# Principally Quasi Ker-Injective Modules

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**Abstract** :

In this paper the concepts of principally quasi-injective modules and pointwise ker-injective modules are generalized to principally quasi ker-injective modules . Many properties and characterizations of principally quasi ker-injective modules are given for example , M is principally quasi ker-injective module if and only if for each $m,n \in M$, such that $ann_R(n) \subseteq ann_R(m)$ ,there exist an R-monomorphism $\alpha: M \to M$ and an R-homomorphism $g:M \to M$ such that $g(n)= \alpha(m)$. Finally some relationships between principally quasi ker-injective modules and another classes of R-modules are given .

## §1: Introduction

Throughout this paper, R will denote an associative, commutative ring with identity, and all R-modules are unitary (left) R-modules. G.F.Birkenmeier proved that an R-module M is ker-injective if and only if for each R-monomorphism $f:A \to B$ (where A and B are R-modules ) and for each R-homomorphism $g:A \to M$ ,there exist an R-monomorphism $\alpha: M \to M$ and R-homomorphism $h: B \to M$ such that $(h \circ f)(a)= (\alpha \circ g)(a)$ for all $a \in A$ [2] .An R-module M is said to be quasi injective if each R-homomorphism of any submodule N of M into M can be extended to an endomorphism of M [7]. An R-module M is called principally N-injective if for any cyclic R-submodule A of N and every R-homomorphism from A into M can be extended to an R-homomorphism from N into M [6]. An R-module M is called principally quasi-injective (or semi-fully stable [1] ) if M is principally M-injective [6] .An R-module M is called pointwise injective if for each R-monomorphism $f:A\to B$ (where A and B are two R-modules), each R-homomorphism $g:A\to M$ and for each $a\in A$ , there exists an R-homomorphism $h_a:B\to M$ ( $h_a$ may depend on a) such that $(h_a\circ f)(a)=g(a)$ [3] . Also an R-module M is pointwise injective if and only if M is principally N-injective for every R-module N [3]. An R-module M is called pointwise ker-injective if for each R-monomorphism $f:A\to B$ (where A and B are R-modules), each R-homomorphism $g:A\to M$ and for each $a\in A$ , there exist an R-monomorphism $\alpha:M\to M$ and R-homomorphism $\beta_a:B\to M$ ($\beta_a$ may depend on a) such that $(\beta_a\circ f)(a)=(\alpha\circ g)(a)$ [5]. An R-monomorphism $f:N\to M$ is called p-split if for each $a\in N$, there exists an R-homomorphism $g_a:M\to N$ ($g_a$ may depend on a ) such that $(g_a\circ f)(a)=a$ [3] . An R-monomorphism $f:N\to M$ is called pointwise ker-split if for each $a\in N$, there exist an R-monomorphism $\alpha:N\to N$ and an R-homomorphism $g_a:M\to N$ ( $g_a$ may depend on a) such that $(g_a\circ f)(a)=\alpha(a)$ [5]. For an R-module M , E(M) and $S=End_R(M)$ will respectively stand for the injective envelope of M and the endomorphism ring of M. $Hom_R(N,M)$ denoted to the set of all R-homomorphism from R-module N into R-module M . For a submodule N of an R-module M and $a\in M$ , $[N:a]_R=\{ r\in R \mid ra\in N \}$. For an R-module M and $a\in M$, then $ann_R(a)$ denoted to the set $[(0):a]_R$ .

## §2 : Principally quasi ker-injective modules

**Definition (2-1)**:- Let M and N be two R-modules , M is said to be principally ker-N-injective ( in short, p-ker-N-injective) if for any cyclic R-submodules A of N and any R-homomorphism f:A→ M ,there exist an R-monomorphism α: M→ M and R-homomorphism g: N→ M such that $(g\circ i)(a)=(\alpha\circ f)(a)$ , for all $a\in A$ , where i is the inclusion R-homomorphism from A to N . An R-module M is called principally quasi ker- injective ( in short, PQ-ker-injective ) if M is p-ker-M-injective . A ring R is called PQ-ker-injective if R is PQ-ker-injective R-module .

