concepts of Fuzzy Set Theory:

In this section, we shall present some basic definitions of fuzzy set theory including the definition of fuzzy numbers and fuzzy functions.

Definition (1)[24]:

Let X be a nonempty set. A fuzzy set A in X is characterized by its membership function $A : X\rightarrow[0; 1]$ and A(x), called the membership function of fuzzy set A, is interpreted as the degree of membership of element x in fuzzy set A
for each $x\in X$.

Definition (2)[24]

Let X be any set of elements, a fuzzy set $\tilde{A}$ is characterized by a membership function

$\mu_{\tilde{A}}(x) : X \longrightarrow [0, 1]$, and may be written as the set of points

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}.$$

Definition (3) [24]***:***

The crisp set of elements that belong to the set $\tilde{A}$ at least to the degree α is called the weak α-level set (or weak α-cut), and is defined by:

$$A_\alpha = \{x \in X : \mu_{\tilde{A}}(x) \ge \alpha\}$$

While the strong α-level set (or strong α-cut) is defined by:

$$A'_\alpha = \{x \in X: \mu_{\tilde{A}}(x) > \alpha\}.$$

Definition (4) [25]***:***

A fuzzy subset $\tilde{A}$ of a universal space X is convex if and only if the sets A_α are convex, $\forall\ \alpha \in [0, 1]$.

or equivalently, we can define convex fuzzy set directly by using its membership function to satisfy:

$$\mu_{\tilde{A}}[\lambda x_1 + (1-\lambda)x_2] \ \ge\ \text{Min}\ \{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$$ for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$.

Remark (1) [25]***:***

A fuzzy number $\tilde{M}$ may be uniquely represented in terms of its α-level sets, as the following closed intervals of the real line:

$$M_\alpha = [m - \sqrt{1-\alpha}, m + \sqrt{1-\alpha}]$$

or

$$M_\alpha = [\alpha m, \frac{1}{\alpha} m]$$

Where m is the mean value of $\tilde{M}$ and $\alpha \in (0, 1]$. This fuzzy number may be written as $\tilde{M} = [\underline{M}, \overline{M}]$, where $\underline{M}$ refers to the greatest lower bound of $\tilde{M}$ and $\overline{M}$ to the least upper bound of $\tilde{M}$.

Remark (2) [25]***:***

Similar to the second approach given in remark (1), one can fuzzyfy any crisp or nonfuzzy function f, by letting:

$$\underline{f}(x) = \alpha f(x), \ \overline{f}(x) = \frac{1}{\alpha} f(x), \ x \in X, \ \alpha \in (0, 1],$$

and hence, the fuzzy function $\tilde{f}$ in terms of its β-levels is given by $f_\alpha = [\underline{f}, \overline{f}]$.

3-Fractional Integration and Derivatives

In this section some definitions and properties related to fractional differentiation and integration are given.

the Riemann-Liouville fractional integration of order q is defined as,

$$f(x) = \frac{1}{\Gamma(q)} \int_0^x (x - t)^{q-1} f(t) dt, \;\; q > 0, x > 0 \quad (2)$$

The next two equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively,

$$D^{\alpha} f(x) = \frac{d^m}{dx^m} (f(x)) \quad (3)$$

$$D_*^{\alpha} f(x) = (\frac{d^m}{dx^m} f(x)) \quad (4)$$

Where $m - 1 < \alpha \leq m$ and $m \in N$.

4- Homotopy Analysis Method (HAM)

In this section the basic ideas of the homotopy analysis method are introduced. Here a description of the method is given by S.J. Liao(16) to handle the general nonlinear problem,

$$N[y(t)] = 0, \; t > 0. \quad (5)$$

where N is a nonlinear operator and $y(t)$ is unknown function of the independent variable t.

4.1- Zero-Order Deformation Equation

Let $y_0(t)$ denote an initial guess of the exact solution of Eq.(5), $h \neq 0$ an auxiliary parameter, $H(t) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property,

$$L[f(t)] = 0, \text{ when } f(t) = 0. \quad (6)$$

The auxiliary parameter h, the auxiliary function $H(t)$, and the auxiliary linear operator L play important roles within the HAM to adjust and control the convergence region of solution series. Liao (16) constructs, using $q \in [0,1]$ as an embedding parameter, the so-called zero-order deformation equation,

$$(1-q) L[\Phi(t;q) - y_0(t)] = q \; h \; H(t) \; N[\Phi(t;q)], \quad (7)$$

Where $\Phi(t;q)$ is the solution which depends on h, $H(t)$, L, $y_0(t)$ and q, when q=0 the zero-order deformation Eq.(7) becomes,

