

## Minimax arc of kind $(0, \delta)$ with weighted points

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### Abstract

In this paper we proved that the existence of  $(k, s; f)$  – arcs of type  $(r, s)$  in  $PG(2, q)$ ; when  $q = nt + 1$ , derived from a minimax arc of kind  $(0, \delta)$ , that is a  $((n - 1)q + 1, n)$  – arc. We find these  $(k, s; f)$  – arcs when  $s - r = q$ , as in the table below. We denote by MIN for minimal, MAX for maximal and X for neither minimal nor maximal.

| Summary for $C = ((n - 1)q + 1, n)$ – arc. [ $q = nt + 1, n > 2$ ] |     |         |             |     |         |         |         |             |
|--------------------------------------------------------------------|-----|---------|-------------|-----|---------|---------|---------|-------------|
| Weights<br>of<br>points                                            | C   | 0       | 0           | 1   | $t + 1$ | $t + 1$ | $n - 1$ | $n - t - 2$ |
|                                                                    | E   | $t$     | $t + 1$     | 0   | 0       | 1       | $n$     | $n - 1$     |
|                                                                    | I   | $t + 1$ | $t - n + 2$ | $n$ | $n - 1$ | 0       | 0       | 0           |
| $W$                                                                | MIN | X       | MAX         | MAX | X       | MAX     | MIN     | X           |

**Keywords:** projective plane of order  $q$ , minimax arcs,  $(k, n; f)$  – arc of type  $(m, n)$ , arcs with weighted points.

### Introduction

In 1991, the author B. J. Wilson <sup>(1)</sup> discussed the existence of the minimax arcs in  $PG(2, q)$ , when  $q$  is an odd prime power. We use the results of <sup>(1)</sup> and combining them with the results of <sup>(2, 3, 4, 5, 6, 7, 8)</sup> to obtain the minimax arc with weighted points.

**Definition 1.**<sup>(2)</sup>

Let  $\pi$  be a projective plane of order  $q$ , and denote by  $\mathcal{P}$  and  $\mathcal{R}$  respectively the set of points and lines of  $\pi$ . Let  $f$  be a function from  $\mathcal{P}$  into the set  $N$  of non – negative integers, and call the weight of  $p \in \mathcal{P}$ , the value  $f(p)$ , and the support of  $f$  the set of points of the plane having non – zero weight. Using  $f$  we can define the function  $F: \mathcal{R} \rightarrow N$  such that for any  $r \in \mathcal{R}$  we have  $F(r) = \sum_{p \in r} f(p)$ . We call  $F(r)$  the weight of the line  $r$ .

**Definition 2.**<sup>(3)</sup>

A  $(k, n; f)$ -arc  $K$  in  $PG(2, q)$  is a function  $f : \mathcal{P} \rightarrow N$  such that  $k = |\text{The support of } f|$  and  $n = \max F$ .

Observe that a  $(k, n)$ -arc  $K$  is a  $(k, n; f)$ -arc if  $f$  is chosen by:

$$f(p) = \begin{cases} 0, & \text{if } p \notin K \\ 1, & \text{if } p \in K \end{cases}.$$

**Definition3.**<sup>(3)</sup>

The type of a  $(k, n; f)$ -arc  $K$  is the set of  $\text{Im } F$ . To write explicitly the type of  $K$  we can use the sequence  $(n_1, \dots, n_\rho)$  where  $n_\lambda \in \text{Im } F$ ,  $\lambda = 1, \dots, \rho$  and  $n_1 < n_2 < \dots < n_\rho = n$ .

**Notation**

- (i)  $\omega = \max f$ .
- (ii)  $[p]$  denote the set of lines passing through  $p$ .

**Lemma 1.**<sup>(4)</sup>

Let  $K$  be a  $(k, n; f)$ -arc in  $PG(2, q)$ . Then for every point  $p$  we have

$$\sum_{r \in [p]} F(r) = W + qf(p).$$

**Lemma 2.**<sup>(4)</sup>

The weight  $W$  of a  $(k, n; f)$ -arc of type  $(m, n)$  satisfies:

$$m(q + 1) \leq W \leq (n - \omega)q + n.$$

**Definition 4.**<sup>(4)</sup>

We call arcs for which the values in lemma (2) are attained, maximal and minimal  $(k, n; f)$ -arcs of type  $(m, n)$  respectively.