**Examples and remarks(2-2):**

1) All principally quasi injective (also pointwise ker-injective modules) are trivial examples of PQ-ker- injective modules .

2) The concept of PQ-ker-injective modules is a proper generalization of both principally quasi injective modules and pointwise ker-injective modules for examples:

i) Let $M=Z\oplus\prod Q$ (where $\prod Q$ is an infinite direct product of copies of Q as Z-module) M is ker-injective Z-module [2] , hence by (1) M is PQ-ker-injective Z-module .If M principally quasi injective Z-module, then by [4,lemma(2,3)], we have that Z is principally quasi injective Z-module and since Z is principally ideal domain , thus Z self injective ring and this a contradiction [7] . Therefore M is PQ-ker injective Z-modules is not principally quasi injective Z-modules ,also this example showed that P-ker-N-injectivity is a proper generalization of principally N-injectivity .

ii) Let M = $Z_p$ as Z-module where p is a prime number . M is PQ-ker-injective Z-module, but by[5,corollary(1.9)] M is not pointwise ker-injective module.

3) P-ker-N-injectivity is an algebraic property .

4) Let M be any R-module and $\prod E(M)$ be infinite direct product of copies of E(M) then :

a) Every R-module of the form $M \oplus \prod E(M)$ is PQ-ker-injective R-module.

b) if M is not PQ-injective R-module , then by [4,lemma(2,3)] , $M \oplus \prod E(M)$ is not PQ-injective R-module.

In the following theorem we give many characterizations of P-ker-N-injective modules

**Theorem (2-3)**: Let M and N be two R-modules and S=$End_R$(M).Then the following statements are equivalent :-

**(1)** M is p-ker-N-injective.

**(2)** For each m$\in$M, n$\in$N such that $ann_R$(n)=$ann_R$(m), there exists an R-monomorphism α:M→M and an R- homomorphism g:N→M such that g(n)=α(m).

**(3)** For each m$\in$ M, n$\in$N such that $ann_R$(n)$\subseteq$ $ann_R$(m), there exist an R-monomorphism α:M→M such that Sα(m)$\subseteq$$Hom_R$(N,M)n.

**(4)** For each R-homomorphism f:A→M (where A be any R-submodule of N ) and each a$\in$A, there exists an R-monomorphism α:M→M and an R-homomorphism g:N→M such that g(a)= (α$\circ$f)(a).

**Proof: (1)⇒(2)** Let M be a p-ker-N-injective R-module. Let $m\in M$, $n\in N$ such that $ann_R(n)\subseteq ann_R(m)$. Define f: Rn→M by f(rn)=rm , for all $r\in R$. It is clear that f is a well-defined R-monomorphism. Since M is p-ker-N-injective R-module, thus there exists an R-monomorphism α:M→M and an R-homomorphism g:N→M such that $g(x)=(\alpha\circ f)(x)$ for all $x\in Rn$. Therefore $g(n)=(\alpha\circ f)(n)=\alpha(f(n))=\alpha(m)$.

**(2)⇒(3)**Let $m\in M$, $n\in N$ such that $ann_R(n)\subseteq ann_R(m)$. By hypothesis, there exists an R-monomorphism α:M→M and an R-homomorphism g:N→M such that g(n)=α(m). Let $\beta\in S$, thus $\beta(\alpha(m))=\beta(g(n))=(\beta\circ g)(n)$. Since $\beta\circ g\in Hom_R(N,M)$ , thus $\beta(\alpha(m))\in Hom_R(N,M)n$ . Therefore $S\alpha(m)\subseteq Hom_R(N,M)(n)$.

