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On Solutions of Second Order Differential Equations

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Abstract

Our aim in this work is to give the general form for finding the general solution to the differential equation which has the form $y''+p(x)y'+Q(x)y=0$ *and its proofs by using the assumption* $y=e^{\int Z(x)dx}$ *which changes the above equation to Riccati equation which has the form*

$$Z'+Z^2+p(x)Z+Q(x)=0 .$$

Introduction

Many researchers in the field of differential equations and others, may face the problem of solving some differential equations whose solutions are difficult to find by using the simple methods . Therefore , they are trying to solve these equations by the power series or the Frobenius method [7] . Mohammed [4] faced difficulties in solving equations with the following forms

$$x^2y''-2pxy'+\left[p(p+1)-b^2x^2\right]y=0$$

$$x^2y''-2pxy'+\left[p(p+1)+b^2x^2\right]y=0 ,$$

By using usual simple methods. So, he tried to solve these two equations which have a regular singular point at x=0, where p is a constant , by using the Frobenius method. Another researcher faced a difficult , in solving the equation $4xy''+2y'+y=0$

By using known methods, therefore, he applied the Frobenius method for solving this equation.

In this work, we will give a method for solving the above equations and which like it. This method depends on finding a function Z(x) such that the assumption $y=e^{\int Z(x)dx}$ *gives the general solution of the differential equations.*

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*1- **Riccati Equation** [5]:-*

The general form of Riccati equation is written as

$$y'=f(x)+g(x)y+h(x)y^2 \ldots (1)$$

Where f(x), g(x) and h(x) are given functions of x (or constants). We can solve it, if one or more particular solutions of (1) can be found by inspection

or otherwise. And the general solution of (1) is easy to be obtained by the following conditions:
i-if y_1 a particular solution is known, then the general solution can be obtained by the assumption:-

$$U = y - y_1$$

Then (1) transformed into Bernoulli equation

$$U'(x) + (g + 2hy_1)U = hU^2$$

So, the general solution of (1) is given by

$$(y - y_1)\left[c - \int h(x)z(x)dx\right] = z(x) \quad ; z(x) = e^{\int (g + 2hy_1)dx}$$

ii –if two particular solutions y_1 and y_2 are known, then the general solution of (1) can be found by the assumption

$$U(y-y_2) = y-y_1,$$

Then the general solution is given by:-

$$y - y_1 = C(y - y_2)e^{\int h(x)\ (y_1 - y_2)dx}$$

Where C is an arbitrary constant.
III-if three particular solutions are known, say y_1 ,y_2 and y_3, then the general solution of equation (1) is given as:

$$\frac{(y - y_1)(y_3 - y_2)}{(y_3 - y_1)(y - y_1)} = C,$$

Where C is an arbitrary constant.

*2- **Some Types of Second Order Differential Equations Reduced to the First Order Differential Equation [3]***

The general formula of the second order differential equation is

$$F\ (x, y, \acute{y}, y'') = 0 \ldots (2)$$

Some of these equations can be reduced to first order equations as follows
*i-*if x *is missing, we assume*

$$\acute{y} = p \quad \Rightarrow \quad y'' = \frac{dp}{dx} = p\frac{dp}{dy}$$

Then the equation (2) becomes of first order equation with variables y and p, and we can solve it by simple methods.
ii-if y is missing, we assume that

$$y'' = p' = \quad \Rightarrow \quad y' = p\ \frac{dp}{dx}$$

Then the equation (2) becomes of the first order equation with variables x and p, and we can also solve it by simple methods.

iii-if x and y are not missing, then the solution of the equation (2) will be shown in the last section.