**Theorem 1.**<sup>(4)</sup>

A necessary condition for the existence of a  $(k, n; f)$ -arc  $K$  of type  $(m, n)$ ,  $m > 0$  is that:

$$(a) \quad q \equiv 0 \pmod{n - m};$$

$$(b) \quad \omega \leq n - m;$$

$$(c) \quad m \leq n - 2.$$

**Minimax arcs**

In  $PG(2, q)$ , where  $q = nt + 1$  is an odd prime power and  $n > 2$ , let  $C$  denote a  $((n - 1)q + 1, n)$ -arc. Let  $P \in C$  and suppose that through  $P$  there pass  $\rho_i$  secant of  $C$  for  $i = 1, 2, \dots, n$ . Then counting lines through  $P$  gives

$$\rho_1 + \rho_2 + \dots + \rho_n = q + 1 \quad (1)$$

And counting points of  $C$  on lines through  $P$  gives

$$\rho_2 + 2\rho_3 + \dots + (n - 1)\rho_n = (n - 1)q. \quad (2)$$

Eliminating  $\rho_n$  gives

$$(n-1)\rho_1 + (n-2)\rho_2 + \cdots + \rho_{n-1} = n-1. (3)$$

The maximum value of  $\rho_n$  occurs when

$$\rho_1 = 1, \quad \rho_2 = \cdots = \rho_{n-1} = 0, \\ \rho_n = q.$$

And the minimum value of  $\rho_n$  occurs when

$$\rho_1 = \rho_2 = \cdots = \rho_{n-2} = 0, \quad \rho_{n-1} \\ = n-1, \\ \rho_n = q-n+2.$$

**Definition 5.**<sup>(1)</sup>

If every point of  $C$  is of one of the above two types then  $C$  is called a minmax arc. The points of  $C$  are called maximum or minimum points accordingly.

Suppose now that  $C$  is a minmax arc having  $\mu$  maximum points and  $\delta$  minimum points. We shall refer to  $C$  as a minmax arc of kind  $(\mu, \delta)$  when appropriate. Then, using the notation  $\tau_i$  to denote the number of  $i$ -secants of  $C$  we have

$$\mu + \delta = (n-1)q + 1 (4)$$

$$\tau_1 = \mu (5)$$

$$\tau_{n-1} = \delta (6)$$

$$\tau_n = \frac{q\mu + (q-n+2)\delta}{n} = q^2 + 3q - 1 - \\ nq + \mu - \frac{q^2 + q + 2\mu - 2}{n} (7)$$

$$\tau_0 = (q^2 + q + 1) - \tau_1 - \tau_{n-1} -$$

$$\tau_n = (1 - q - \mu) + \frac{q^2 + q + 2\mu - 2}{n} (8)$$

Let  $Q$  be a point not on  $C$  and suppose that through  $Q$  there pass  $\sigma_i i$ -secants for  $i = 0, 1, n-1, n$ . Then

$$\sigma_0 + \sigma_1 + \sigma_{n-1} + \sigma_n = q + 1. (9) \text{ and}$$

$$\sigma_1 + (n-1)\sigma_{n-1} + n\sigma_n = nq - q + 1. (10)$$

From these we may obtain

$$n\sigma_0 + (n-1)\sigma_1 + \sigma_{n-1} = n + q - 1. \\ (11)$$

Now, suppose that  $C$  be of kind  $(0, \delta)$  in  $\text{PG}(2, q)$  with  $q = nt + 1$ . Through each point of  $C$  are  $q - n + 2$ ,  $n$ -secant and  $n-1, (n-1)$ -secants. Hence the equations (4) – (8), become:

$$\delta = (n-1)q + 1 = (n-1)nt + n. (12)$$

$$\tau_1 = 0. (13) \quad \tau_{n-1} = \delta = (n-1)nt + n. (14)$$

$$\tau_n = \frac{(q-n+2)\delta}{n} \\ = \frac{(q-n+2)((n-1)nt+n)}{n}$$

$$= (q-n+2)((n-1)t+1) = \\ (nt-n+3)((n-1)t+1), (15)$$

$$\tau_0 = (1 - q) + \frac{q^2 + q - 2}{n} =$$

$$\frac{(nt+1)^2 + nt - 1}{n} - nt = nt^2 + 3t - nt .$$

(16)