**(3)⇒(4)**Let f:A→M be any R-homomorphism where A be any R-submodule of N, and let $a\in A$. Put m=f(a), since $m\in M$ and $ann_R(m)\subseteq ann_R(a)$, thus there exists an R-monomorphism α:M→M such that $S\alpha(m)Hom_R(N,M)a$. Let $I_M$:M→M be the identity R-homomorphism. Since $I_M\in S$, thus there exists an R-homomorphism g:N→M such that $I_M(\alpha(m))=g(a)$.Thus $g(a)=\alpha(m)=\alpha(f(a))=(\alpha\circ f)(a)$.

**(4)⇒(1)**Let A=Ra be any cyclic R-submodule of N and f:A→M be any R-homomorphism. Since $a\in A$, thus by hypothesis there exists an R-monomorphism α:M→M and an R-homomorphism g:N→M such that $g(a)=(\alpha\circ f)(a)$. For each $x\in A$, x=ra for some $r\in R$ , we have that $g(x)=g(ra)=rg(a)=r(\alpha\circ f)(a)=(\alpha\circ f)(ra)=(\alpha\circ f)(x)$. Therefore M is p-ker-N-injective R- module. □

As an immediate consequence of Theorem (2.3) we have the following corollary in which we give many characterizations of PQ-ker-injective modules.

**Corollary (2.4):-**The following statements are equivalent for an R-module M :-

**(1)** M is PQ-ker-injective.

**(2)** For each n,m∈M, such that $ann_R(n) \subseteq ann_R(m)$, there exists an R-monomorphism α:M→M and an R-homomorphism g:M→M such that g(n)=α(m).

**(3)** For each n,m∈ M such that $ann_R(n) \subseteq ann_R(m)$,there exist an R-monomorphism α:M→M such that Sα(m)⊆ Sn.

**(4)** For each R-homomorphism f:A→M (where A be any R-submodule of M ) and each a∈A, there exists an R-monomorphism α:M→M and an R-homomorphism g:M→M such that g(a)= (α ∘ f)(a).

**Corollary (2.5):-** The following statements are equivalent for an R-module M :

**(1)** M is P-ker-R-injective.

**(2)** For each m∈M, n∈R such that $ann_R(n)=ann_R(m)$, there exists an R-monomorphism α:M→M and an R- homomorphism g:R→M such that g(n)=α(m).

**(3)** For each m∈ M, n∈R such that $ann_R(n)= ann_R(m)$,there exist an R-monomorphism α:M→M such that $S\alpha(m) \subseteq Hom_R(R,M)n$.

**(4)** For each R-homomorphism f:A→M (where A be any ideal of R) and each a∈A, there exists an R-monomorphism α:M→M and an R-homomorphism g:R→M such that g(a)= (α ∘ f)(a).

**Proposition (2-6)**:-Every integral domain R is PQ-ker-injective ring.

**Proof**: let R be any integral domain and let $n,m \in R$ such that $ann_R(n) \subseteq ann_R(m)$ .Since R is an integral domain , thus $ann_R(r)=0$ for all $r \in R$ ,$r \neq 0$.

i) if $n = 0$ , thus $ann_R(n)=R$ ,since $ann_R(n) \subseteq ann_R(m)$ , then $ann_R(m)= R$ and this implies that $m = 0$. Define $g: R \to R$ and $\alpha : R \to R$ by $g(x)=x$ and $\alpha(x)=x$ for all $x \in R$. It clear that g is an R-homomorphism and $\alpha$ is an R-monomorphism and $g(n)=\alpha(m)$ .

ii) if $n \neq 0$ , define $g:R \to R$ by $g(x)=mx$ for all $x \in R$, And $\alpha: R \to R$ by $\alpha(x)=nx$ for all $x \in R$ , It is clear that g and $\alpha$ are R-homomorphisms .

for each $x,y \in R$ if $\alpha(x)=\alpha(y)$ then $nx=ny$ and since $n \neq 0$ and R is an integral domain , thus $x=y$ ,therefore $\alpha$ is an R-monomorphism and $g(n)= m\, n = n\, m =\alpha(m)$ . From i and ii we have R is a PQ-ker- injective ring by corollary (2-4) . □

**Example (2-7)** : Z as Z-modules (by proposition 2-6) is PQ-ker-injective but Z is not PQ-injective Z-module and not pointwise-ker-injective module.