3-How to find the general solution for the second order differential equations which have the standard form

$$y''+p(x)\,\acute{y}+Q(x)\,y=0.$$

In this section, we discuss a method for solving some of the second order differential equations, which have the general form

$$y''+p(x)\,\acute{y}+Q(x)\,y=0\ldots(3)$$

Where p(x) and Q(x) are functions of x. In order to find the general solution of this equation, we search a new function Z(x), such that the assumption

$$y=e^{\int Z(x)dx}\quad\ldots(4)$$

Represents, the general solution of the equation (3). This assumption will transform the equation (3) to the first order differential equation through finding ý, y'' from equation (4) where as,

$$y'=Z(x)e^{\int Z(x)dx}$$

$$y''=Z'(x)e^{\int Z(x)dx}+Z^2(x)e^{\int Z(x)dx}=(Z'(x)+Z^2(x))e^{\int Z(x)dx},$$

And by substituting (y, ý, y'') in equation (3), we get

$$\left(Z'(x)+Z^2(x)\right)e^{\int Z(x)dx}+p(x)Z(x)e^{\int Z(x)dx}+Q(x)e^{\int Z(x)dx}=0$$

Since $e^{\int Z(x)dx}\neq 0$,

$$Z'(x)+Z^2(x)+p(x)Z(x)+Q(x)=0\ldots(5),$$

(5) is an equation of the first order which is similar to Riccati equation. In order to find the solution of (5), we go back to the form of the functions P(x) and Q(x). Now

*i-**if** p(x) **and** Q(x) **are constants say** p(x) =a **and**Q(x) =b **then the equation** (5) **becomes***

$$Z'(x)+Z^2(x)+aZ(x)+b=0$$

And the solution of (3) is given by:-

a)

$$\boxed{y=e^{-\frac{a}{2}x}\left(A\cos\sqrt{b-\frac{a^2}{4}}\,x+B\sin\sqrt{b-\frac{a^2}{4}}\,x\right)}$$

$if\ \ b \neq \frac{a^2}{4}$, where A and B are arbitrary constants

b) $$\boxed{y = Ae^{-\frac{a}{2}x}(x+c)}$$,

$if\ b = \frac{a^2}{4}$, where A and C are arbirary constants.

Proof:

a) Since $Z'(x) + Z^2(x) + aZ(x) + b = 0$, so

$$\frac{dz}{\left(z+\frac{a}{2}\right)^2 + b - \frac{a^2}{4}} + dx = 0 \Rightarrow \int \frac{dz}{\left(z+\frac{a}{2}\right)^2 + d^2} = -x + C; d^2 = b - \frac{a^2}{4}$$

$$\Rightarrow \frac{1}{d}\tan^{-1}\left[\frac{z+a/2}{d}\right] = -x + c \Rightarrow z = d\tan(f - dx) - \frac{a}{2};\ f = dc$$

$$y = e^{\int \left[d\tan(f-dx) - \frac{a}{2}\right]dx} = e^{\ln cos(f-dx) - \frac{a}{2}x + g}$$

$$y = A_1 e^{-\frac{a}{2}x}\cos(f - dx); A_1 = e^g$$

$$y = e^{-\frac{a}{2}x}\left(A\cos\sqrt{b-\frac{a^2}{4}}x + B\sin\sqrt{b-\frac{a^2}{4}}x\right); where\ \ A = A_1 \cos f\ \ and\ \ B = A_1 \sin f$$

$$b) If\quad b = \frac{a^2}{4} \Rightarrow \frac{dz}{\left(z+\frac{a}{2}\right)^2} + dx = 0$$

$$\Rightarrow -\frac{1}{z+\frac{a}{2}} = C_1 - x \Rightarrow z + \frac{a}{2} = \frac{1}{x+C}; C = -C_1$$

$$Since\qquad y = e^{\int z(x)dx} \Rightarrow y = e^{\int\left(\frac{1}{x+c} - \frac{a}{2}\right)dx}$$

$$\Longrightarrow \quad y = e^{\ln(x+c)-\frac{a}{2}x+C_2}$$

$$\Rightarrow \quad y = Ae^{-\frac{a}{2}x}(x+c) \quad ; A = e^{C_2}$$

ii – If $Q(x)=0$, then the general solution of (3) is given by :-

$$y = A\int e^{-\int p\,dx}\,dx + B$$

where A and B are arbitrary constants

*P***roof***:-since*

$$z'(x) + z^2(x) + pz(x) + Q = 0 \Longrightarrow$$

$$z' + z^2 + pz = 0 \Longrightarrow z' + pz = -z^2$$

This is similar to Bernoulli equation, to solve it, we put

$$z^{-1} = t \Rightarrow t' - pt = 1$$

$$I.F = e^{-\int p(x)dx} \Rightarrow e^{-\int p(x)dx}dt - p(x)e^{-\int p(x)dx}dx = e^{-\int p(x)dx}dx$$