Since  $C$  be of kind ( 0, δ ), then the equations (9), (10) and (11) becomes:

$$\sigma_0 + \sigma_{n-1} + \sigma_n = nt + 2. \quad (17)$$

$$(n - 1)\sigma_{n-1} + n\sigma_n = (n - 1)nt + n .$$

(18)

$$n\sigma_0 + \sigma_{n-1} = n + nt. \quad (19)$$

From equation (18) and (19) we must have  $n|\sigma_{n-1}|$ . Let  $l$  be a  $(n - 1) -$  secant.  $l$  has to meet  $(\tau_{n-1} - 1 = (n - 1)q)$  other,  $(n - 1) -$  secant. Since  $l$  contain  $(n - 1)$  points of  $C$  and through each point there pass  $(n - 2)$  lines differ from  $l$ , then it meets  $(n - 1)(n - 2)$  lines on  $C$  and  $(n - 1)(q - n + 2)$  in  $(n - 1)$  ones in its remaining  $(q - n + 2)$  points. Hence  $\sigma_{n-1} = n$  on  $l \setminus C$ . Thus the points not on  $C$  may be classified as:

Table (1)

| Types | $\sigma_0$ |  |
|-------|------------|--|
| I     | $t + 1$    |  |
| E     | $t$        |  |

$$|C| + |E| + |I| = q^2 + q + 1$$

$$= n^2 t^2 + 3nt + 3.$$

Since  $|C| = (n - 1)q + 1 = n^2 t - nt + n$ .

$|E| + |I| = q^2 + 2q - nq = n^2 t^2 - n^2 t + 4nt - n + 3$ . From equation (12.9) of Lemma (12.1) of <sup>(9)</sup>, we have:

$$(q + 1 - i)\tau_i = \sum_{Q \in PG(2, q) \setminus C} \sigma_i$$

$$\Rightarrow (q - n + 2)\tau_{n-1}$$

$$= \sum_{Q \in PG(2, q) \setminus C} \sigma_{n-1} .$$

$$(q - n + 2)((n - 1)nt + n)$$

$$= \sum_{Q \in E} \sigma_{n-1} = n|E| .$$

$$n|E| = (nt - n + 3)((n - 1)nt + n).$$

$$|E| = (nt - n + 3)((n - 1)t + 1)$$

$$= n^2 t^2 - nt^2 - n^2 t + 5nt - 3t - n + 3 .$$

Then

$$|I| = nt^2 - nt + 3t .$$

A  $n -$  secant meets  $n(n - 1), (n - 1) -$  secants on  $C$  and hence has to meet  $(\tau_{n-1} - n(n - 1)) = n((n - 1)t - n + 2)$ , elsewhere in  $n$ 's. Then on a  $n$

$n -$  secant are  $((n - 1)t - n + 2) - E$  points and hence  $(n^2 t^2 - nt^2 - n^2 t + 5nt - 3t - n + 3) = n((n - 1)t - n + 2)$  points. On a  $(n - 1) -$  secant are  $q - n + 2 = nt - n + 3, E$  points. A  $0 -$  secant has to meet  $(n - 1)q + 1, (n - 1) -$  secants in  $n$ 's and hence on it are

$$\frac{(n-1)q+1}{n} = \frac{(n-1)nt+n}{n} = (n-1)t + 1, E \text{ points and hence } q+1 -$$

$$\frac{(n-1)q+1}{n} = \frac{q+n-1}{n} = t+1, \quad I \text{ points.}$$

Hence we obtain the following table:

Table (2)

| Lines                   | Points on $C$ | Points on $E_+ (t+1)\alpha$ . | Points on $I$ |
|-------------------------|---------------|-------------------------------|---------------|
| $n - \text{secant}$     | $n$           | $((n-1)t - n + 2)$            | $t$           |
| $(n-1) - \text{secant}$ | $n-1$         | $(nt - n + 3)$                | $0$           |
| $0 - \text{secant}$     | $0$           | $(n-1)t + 1$                  | $t+1$         |

$$\begin{aligned} & ((n-1)t - n + 2) \beta + t \alpha \\ & = (nt - n + 3) \beta. \end{aligned}$$

$$t\alpha = (t+1)\beta.$$

$$\text{Set } \alpha = t+1, \quad \beta = t.$$

### Minimax arc with weighted points

In this section, we discuss  $(k, n; f) - \text{arc}$  of type  $(r, s)$ ,  $\text{Im}(f) = \{0, \alpha, \beta\}$  where  $\alpha$  and  $\beta$  are determined according to the weights of the lines and Table (2). We will divide this section into three cases and each case partition into three subcases:

#### Case (1):

Now, in this case assign weights  $\alpha$  to  $I$  points,  $0$  to points of  $C$  and  $\beta$  to  $E$  points. Weights of lines are

$$\begin{aligned} & F(n - \text{secant}) \\ & = ((n-1)t - n + 2) \beta + t \alpha. \end{aligned}$$

$$\begin{aligned} & F((n-1) - \text{secant}) \\ & = (nt - n + 3) \beta. \end{aligned}$$

$$\begin{aligned} & F(0 - \text{secant}) \\ & = ((n-1)t + 1) \beta \end{aligned}$$

$$\begin{aligned} & \text{(i) Let } F(n - \text{secant}) = F((n-1) - \text{secant}). \text{ Then} \\ & ((n-1)t - n + 2) \beta + t \alpha = (nt - n + 3) \beta. \end{aligned}$$

$$\begin{aligned} & ((n-1)t - n + 2) \beta + t \alpha \\ & = (nt - n + 3) \beta. \end{aligned}$$

$$t\alpha = (t+1)\beta.$$

$$\text{Set } \alpha = t+1, \quad \beta = t.$$

$$\begin{aligned} & F(n - \text{secant}) \\ & = ((n-1)t - n + 2) t \\ & + t(t+1) \\ & = nt^2 - nt + 3t. \end{aligned}$$

$$\begin{aligned} & F((n-1) - \text{secant}) \\ & = (nt - n + 3) t \\ & = nt^2 - nt + 3t. \end{aligned}$$

$$\begin{aligned} & F(0 - \text{secant}) \\ & = (t+1)^2 \\ & + ((n-1)t + 1) t \\ & = nt^2 + 3t + 1. \end{aligned}$$

$$r = nt^2 - nt + 3t,$$

$$s = nt^2 + 3t + 1.$$

Clearly that  $(s - r)|q$ , because  $s - r = nt + 1 = q$ .

Only  $r$ -weighting lines pass through points of  $C$ = points of weight 0. Then we expect this arc to be minimal with  $\omega = \alpha = t + 1$  and

$$\begin{aligned} W &= r(q + 1) = (nt^2 - nt + 3t)(q + 1) \\ &= (nt^2 - nt + 3t)(nt + 2) \\ &= n^2t^3 + 5nt^2 - n^2t^2 - 2nt + 6t. \end{aligned}$$

**Result 1.** There exists  $(q^2 + 2q - nq, nt^2 + 3t + 1; f)$ -arc of type  $(nt^2 - nt + 3t, nt^2 + 3t + 1)$ , when  $W$  is minimal and the points of weight 0 are the points of  $C$ .

(ii) Let  $F(n - \text{secant}) = F(0 - \text{secant})$ . Then

$$\begin{aligned} &((n - 1)t - n + 2)\beta + t\alpha \\ &= ((n - 1)t + 1)\beta + (t + 1)\alpha. \end{aligned}$$

$(n - 1)\beta + \alpha = 0$ , which is impossible. Because  $\alpha > 0$ ,  $\beta > 0$  and  $n > 2$ .

(iii) Let  $F((n - 1) - \text{secant}) = F(0 - \text{secant})$ .

$$\begin{aligned} &(nt - n + 3)\beta \\ &= ((n - 1)t + 1)\beta + (t + 1)\alpha. \end{aligned}$$

$$(t - n + 2)\beta = (t + 1)\alpha$$

Set  $\alpha = t - n + 2$ ,  $\beta = t + 1$ , provides that  $n < t + 2 \Rightarrow t > n - 2$ .

$$\begin{aligned} F(n - \text{secant}) &= ((n - 1)t - n + 2)(t + 1) + t(t - n + 2) \\ &= nt^2 - nt + 3t - n + 2. \end{aligned}$$

$$\begin{aligned} F((n - 1) - \text{secant}) &= (nt - n + 3)(t + 1) \\ &= nt^2 + 3t - n + 3. \end{aligned}$$

$$\begin{aligned} F(0 - \text{secant}) &= ((n - 1)t + 1)(t + 1) + (t + 1)(t - n + 2) \\ &= nt^2 + 3t - n + 3. \end{aligned}$$

$$\begin{aligned} r &= nt^2 - nt + 3t - n + 2, \quad s \\ &= nt^2 + 3t - n + 3. \end{aligned}$$

Also, we have  $s - r = nt + 1 = q$ .