$$\Phi(t;0) = y_0(t), \quad (8)$$

and when q=1, since $h \neq 0$ and $H(t) \neq 0$, the zero-order Eq.(7) reduces to,

$$N[\Phi(t;1)] = 0. \quad (9)$$

So, $\Phi(t;1)$ is exactly the solution of the nonlinear Eq.(5). Define the so-called m^{th}-order deformation derivatives,

$$y_m(t) = \frac{1}{m!} \frac{\partial^m \Phi(t;q)}{\partial q^m} \bigg|_{q=0}. \quad (10)$$

If the power series (10) of $\Phi(t;q)$ converge at q=1, then we gets the following series solution:

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) ,$$

where the $y_m(t)$ terms can be determined by the so-called high-order deformation equations described below.

4.2 High-Order Deformation Equation

Define the vector,

$$\vec{y}_n = \{y_0(t), y_1(t), y_2(t), ..., y_n(t)\}. (11)$$

Differentiating Eq.(7) m-times with respect to embedding parameter q, then setting q=0 and dividing them by m!, we have the so-called m^{th}-order deformation equation,

$$L[y_m(t) - X_m y_{m-1}(t)] = h\ H(t)\ R_m(\vec{y}_m, t) \quad , \quad (12)$$

Where,

$$X_m = \begin{cases} 0, & m \le 1. \\ 1, & \text{otherwise.} \end{cases}$$

and

$$R_m(\vec{y}_{m-1}, t) = \frac{1}{(m-1)!} \left. \frac{\partial^{m-1} N[\Phi(t;q)]}{\partial q^{m-1}} \right|_{q=0} . \quad (13)$$

For any given nonlinear operator N, the term $R_m(\vec{y}_m, t)$ can be easily expressed by Eq.(13). Thus we can gain $y_1(t), y_2(t), y_3(t), ...$ by means of solving the linear high-order deformation Eq.(12) one after the other in order. The m^{th}-orderapproximation of $y(t)$ is given by:

$$y(t) = \sum_{k=0}^{m} y_k(t) \ .(14)$$

5-The Reliable Algorithm of HAM

In this section, we present a reliable approach of the homotopyanalysis method that given by Odibat Z., et al. [(15)]This modification can be implemented for integer order and fractional order nonlinear equations.

Now, to illustrate the basic ideas of this algorithm, we consider the following nonlinear integral equation of fractional order:

$$D_*^{\alpha} y(t) = N(y) + g(t), \quad t > 0. (15)$$

Where N is a nonlinear operator which might include integer order or fractional order integration, g(t) is a known analytic function.

In view of the homotopy technique, we can construct the following homotopy:

$$(1-q)L[\Phi(t,q) - \Phi_0(t)] = qhH(\Phi(t,q) - N[\Phi(t,q)] - g(t)](16)$$

Where qϵ[0,1] is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, H(t) $\neq 0$ is an auxiliary function,$\Phi_0(t)$ is an initial guess of y(t).

When q=0, Eq.(16) becomes:

$$\Phi(t,0) - \Phi_0(t) = 0(17)$$

It is obvious that when q=1, Eq.(16) becomes the original nonlinear Eq.(15). Thus as q various from 0 to 1, the solution y(x,q) varies from the initial guess $y_0(t)$ to the solution $\Phi(t,1)$. The basic assumption ofthis approach is that the solution of Eq. (15) can be expressed as a power series in q,

$$\Phi = \Phi_0 + q\Phi_1 + q^2\Phi_2 + \cdots (18)$$

Substituting the series (18) into the homotopy (16) and then equating the coefficient of the like powers of q, we obtain the high-order deformation equations, $\Phi_1 = h\,H(\Phi_0 - N(\Phi_0) - g(t))$

$$\Phi_2 = \Phi_1 + h\,H(\Phi_1\text{-}N_1(\Phi_0,\Phi_1))$$

$$\Phi_3 = \Phi_2 + h\,H(\Phi_2\text{-}N_2(\Phi_0,\Phi_1,\Phi_2))(19)$$

$$\Phi_4 = \Phi_3 + h\,H(\Phi_3\text{-}N_3(\Phi_0,\Phi_1,\Phi_2,\Phi_3))$$

Where,

$$N(\Phi_0 + q\Phi_1 + q^2\Phi_2 + \cdots) = N_0(\Phi_0) + qN_1(\Phi_0,\Phi_1) + q^2N_2(\Phi_0,\Phi_1,\Phi_2) + \ldots$$

The approximate solution of Eq.(15), therefore, can be readily obtained,

$$y = \lim_{q\to 1}\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots .(20)$$

The success of the technique is based on the proper selection of the initial guessΦ_0.