$$\Rightarrow e^{-\int p(x)dx}t = \int e^{-\int p(x)dx}dx \Rightarrow Z = \frac{e^{-\int p(x)dx}}{\int e^{-\int p(x)dx}dx}$$

$$y = e^{\int \frac{e^{-\int p(x)dx}}{\int e^{-\int p(x)dx}dx}dx}$$

$$y = e^{\ln\int e^{-\int p(x)dx}dx + C_1} = A\int e^{-\int p(x)dx}dx + B \qquad , A = e^{c_1}$$

iii − if $p(x) = 2\sqrt{Q(x)}$, *then the equation (5) can be solved by the assumption* $u = Z(x) + \sqrt{Q(x)}$,

since $z' + z^2 + pz + Q = 0 \Rightarrow z' + z^2 + 2\sqrt{Q} + Q = 0$

$$\Rightarrow z' + (z + \sqrt{Q})^2 = 0$$

to solve this equation, let

$$u = Z + \sqrt{Q} \Rightarrow Z' = u' - \frac{Q'}{2\sqrt{Q}} = u' - \frac{Q'}{p}$$

$$\Rightarrow u' - \frac{Q'}{p} + u^2 = 0 \Rightarrow u' + u^2 = \frac{Q'}{p}$$

This is Riccati equation, with $f(x) = 1, g(x) = 0$ *and* $k(x) = \dfrac{Q'}{p}$

Now, there are many cases

1 − if u_1 *is known solution to the last equation, then the general solution is given by*

$$y = Ae^{\int\left(u_1 - \sqrt{Q(x)}\right)dx} \int e^{-2\int u_1 dx}\, dx$$

Proof:-

The assumption $d = u - u_1$ *transforms the equation to Bernoulli equation which has the form* $d' + d^2 + 2u_1 d = 0 \Rightarrow d' + 2u_1 d = -d^2$

to solve it, we put

$d^{-1} = t \Rightarrow -d^{-2} d' = t' \Rightarrow d^{-2} d' = -t' \Rightarrow t' - 2u_1 t = 1$, *this is linear equation, and its integrating factor is given by:−*

$I.f = e^{-\int 2u_1 dx}$, *so the general solution of the last equation is given by:−*

$$e^{-\int 2u_1 dx}\, t = \int e^{-\int 2u_1 dx}\, dx \Rightarrow \frac{e^{-\int 2u_1 dx}}{d} = \int e^{-\int 2u_1 dx}\, dx$$

$$\Rightarrow u - u_1 = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}\, dx} \Rightarrow u = \frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}\, dx} + u_1$$

$$\Rightarrow Z+\sqrt{Q(x)}=\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}dx}+u_1 \Rightarrow Z=\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}dx}+u_1-\sqrt{Q(x)}$$

$I.f=e^{-\int 2u_1 dx}$, *so the general solution of the last equation is given by :–*

$$e^{-\int 2u_1 dx}t=\int e^{-\int 2u_1 dx}dx \Rightarrow \frac{e^{-\int 2u_1 dx}}{d}=\int e^{-\int 2u_1 dx}dx$$

$$\Rightarrow u-u_1=\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}dx} \Rightarrow u=\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}dx}+u_1$$

$$\Rightarrow Z+\sqrt{Q(x)}=\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}dx}+u_1 \Rightarrow Z=\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}dx}+u_1-\sqrt{Q(x)}$$

$$\Rightarrow y=e^{\int\left(\frac{e^{-\int 2u_1 dx}}{\int e^{-\int 2u_1 dx}dx}+u_1-\sqrt{Q(x)}\right)dx}$$

$$\Rightarrow y=e^{In\int e^{-\int 2u_1 dx}dx+\int\left(u_1-\sqrt{Q(x)}\right)dx+c}$$

$$\Rightarrow y=A e^{\int\left(u_1-\sqrt{Q(x)}\right)dx}\int e^{-\int 2u_1 dx}dx \;\; ;A=e^{C}$$

We can also get the above general formula by the assumption $u=u_1+\dfrac{1}{d}$.