This arc is neither maximal nor minimal. Since

$$\sum_{l \in [P]} F(l) = W + q f(P).$$

If  $P \in C \Rightarrow f(P) = 0$ .

$$W = \sum_{l \in [P]} F(l)$$

$$W = (q - n + 2)F(n - \text{secant}) + (n - 1)F((n - 1) - \text{secant})$$

$$= n^2 t^3 - n^2 t^2 + 5nt^2 - nt + 6t - n + 3.$$

**Result 2.** There exists  $(q^2 + 2q - nq, nt^2 + 3t - n + 3; f)$  – arc of type  $(nt^2 - nt + 3t - n + 2, nt^2 + 3t - n + 3)$ , when  $W$  is neither minimal nor maximal and the points of weight 0 are the points of  $C$ .

### Case (2):

Assign weights  $\alpha$  to  $I$  points  $\beta$  to points of  $C$  and 0 to  $E$  points. Weights of lines are

$$F(n - \text{secant}) = t\alpha + n\beta$$

$$F((n - 1) - \text{secant}) = (n - 1)\beta$$

$$F(0 - \text{secant}) = (t + 1)\alpha.$$

(i) Let  $F(n - \text{secant}) = F((n - 1) - \text{secant})$

$t\alpha + n\beta = (n - 1)\beta \Rightarrow t\alpha = -\beta$ , which is impossible.

(ii) Let  $F(n - \text{secant}) = F(0 - \text{secant})$ . Then

$$t\alpha + n\beta = (t + 1)\alpha \Rightarrow n\beta = \alpha.$$

$$\text{Set } \alpha = n, \beta = 1.$$

$$F(n - \text{secant}) = nt + n = n(t + 1).$$

$$F((n - 1) - \text{secant}) = n - 1.$$

$$F(0 - \text{secant}) = n(t + 1).$$

$$r = n - 1, \quad s = n(t + 1).$$

Since through every point of maximum weight ( $\alpha = n$ ) there pass only – weighting lines, then this arc is maximal. Then

$$\begin{aligned} W &= (s - \alpha)q + s = ntq + n(t + 1) \\ &= nt(nt + 1) \\ &\quad + n(t + 1) \end{aligned}$$

$$= n^2 t^2 + 2nt + n.$$

**Result 3.** There exists  $(nt^2 + n^2 t - 2nt + 3t + n, n(t + 1); f)$  – arc of type  $(n - 1, n(t + 1))$ , when  $W$  is maximal and the points of weight 0 are the points of  $E$ .

(iii) Let  $F((n - 1) - \text{secant}) = F(0 - \text{secant})$ . Then

$$(n - 1)\beta = (t + 1)\alpha$$

Set  $\alpha = n - 1, \beta = t + 1$ , provides that  $n \neq t + 2$ .

$$\begin{aligned} F(n - \text{secant}) &= t(n - 1) + n(t + 1) \\ &= 2nt - t + n = s. \end{aligned}$$

$$\begin{aligned} F((n-1) - \text{secant}) \\ &= (n-1)(t+1) \\ &= nt - t + n - 1 = r. \end{aligned}$$

$$\begin{aligned} F(0 - \text{secant}) &= (n-1)(t+1) \\ &= nt - t + n - 1 = r. \end{aligned}$$

$$\begin{aligned} s &= 2nt - t + n, \quad r \\ &= nt - t + n - 1. \end{aligned}$$

This arc is neither maximal nor minimal. Then

$$\sum_{l \in [P]} F(l) = W + q f(P).$$

$$\begin{aligned} \text{If } P \in E \Rightarrow f(P) = 0 &\Rightarrow W \\ &= \sum_{l \in [P]} F(l) \end{aligned}$$

$$\begin{aligned} W &= ((n-1)t - n + 2)F(n \\ &\quad - \text{secant}) \\ &\quad + n F((n-1) \\ &\quad - \text{secant}) \\ &\quad + t F(0 - \text{secant}) \end{aligned}$$

$$= 2n^2t^2 - 2nt^2 + 4nt - 3t + n.$$

**Result 4.** There exists  $(nt^2 + n^2t - 2nt + 3t + n, 2nt - t + n; f)$  - arc of type  $(nt - t + n - 1, 2nt - t + n)$ , when  $W$  is neither maximal nor minimal and the points of weight 0 are the points of  $E$ .