Applying the operator I^α to both sides of equation (15) gives,

$$y(t) = \sum_{k=0}^{m-1} y^k(0^+)\frac{t^k}{k!} + I^\alpha N(y) + I^\alpha g(t), t > 0(21)$$

Neglecting the nonlinear term $I^\alpha N(y)$on the right hand side, we can use the remaining part as the initial guess of the solution. That is

$$\Phi_0(t) = \sum_{k=0}^{m-1} y^k(0^+)\frac{t^k}{k!} + I^\alpha g(t)(22)$$

<u>6-The Reliable Algorithm of HAM for Solving Fuzzy Integral Equation of Fractional Order:</u>

In this section we shall use the reliable algorithm of the HAM that was given in section (5) in order to find the approximate solution of the fuzzy fractional integral equations given by:

$$\tilde{y}(t) = \tilde{f}(t) + N(\tilde{y}) (23)$$

Where

$$\tilde{N}(\tilde{y}) = I^q K[\tilde{y}(t)] \text{ and}$$

$$I^q K[\tilde{y}(t)] = \frac{1}{\Gamma(q)}\int_0^t (x-s)^{q-1} K(\tilde{y}(s))\,ds$$

(24)

The approximate solution of Eq.(23) can be written as $\tilde{y} = [\underline{y}, \overline{y}]$ and in order to find $\overline{y}$ and $\underline{y}$ we must solve the problems:

$$\underline{y} = \underline{f} + N(\underline{y})$$

Where

$$N(\underline{y}) = I^q\left[K(\underline{y}(t))\right] = \frac{1}{\Gamma(q)}\int_0^t (x-s)^{q-1} K(\underline{y}(t))ds$$

and

$$\overline{y} = \overline{f} + N(\overline{y})$$

Where

$$N(\overline{y}) = I^q\left[K(\overline{y}(t))\right] = \frac{1}{\Gamma(q)}\int_0^t (x-s)^{q-1} K(\overline{y}(t))ds,$$

respectively. Applying the reliable algorithm of HAM for (25) to find $\underline{y}$ we need to construct the following homotopy:

$$(1-q)\left[\underline{\Phi}(t,q) - \underline{\Phi}_0(t)\right] = qh\,H(\underline{\Phi}(t,q) - N\left(\underline{\Phi}(t,q)\right) - \underline{f}(t))(27)$$

Where $q\epsilon[0,1]$, $h\neq 0$ is anonzero auxiliary parameter, H(t) $\neq 0$ is an auxiliary function,$\underline{\Phi}_0$(t) is an initial guess of $\underset{\sim}{y}(t)$.

Substituting Eq.(18) into Eq.(27) and then equating the coefficient of like powers of q. Hence, after seeking h=-1 and H=1; therefore, Eq.(19) becomes:

$$\underline{\Phi}_1 = N_0(\underline{\Phi}_0) = \frac{1}{\Gamma(q)}\int_0^t (x-s)^{q-1}\underline{A}_0(s)ds$$

$$\underline{\Phi}_2 = N_1(\underline{\Phi}_0, \underline{\Phi}_1) = \frac{1}{\Gamma(q)}\int_0^t (x-s)^{q-1}\underline{A}_1(s)ds$$

$$\underline{\Phi}_3 = N_2(\underline{\Phi}_0, \underline{\Phi}_1, \underline{\Phi}_2) = \frac{1}{\Gamma(q)}\int_0^t (x-s)^{q-1}\underline{A}_2\,(s)ds \quad (28)$$

$$\vdots$$

(26)

$$\underline{\Phi}_n = N_n(\underline{\Phi}_0, \underline{\Phi}_1, \underline{\Phi}_2, \dots, \underline{\Phi}_n) = \frac{1}{\Gamma(q)}\int_0^t (x-s)^{q-1}\underline{A}_n(s)ds$$

where,

$$\underline{A}_n = \left[\frac{1}{n!}\frac{d^n}{d\lambda^n}k\left(\sum_{i=0}^{n}\lambda^i \underline{y}_i\right)\right]_{\lambda=0}, \; n = 0,1,2,\dots$$

and according to Eq.(22), the$\underline{\Phi}_0$(t) will be take the form

$$\underline{\Phi}_0(t) = \sum_{k=0}^{m-1}\underline{y}^k(0^+)\frac{t^k}{k!} + I^\alpha \underline{f}(t),(29)$$

and similarly for the case of $\overline{y}$.