$2-$*if* u_1 *and* u_2 *are two known solutions ,then the general solution of the last equation is given by ;–*

$$\boxed{y=e^{\int\left(\frac{u_1-Cu_2\ e^{\int(u_1-u_2)dx}}{1-C\ e^{\int(u_1-u_2)dx}}-\sqrt{Q}\right)dx} \quad ;C=constant}$$

proof :–From Riccati equation we get

$$u-u_1 = C(u-u_2)e^{\int (u_1-u_2)dx} \quad ;C \text{ is any arbitrary constant}$$

$$\Rightarrow u = \frac{u_1 - Cu_2 e^{\int (u_1-u_2)dx}}{1-C\ e^{\int (u_1-u_2)dx}}$$

$$\Rightarrow y = e^{\int \left(\frac{u_1 - Cu_2\ e^{\int (u_1-u_2)dx}}{1-C\ e^{\int (u_1-u_2)dx}} \sqrt{Q} \right) dx}$$

$3-$ *if* u_1, u_2 *and* u_3 *are three known solutions ,then the general solution of the last equation is given by;–*

$$y = e^{\int \left(\frac{u_1 - CJ(x)u_2}{1-CJ(x)} - \sqrt{Q(x)} \right) dx} \quad ; J(x) = \left(\frac{u_3-u_1}{u_3-u_2} \right) \quad ;C = constant$$

Proof: - From Riccati equation we get:

$$\frac{u-u_1}{u-u_2} = C\left(\frac{u_3-u_1}{u_3-u_2} \right) ;C \quad \text{is any arbitrary constant}$$

$$\Rightarrow \frac{u-u_1}{u-u_2} = CJ(x) \quad ; J(x) = \left(\frac{u_3-u_1}{u_3-u_2} \right)$$

$$\Rightarrow u-u_1 = CJ(x)u - CJ(x)u_2 \Rightarrow u = \frac{u_1 - CJ(x)u_2}{1-CJ(x)}$$

$$\Rightarrow Z = \frac{u_1 - CJ(x)u_2}{1-CJ(x)} - \sqrt{Q(x)}$$

$$\Rightarrow y = e^{\int\left(\frac{u_1 - CJ(x)u_2}{1-CJ(x)} - \sqrt{Q(x)}\right)dx} \quad ; J(x) = \left(\frac{u_3-u_1}{u_3-u_2}\right)$$

Note*: - some of these equations can be transformed into variable separable equations and don't need the above formula to find the general solution*

Example *(1):- for solving the differential equation*

$y''+2xy'+x^2y=0 \quad ; P(x)=2x, Q(x)=x^2 \quad ,$

by using the equation (5) we get

$$Z'+Z^2+2xZ+x^2=0$$
$$Z'+(Z+x)^2=0$$

let

$$Z+x=t \Rightarrow Z'=t'-1$$

$$\Rightarrow t'-1+t^2=0 \Rightarrow \frac{dt}{1-t^2} - dx = 0 \Rightarrow tanh^{-1}t = x+C \Rightarrow t = tanh(x+C)$$

$$\Rightarrow z = tanh\ (x+C) - x$$

Since

$$y = e^{\int Z(x)dx}$$

$$\Rightarrow y = e^{\int (tanh(x+C)-x)dx}$$

$$\Rightarrow y = e^{In\ cosh(x+C) - \frac{1}{2}x^2 + a}$$

$$\Rightarrow y = e^{-\frac{1}{2}x^2+a} \cosh(x+C)$$

$$\Rightarrow y = e^{-\frac{1}{2}x^2}(A\cosh x + B\sinh x) \quad ; A = e^a\cosh C, B = e^a \sinh C$$