**Case (3):**

Assign weights 0 to  $I$  points,  $\alpha$  to points of  $C$  and  $\beta$  to  $E$  points. Weights of lines are

$$\begin{aligned} F(n - \text{secant}) \\ &= n\alpha \\ &\quad + ((n-1)t - n \\ &\quad + 2)\beta. \end{aligned}$$

$$\begin{aligned} F((n-1) - \text{secant}) \\ &= (n-1)\alpha \\ &\quad + (nt - n + 3)\beta. \end{aligned}$$

$$F(0 - \text{secant}) = ((n-1)t + 1)\beta.$$

(i) Let  $F(n - \text{secant}) = F((n-1) - \text{secant})$ . Then

$$\begin{aligned} n\alpha + ((n-1)t - n + 2)\beta \\ &= (n-1)\alpha \\ &\quad + (nt - n + 3)\beta. \end{aligned}$$

$$\alpha = (t+1)\beta.$$

$$\text{Set } \alpha = t+1, \quad \beta = 1.$$

$$\begin{aligned} F(n - \text{secant}) \\ &= n(t+1) \\ &\quad + ((n-1)t - n + 2) \\ &= (2n-1)t + 2. \end{aligned}$$

$$\begin{aligned} F((n-1) - \text{secant}) \\ &= (n-1)(t+1) + nt \\ &\quad - n + 3 \\ &= (2n-1)t + 2. \end{aligned}$$

$$F(0 - \text{secant}) = (n-1)t + 1.$$

$$\begin{aligned} s &= (2n - 1)t + 2, \quad r \\ &= (n - 1)t + 1, \quad \omega \\ &= t + 1. \end{aligned}$$

This arc is maximal.

$$\begin{aligned} W &= (q + 1)(s - \omega) + \omega \\ &= (q + 1)((2n - 2)t \\ &\quad + 1) + (t + 1) \\ &= (nt + 2)((2n - 2)t + 1) + (t + 1) \\ &= n(2n - 2)t^2 + (5n - 3)t + 3. \end{aligned}$$

**Result 5.** There exists  $(n^2t^2 - nt^2 + 4nt - 3t + 3, (2n - 1)t + 2; f)$  - arc of type  $((n - 1)t + 1, (2n - 1)t + 2)$ , when  $\text{Im}(f) = \{0, 1, t + 1\}$ ,  $W$  is maximal and the points of weight 0 are the points of  $I$ .

(ii) Let  $F(n - \text{secant}) = F(0 - \text{secant})$ . Then

$$\begin{aligned} n\alpha + ((n - 1)t - n + 2)\beta \\ = ((n - 1)t + 1)\beta. \end{aligned}$$

$$n\alpha = (n - 1)\beta.$$

$$\text{Set } \alpha = n - 1, \quad \beta = n.$$

$$\begin{aligned} F(n - \text{secant}) \\ &= n(n - 1) \\ &\quad + n((n - 1)t - n + 2) \\ &= n(n - 1)t + n. \end{aligned}$$

$$\begin{aligned} F((n - 1) - \text{secant}) \\ &= (n - 1)^2 \\ &\quad + n(nt - n + 3) \\ &= n^2t + n + 1. \end{aligned}$$

$$\begin{aligned} F(0 - \text{secant}) &= n((n - 1)t + 1) \\ &= n(n - 1)t + n. \end{aligned}$$

$$\begin{aligned} s &= n^2t + n + 1, \quad r \\ &= n(n - 1)t + n, \quad \omega \\ &= n. \end{aligned}$$

This arc is minimal.

$$\begin{aligned} W &= r(q + 1) \\ &= (n(n - 1)t + n)(nt \\ &\quad + 2) \\ &= n^2(n - 1)t^2 \\ &\quad + (3n^2 - 2n)t + 2n. \end{aligned}$$

**Result 6.** There exists  $(n^2t^2 - nt^2 + 4nt - 3t + 3, n^2t + n + 1; f)$  - arc of type  $(n(n - 1)t + n, n^2t + n + 1)$ , when  $\text{Im}(f) = \{0, n - 1, n\}$ ,  $W$  is minimal and the points of weight 0 are the points of  $I$ .