7-Illustrative Examples:

In this section we shall give some non-linear problems in order to illustrate the validity and accuracy of the proposed method.

Example (1): Consider the linear fuzzy integral equation of fractional order.

$$\tilde{y} = \tilde{f}(t) + I^{1/2}\,\tilde{y}(t);\; t\in[0,1].\,(30)$$

In this case the fuzzy function $\tilde{f}$ will be given as $\tilde{f} = [\overline{f}, \underline{f}]$,

where,

$$\underline{f}=\beta\left[e^{2t}-\frac{2\sqrt{t}}{\sqrt{\pi}}-\frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}-\frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}}-\frac{128t^{\frac{7}{2}}}{105\sqrt{\pi}}\right]$$

and

$$\overline{f}=\frac{1}{\beta}\left[e^{2t}-\frac{2\sqrt{t}}{\sqrt{\pi}}-\frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}-\frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}}-\frac{128t^{\frac{7}{2}}}{105\sqrt{\pi}}\right],$$

$0<\beta\leq 1,$

and the exact solution in the case β=1is $y(t)=e^{2t}$.

and similar to Eq. (29), the $\overline{\Phi}_0(t)$ and $\underline{\Phi}_0(t)$ will be of the forms:

$$\overline{\Phi}_0(t)=\frac{1}{\beta}\left[e^{2t}-\frac{2\sqrt{t}}{\sqrt{\pi}}-\frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}-\frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}}-\frac{128t^{\frac{7}{2}}}{105\sqrt{\pi}}\right]$$

and

$$\underline{\Phi}_0(t)=\beta\left[e^{2t}-\frac{2\sqrt{t}}{\sqrt{\pi}}-\frac{8t^{\frac{3}{2}}}{3\sqrt{\pi}}-\frac{32t^{\frac{5}{2}}}{15\sqrt{\pi}}-\frac{128t^{\frac{7}{2}}}{105\sqrt{\pi}}\right],$$

respectively.

Now, the upper and lower equations become:

$$\begin{aligned}
\overline{\Phi}_1(t)&=I^{1/2}\overline{\Phi}_0(t), &\quad \underline{\Phi}_1(t)&=I^{1/2}\underline{\Phi}_0(t)\\
\overline{\Phi}_2(t)&=I^{1/2}\overline{\Phi}_1(t), &\quad \underline{\Phi}_2(t)&=I^{1/2}\underline{\Phi}_1(t)\\
\overline{\Phi}_3(t)&=I^{1/2}\overline{\Phi}_2(t), &\quad \underline{\Phi}_3(t)&=I^{1/2}\underline{\Phi}_2(t)
\end{aligned}$$

Followingtable (1) represents the comparison between the approximate solutions of example (1)using the reliable algorithm of HAM up to three termsand the exact solution for the case $\beta=1$.

Table (1)

Comparison of the approximate solution ofexample (1) using the Reliable HAMwith the exact solution for $\beta = 1$.

The solution with β=1		**The solution with β =0.5**		**The solution with β =0.75**		**The solution with β =0.25**	
Approximate Solution	**Exact Solution**	**Approximate Solution With Lower Bound**	**Approximate Solution With Upper Bound**	**Approximate Solution With Lower Bound**	**Approximate Solution With Upper Bound**	**Approximate Solution With Lower Bound**	**Approximate Solution With Upper Bound**
1	1	0.5	2	0.75	1.333	0.25	4
1.216	1.221	0.608	2.432	0.912	1.621	0.304	4.864
1.469	1.492	0.735	2.938	1.102	1.959	0.367	5.877
1.769	1.822	0.884	3.537	1.326	2.358	0.442	7.074
2.127	2.226	1.064	4.254	1.595	2.836	0.532	8.508
2.562	2.718	1.281	5.124	1.921	3.416	0.64	10.248

Example (2): Consider the nonlinear fuzzy integral equation of fractional order.

$\tilde{y} = \tilde{f}(t) + I^{1/2}\, \tilde{y}^2(t);\ t \in [0,1].$ (31)

Where the exact solution for the case $\beta = 1$is $y(t) = t$.