iv)if P(x) and Q(x) are not any one of the above cases ,then

the eqution $Z'+Z^2+P(x)Z+Q(x)=0$ *is similar to Riccati equation.*

As a result then there are three cases :–

1-if Z_1 is known solution to it, then the general solution of (3) is given by:-

$$\boxed{y = A e^{\int Z_1 dx} \int e^{-\int (p+2Z_1)dx} dx \quad ; A = e^a}$$

The assumption $Z = Z_1 + u$ *,transforms the equation to Bernolli equation which has the form:–*

$$u' + (p+2Z_1)u + u^2 = 0 \quad ,$$

to solve it ,we assume $u^{-1} = t$

$\Rightarrow t' - (P+2Z_1)t = 1$ *,this is a linear equation ,and its integrating factor is given by :–*

$$I.F = e^{-\int (P+2Z_1)dx}$$

$$\Rightarrow t.e^{-\int (p+2Z_1)dx} = \int e^{-\int (P+2Z_1)dx} dx$$

$$\Rightarrow Z = \frac{e^{-\int (P+2Z_1)dx}}{\int e^{-\int (P+2Z_1)dx} dx} + Z_1$$

$$\Rightarrow y = e^{\int \left(\frac{e^{-\int (P+2Z_1)dx}}{\int e^{-\int (P+2Z_1)dx} dx} + Z_1 \right) dx}$$

$$\Rightarrow y = e^{\ln \int e^{-\int (P+2Z_1)dx} dx} e^{\int Z_1 dx + a}$$

$$\Rightarrow y = A\ e^{\int Z_1\, dx} \int e^{-\int (p+2Z_1)dx} dx \qquad ; A = e^a$$

Note: - we can get the above formula in 1 by using the assumption $Z = Z_1 + \frac{1}{u}$.

$2-$ *If* Z_1 *and* Z_2 *are two known solutions of it, then the general solution of this equation is given by :–*

$$y=e^{\int\left(\frac{Z_1-CZ_2e^{\int(Z_1-Z_2)dx}}{1-Ce^{\int(Z_1-Z_2)dx}}\right)dx} \quad ; C=\text{constant}$$

proof :–From Riccati Equation, we can write

$$Z\left(1-Ce^{\int(Z_1-Z_2)dx}\right)=Z_1-CZ_2e^{\int(z_1-z_2)dx}$$

$$\Rightarrow Z=\frac{Z_1-CZ_2e^{\int(Z_1-Z_2)dx}}{1-Ce^{\int(Z_1-Z_2)dx}}$$

$$\Rightarrow y=e^{\int\left(\frac{Z_1-CZ_2e^{\int(Z_1-Z_2)dx}}{1-C\,e^{\int(Z_1-Z_2)dx}}\right)dx}$$

$3-$ *If* Z_1, Z_2 *and* Z_3 *are three known solutions of it, then the general solution of this equation is given by :–*

$$y=e^{\int\left(\frac{Z_1-C\,J(x)Z_2}{1-C\,J(x)}\right)dx} \quad ; C=\text{constant and } J(x)=\frac{Z_3-Z_1}{Z_3-Z_2}$$

proof :–From Riccati equation, we can write

$$\frac{Z-Z_1}{Z-Z_2}=C\left(\frac{Z_3-Z_1}{Z_3-Z_2}\right) \quad ; C \text{ be any arbitrary constant}$$

$$\Rightarrow \frac{Z-Z_1}{Z-Z_2}=C\,J(x) \quad ; J(x)=\frac{Z_3-Z_1}{Z_3-Z_2}$$

$$\Rightarrow Z-Z_1=C\,J(x)Z-C\,J(x)Z_2$$

$$\Rightarrow Z=\frac{Z_1-C\,J(x)\,Z_2}{1-C\,J(x)}$$

$$\Rightarrow y = e^{\int\left(\frac{Z_1 - C\,J(x)Z_2}{1 - C\,J(x)}\right)dx}$$

Example(2): – For solving the differential equation

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0,$$

we use the general form in the above formula, which is

$$y = A\,e^{\int z_1 dx}\int e^{-\int(p + 2z_1)dx}dx \quad ; A = e^a$$

now, let $Z_1 = \frac{1}{x}$ *(which is a particular solution of Riccati equation)*