(iii) Let  $F((n - 1) - \text{secant}) = F(0 - \text{secant})$ . Then

$$\begin{aligned} (n - 1)\alpha + (nt - n + 3)\beta \\ = ((n - 1)t + 1)\beta. \end{aligned}$$

$(n - 1)\alpha = (n - (t + 2))\beta$ , which is impossible if  $n \leq (t + 2)$ .

Now, we discuss the case in which  $n > (t + 2)$ . Then we suppose that  $n = m + (t + 2)$ ,  $m \in \mathbb{Z}^+$ .

Set  $\alpha = n - (t + 2) = m$ ,  $\beta = n - 1 = m + t + 1$ .

$$\begin{aligned} F(n - \text{secant}) &= nm \\ &+ ((n - 1)t - n \\ &+ 2)(m + t + 1) \\ &= (n - 1)^2 t - nt + n - 2. \end{aligned}$$

$$\begin{aligned} F((n - 1) - \text{secant}) &= (n - 1)m \\ &+ (nt - n + 3)(m + t \\ &+ 1) \\ &= (n - 1)^2 t + n - 1. \end{aligned}$$

$$\begin{aligned} F(0 - \text{secant}) &= ((n - 1)t + 1)(n \\ &- 1) \\ &= (n - 1)^2 t + n - 1. \end{aligned}$$

Since  $(n - 1)^2 t + n - 1 - (n - 1)^2 t + nt - n + 2 = nt + 1 = q$ , then  $s = (n - 1)^2 t + n - 1$ ,  $r = (n - 1)^2 t - nt + n - 2$ . Also this arc is neither minimal nor maximal. Then we have

$$\sum_{l \in [P]} F(l) = W + q f(P).$$

$$\text{If } P \in I \Rightarrow f(P) = 0 \Rightarrow W$$

$$= \sum_{l \in [P]} F(l)$$

$$\begin{aligned} W &= ((n - 1)t + 1)F(n - \text{secant}) \\ &+ (t + 1)F(0 \\ &- \text{secant}) \end{aligned}$$

$$\begin{aligned} W &= ((n - 1)t + 1)((n - 1)^2 t - nt \\ &+ n - 2) \\ &+ (t + 1)((n - 1)^2 t \\ &+ n - 1) \end{aligned}$$

$$\begin{aligned} &= (n^3 - 3n^2 + 2n)t^2 \\ &+ (3n^2 - 7n + 3)t \\ &+ 2n - 3. \end{aligned}$$

**Result 7.** There exists  $(n^2 t^2 - nt^2 + 4nt - 3t + 3, (n - 1)^2 t + n - 1; f) -$  arc of type  $((n - 1)^2 t - nt + n - 2, (n - 1)^2 t + n - 1)$ , when  $\text{Im}(f) = \{0, m, n - 1\}$ , where  $m = n - (t + 2) \in \mathbb{Z}^+$ . Also  $W$  is neither minimal nor maximal and the points of weight 0 are the points of  $I$ .

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القوس الأعظم الأصغر من النوع  $(0, \delta)$  مع النقاط الموزونة

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### الخلاصة

في هذا البحث برهنا وجود الأقواس  $(k, s; f)$  من النوع  $(r, s)$  في  $PG(2, q)$  ، عندما  $q = nt + 1$  ، مشتقاً من القوس الأعظم الأصغر  $C$  من النوع  $(0, \delta)$  أي انه القوس  $((n-1)q + 1, n)$  . نحن وجدنا الأقواس  $(k, s; f)$  عندما  $r = q$  و كما مبين في الجدول أدناه . رمزنا بالرمز MAX للقيمة العظمى و MIN للقيمة الصغرى و X للتي ليست عظمى ولا صغرى.

| Summary for $C = ((n-1)q + 1, n) - \text{arc. } [q = nt + 1, n > 2]$ |   |         |             |     |         |         |         |             |
|----------------------------------------------------------------------|---|---------|-------------|-----|---------|---------|---------|-------------|
| Weights<br>of<br>points                                              | C | 0       | 0           | 1   | $t + 1$ | $t + 1$ | $n - 1$ | $n - t - 2$ |
|                                                                      | E | $t$     | $t + 1$     | 0   | 0       | 1       | $n$     | $n - 1$     |
|                                                                      | I | $t + 1$ | $t - n + 2$ | $n$ | $n - 1$ | 0       | 0       | 0           |
| W                                                                    |   | MIN     | X           | MAX | X       | MAX     | MIN     | X           |