In this case the fuzzy function $\tilde{f}$ will be given as$\tilde{f} = [\overline{f}, \underline{f}]$,

Where,

$$\underline{f}=\beta\left[t-\frac{2}{\Gamma\left(\frac{7}{2}\right)}t^{\frac{5}{2}}\right] \text{ and } \overline{f}=\frac{1}{\beta}\left[t-\frac{2}{\Gamma\left(\frac{7}{2}\right)}t^{\frac{5}{2}}\right];\ 0<\beta\le 1,$$

and similar to Eq. (26), the $\overline{\Phi}_0(t)$ and $\underline{\Phi}_0(t)$ will be of the forms:

$$\overline{\Phi}_0(t)=\frac{1}{\beta}\left[t-\frac{2}{\Gamma\left(\frac{7}{2}\right)}t^{\frac{5}{2}}\right] \text{ and}$$

$$\underline{\Phi}_0(t)=\beta\left[t-\frac{2}{\Gamma\left(\frac{7}{2}\right)}t^{\frac{5}{2}}\right].$$

as given in Eq.(28) the upper solutions become:

$$\begin{aligned}
\overline{\Phi}_1&=hH\left(\overline{\Phi}_0(t)-\overline{\Phi}_0(t)-N_0\left(\overline{\Phi}_0\right)\right)\\
\overline{\Phi}_2&=\overline{\Phi}_1+hH\left(\overline{\Phi}_1(t)-N_1\left(\overline{\Phi}_0,\overline{\Phi}_1\right)\right)\\
\overline{\Phi}_3&=\overline{\Phi}_2+hH\left(\overline{\Phi}_2(t)-N_2\left(\overline{\Phi}_0,\overline{\Phi}_1,\overline{\Phi}_2\right)\right)\\
&\vdots\\
\overline{\Phi}_n&=\overline{\Phi}_{n-1}+hH\left(\overline{\Phi}_{n-1}(t)-N_{n-1}\left(\overline{\Phi}_0,\overline{\Phi}_1,\overline{\Phi}_2,\dots,\overline{\Phi}_{n-1}\right)\right)
\end{aligned}$$

Hence, after seeking $h=-1$ and H=1; therefore, the above equations becomes

$$\begin{aligned}
\overline{\Phi}_1(t)&=\left(N_0\left(\overline{\Phi}_0\right)\right)\\
\overline{\Phi}_2(t)&=\left(N_1\left(\overline{\Phi}_0,\overline{\Phi}_1\right)\right)\\
\overline{\Phi}_3(t)&=\left(N_2\left(\overline{\Phi}_0,\overline{\Phi}_1,\overline{\Phi}_2\right)\right)\\
&\vdots\\
\overline{\Phi}_n(t)&=\left(N_{n-1}\left(\overline{\Phi}_0,\overline{\Phi}_1,\overline{\Phi}_2,\dots,\overline{\Phi}_{n-1}\right)\right)
\end{aligned}$$

Where,

$$\begin{aligned}
N_0\left(\overline{\Phi}_0\right)&=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\bar{A}_0(s)ds\\
N_1\left(\overline{\Phi}_0,\overline{\Phi}_1\right)&=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\bar{A}_1(s)ds\\
N_2\left(\overline{\Phi}_0,\overline{\Phi}_1,\overline{\Phi}_2\right)&=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\bar{A}_2(s)ds \quad (32)\\
&\vdots\\
N_n\left(\overline{\Phi}_0,\overline{\Phi}_1,\overline{\Phi}_2,\dots,\overline{\Phi}_{n-1}\right)&=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\bar{A}_n(s)\,ds
\end{aligned}$$

Where,

$$\bar{A}_n=\left[\frac{1}{n!}\frac{d^n}{d\lambda^n}k\left(\sum_{i=0}^{n}\lambda^i\bar{y}_i\right)\right]_{\lambda=0},\ n=0,1,2,\dots$$

and in the same manner the lower solutions become:

$$\begin{aligned}
\underline{\Phi}_1&=hH\left(\underline{\Phi}_0(t)-\underline{\Phi}_0(t)-N_0\left(\underline{\Phi}_0\right)\right)\\
\underline{\Phi}_2&=\underline{\Phi}_1+hH\left(\underline{\Phi}_1(t)-N_1\left(\underline{\Phi}_0,\underline{\Phi}_1\right)\right)\\
\underline{\Phi}_3&=\underline{\Phi}_2+hH\left(\underline{\Phi}_2(t)-N_2\left(\underline{\Phi}_0,\underline{\Phi}_1,\underline{\Phi}_2\right)\right)\\
&\vdots\\
\underline{\Phi}_n&=\underline{\Phi}_{n-1}+hH\left(\underline{\Phi}_{n-1}(t)-N_{n-1}\left(\underline{\Phi}_0,\underline{\Phi}_1,\underline{\Phi}_2,\dots,\underline{\Phi}_{n-1}\right)\right)
\end{aligned}$$