$$y = A\,e^{\int \frac{1}{x}dx}\int e^{-\int \frac{4}{x}dx}dx$$

$$y = A\,e^{In\,x}\int e^{-4\,In\,x}dx = Ax\left(\frac{-x^{-3}}{3} + C_1\right) = -\frac{A}{3}x^{-2} + Bx \quad ; B = AC_1$$

Note: – when the eqution i transformed into Riccati equation, we must know at least one particular solution in order to find its general solution. so the assumption (4) may not be useful to find the general solution when at least one particular solutrion of Riccati equation is unknown.

> *Corollary: – The assumption*
>
> $$y = e^{\int \frac{Z(x)}{x}dx},$$
>
> *is useful to solve the differential eqution,*
>
> $$x^2 y'' + axy' + by = 0 \ldots (6),$$
>
> *where a and b are constants.*

proof: – since $\quad y = e^{\int \frac{Z(x)}{x}dx},$

so

$$y'=\frac{Z(x)}{x}e^{\int\frac{z(x)}{x}dx} \quad ...(7)$$

$$y''=\left(\frac{Z(x)}{x}\right)^2 e^{\int\frac{z(x)}{x}dx}+\left(\frac{xZ'(x)-Z(x)}{x^2}\right)e^{\int\frac{z(x)}{x}dx} \quad ...(8) \qquad ,$$

by substitution (7) and (7) in equation (6), we get

$$x^2\left(\left(\frac{Z(x)}{x}\right)^2 e^{\int\frac{z(x)}{x}dx}+\left(\frac{xZ'(x)-Z(x)}{x^2}\right)e^{\int\frac{z(x)}{x}dx}\right)+ax\left(\frac{Z(x)}{x}e^{\int\frac{z(x)}{x}dx}\right)+b\ e^{\int\frac{z(x)}{x}dx}=0,$$

since $e^{\int\frac{z(x)}{x}dx}\neq 0,$

so

$$\Rightarrow Z^2(x)+xZ'(x)-Z)(x)+aZ(x)+b=o$$

$$\Rightarrow Z'(x)+\frac{1}{x}Z^2(x)+\left(\frac{a-1}{x}\right)(x)+\frac{b}{x}=0...(9)$$

This is similar to Riccati equation. Now, there are many cases:-

$1-$ *If* Z_1 *is a known solution to it, then the assumption*

$Z = Z_1 + u$ *transforms the equation to the Bernolli*

equation as follows : $-$

$$u'+Z_1'+\frac{1}{x}\left(u^2+2uZ_1+Z_1^2\right)+\left(\frac{a-1}{x}\right)(u+Z_1)+\frac{b}{x}=0$$

$$\Rightarrow u'+\frac{1}{x}u^2+\left(\frac{a-1+2Z_1}{x}\right)u=0$$

$$u'+\left(\frac{a-1+2Z_1}{x}\right)u=-\frac{1}{x}u^2$$

this is similar to Bernolli equation .To solve it ,

we assume $u^{-1} = t \Rightarrow$

$t' - \left(\frac{a-1+2Z_1}{x}\right)t = \frac{1}{x}$, *this is a linear equation ,and its integrating is factor given by :–*

$$I.F = e^{-\int\left(\frac{a-1+2Z_1}{x}\right)dx},$$

the general solution of the last equation is given by :–

$$t.e^{-\int\left(\frac{a-1+2Z_1}{x}\right)dx} = \int \frac{1}{x} e^{-\int\left(\frac{a-1+2Z_1}{x}\right)dx} dx$$

$$\Rightarrow u = \frac{e^{-\int\left(\frac{a-1+2z_1}{x}\right)dx}}{\int \frac{1}{x} e^{-\int\left(\frac{a-1+2z_1}{x}\right)dx} dx}$$