Hence, after seeking $h=-1$ and H=1; therefore, the above equations becomes

$$\begin{aligned}
\underline{\Phi}_1(t)&=\left(N_0\left(\underline{\Phi}_0\right)\right)\\
\underline{\Phi}_2(t)&=\left(N_1\left(\underline{\Phi}_0,\underline{\Phi}_1\right)\right)\\
\underline{\Phi}_3(t)&=\left(N_2\left(\underline{\Phi}_0,\underline{\Phi}_1,\underline{\Phi}_2\right)\right)\\
&\vdots\\
\underline{\Phi}_n(t)&=\left(N_{n-1}\left(\underline{\Phi}_0,\underline{\Phi}_1,\underline{\Phi}_2,\dots,\underline{\Phi}_{n-1}\right)\right)
\end{aligned}$$

Where,

$$N_0(\underline{\Phi}_0)=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\left(\underline{A}_0(s)\right)ds$$

$$N_1(\underline{\Phi}_0,\underline{\Phi}_1)=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\left(\underline{A}_1(s)\right)ds$$

$$N_2(\underline{\Phi}_0,\underline{\Phi}_1,\underline{\Phi}_2)=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\left(\underline{A}_2(s)\right)ds$$

$$\vdots$$

$$N_n(\underline{\Phi}_0,\underline{\Phi}_1,\underline{\Phi}_2,\ldots,\underline{\Phi}_{n-1})=\frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5}\,\underline{A}_n(s)\,ds \tag{33}$$

Where,

$$\underline{A}_n=\left[\frac{1}{n!}\frac{d^n}{d\lambda^n}k\left(\sum_{i=0}^{n}\lambda^i\underline{y}_i\right)\right]_{\lambda=0},\ n=0,1,2,\ldots$$

The following table (2) represents the comparison between the approximate solutions of example (2) using the reliable algorithm of HAM up to three terms and the exact solution for the case $\beta=1$.

Table(2)

Comparison of the approximate solution of example (2) using the reliable HAM with the exact solution for $\beta = 1$.

The solution with β=1		The solution with β =0.25		The solution with β =0.5		The solution with β =0.75	
Approximate Solution	Exact Solution	Approximate Solution With lower bound	Approximate Solution With upper bound	Approximate Solution With lower bound	Approximate Solution With upper bound	Approximate Solution With lower bound	Approximate Solution With upper bound
0	0	0	0	0	0	0	0
0.1	0.1	0.025	0.426	0.05	0.204	0.075	0.134
0.2	0.2	0.048	0.987	0.097	0.426	0.148	0.272
0.3	0.3	0.069	1.886	0.142	0.681	0.219	0.416
0.399	0.4	0.088	3.44	0.183	0.991	0.286	0.568
0.496	0.5	0.103	6.006	0.218	1.371	0.348	0.73

8- Conclusion:

In this paper we have been describe a methodology of reliable algorithm of HAM which has been applied for determining approximation solution of fuzzy integral equation of fractional order. The numerical results showed that this algorithm has good accuracy and reduces the calculations. This technique can be considered as an easy efficient for solving various kinds of non-linear problems in science and engineering without any assumptions restrictions. The accuracy of the obtained solution can be improved by taking more terms in the solution.

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طريقة تحليلالهوموتوبي الموثوقة لحل المعادلات التكاملية الضبابية ذات الرتب الكسورية

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الخلاصة

في هذا البحث قدمنا الحل التقريبي لمسائل المعادلات التكاملية الضبابية ذات الرتب الكسورية بالاعتماد على خوارزمية مطورة لطريقة تحليل الهوموتوبي والتي استخدمت لحلالمعادلات التكاملية الضبابية ذات الرتب الكسورية وان التكامل الكسري هو من نوع ريمان ليوفيلي. تم اعطاء بعض الأمثلةالخطية واللاخطية لتوضيح دقة وفعالية الأسلوب المقترحوالنتائج التي تم الحصول عليها تظهر ان الطريقة سهلة التطبيق وعالية الدقة بالإضافة الى سرعة الحل عند التطبيق لحل هكذا نوع منالمعادلات التكاملية الضبابية ذات الرتب الكسرية.