$$\Rightarrow Z = \frac{e^{-\int\left(\frac{a-1+2z_1}{x}\right)dx}}{\int \frac{1}{x} e^{-\int\left(\frac{a-1+2z_1}{x}\right)dx} dx} + Z_1$$

since $\quad y = e^{\int \frac{z(x)}{x} dx}$

$$\boxed{y = e^{\int\left(\frac{e^{-\int\left(\frac{a-1+2Z_1}{x}\right)dx}}{\int \frac{1}{x} e^{-\int\left(\frac{a-1+2Z_1}{x}\right)dx} dx} + Z_1/x\right)dx}}$$

Note – we can also get the above general formula by the assumption $Z = Z_1 + \frac{1}{u}$

2 – *If* Z_1 *and* Z_2 *are two known solutions of it, then the general solution of the above equation can be got, as follows :* –

$$Z - Z_1 = C(Z - Z_2)e^{\int (Z_1 - Z_2)dx} \quad \text{(by using case (2)of Riccati equation}$$

so

$$Z\left(1 - C\, e^{\int (Z_1 - Z_2)dx}\right) = Z_1 - C\, Z_2\, e^{\int (Z_1 - Z_2)dx}$$

$$\Rightarrow Z = \frac{Z_1 - C\, Z_2\, e^{\int (Z_1 - Z_2)dx}}{1 - C\, e^{\int (Z_1 - Z_2)dx}}$$

$$y = e^{\int \left(\frac{Z_1 - C\, Z_2\, e^{\int (z_1 - z_2)dx}}{1 - C\, e^{\int (z_1 - z_2)}} / x \right)dx}$$

3 – *If* $Z_1 - Z_2$ *and* Z_3 *are three known solutions of it, then the general solution of the above equation is given as follows :* –

$$\frac{Z - Z_1}{Z - Z_2} = C\left(\frac{Z_3 - Z_1}{Z_3 - Z_2}\right) \quad ;C \text{ be an arbitrary constant}$$

$$\Rightarrow \frac{Z - Z_1}{Z - Z_2} = C\ J(x) \quad ; J(x) = \frac{Z_3 - Z_1}{Z_3 - Z_2}$$

$$\Rightarrow Z - Z_1 = C\, J(x)\, Z - C\, J(x)\, Z_2$$

$$\Rightarrow Z = \frac{Z_1 - C\, J(x)\, Z_2}{1 - C\, J(x)}$$

$$y = e^{\int \left(\frac{Z_1 - C\, J(x) Z_2}{1 - C\, J(x)} / x \right)dx}$$

Note : – *some types of these kinds of differential equations of the above formula need not find its general solution to know any particular solution and we can solve them by a simple method.*

Example(3):– for solving the differential equation

$$x^2y''+xy'-y=0 \quad ,$$

from the formula(9) we get

$$\Rightarrow xZ'-(1-Z^2)=0$$

$$\Rightarrow Z=\frac{Ax^2-1}{Ax^2+1}$$

$$\Rightarrow y=A_1x+\frac{c}{x} \quad ;A_1=AC$$

Example (4):– for solving the differential equation

$$x^2y''-2pxy'+\left[p(p+1)+b^2x^2\right]y=0$$

We would divide by x^2 to get the equation in the standard form

$$y''-\frac{2p}{x}y'+\left[\frac{p(p+1)+b^2x^2}{x^2}\right]y=0\,,$$

now, by using the assumption (4) we get

$$Z'+Z^2-\frac{2p}{x}z+\left[\frac{p(p+1)+b^2x^2}{x^2}\right]=0$$

which is Riccati equation

Now $$y=A\int e^{-\int(2Z_1+P)dx}dx \quad .e^{\int z_1dx} \quad ,$$

where $Z_1=\frac{p}{x}+ib$ *is known particular solution*

$$\Rightarrow y=x^p\left(A\cos bx+B\sin bx\right)$$

Where A and B are constants