**proposition(2-8)**: Let M , N and K are R-modules , if M is P-ker-K-injective R-module and there exist an R-monomorphism from N into K, then M is P-ker-N-injective R-module.

**Proof**: Let $f:N \to K$ be any R-monomorphism and let M be a P-ker-K-injective R-module . Let $m \in M$ , $n \in N$ such that $ann_R(n) \subseteq ann_R(m)$ .Let $x \in ann_R(f(n))$ , thus $xf(n)= 0$ and hence $f(xn)=0$ , since f is an R-monomorphism ,thus $xn=0$ and this implies that $x \in ann_R(n)$,since $ann_R(n) \subseteq ann_R(m)$ then $x \in ann_R(m)$, Therefore

$ann_R(f(n)) \subseteq ann_R(m)$. Since M is P-ker-K-injective , thus by theorem (2-3) there exist an R-homomorphism g: K→ M and an R-monomorphism α :M→M such that g(f(n))=α(m). Put $g_1= g\circ f$ :N→M $g_1$ is an R-homomorphism and $g_1(n)=(g\circ f)(n)$= g(f(n))=α(m). Therefore M is P-ker-N-injective R-module (by theorem 2-3).□

**Corollary(2-9)**: Let M and N be two R-modules, if M is P-ker-N-injective , then M is P-ker-A-injective for each R-submodule A of N.

**Proof** :- Let M be a P-ker-N-injective R-module and let A be any R-submodule of N , let i:A→N be the inclusion R-homomorphism , it is clear that i is an R-monomorphism .Thus by proposition (2-8) , M is P-ker-A-injective R-module .□

As an immediate consequence of corollary (2-9) we have the following corollary.

**Corollary(2-10)**: Let N be any R-submodule of an R-module M , if N is P-ker-M-injective ,then N is PQ-ker-injective R-module .

As an immediate consequence of proposition(2-8) we have the following corollary.

**Corollary(2-11)**: If $N_1$ and $N_2$ are isomorphic R-modules and if M is P-ker- $N_i$-injective then M is P-ker-$N_j$-injective , for each i,j=1,2 and $i\neq j$ .

**Proposition(2-12)**: Any direct summand invariant R-submodule of P-ker-N-injective R-module is P-ker-N-injective .

**Proof**: let M be any P-ker-N-injective R-module and A be any direct summand invariant R-submodule of M , Thus there exist an R-submodule $A_1$ of M such that $M=A\oplus A_1$ . Let $a\in A$ , $n\in N$ such that $ann_R(a)\subseteq ann_R(n)$ , since $a\in M$ and M is P-ker-N-injective R-module , thus by theorem (2-3) there exists an R-homomorphism $\alpha:N\rightarrow M$ and an R-monomorphism $\alpha:M\rightarrow M$ such that $g(n)=\alpha(0)$.Since A is an invariant R-sub module of M, thus $\alpha(A)\subseteq A$ . Define $\alpha':A\rightarrow A$ by $\alpha'(x)=\alpha(x)$ for all $x\in A$. It is clear that $\alpha'$ is an R-monomorphism , Put $g_1=\pi_1\circ g$: $N\rightarrow A$ where $\pi_1$ is the natural projection from $M=A\oplus A_1$ into A . It is clear that $g_1$is an R-homomorphism and $g_1(n)=(\pi_1\circ g)(n)=\pi_1(g(n)) = \pi_1(\alpha(a)) = \pi_1(\alpha'(a)) =\alpha'(a)$ Therefore A is P-ker-N-injective R-module by theorem (2-3).□

By proposition(2-12) and corollary(2-10) we have the following **corollary**.

**Corollary(2-13)**:Any direct summand invariant R-submodule of PQ-ker-N-injective R-module is PQ-ker-N-injective R-module .□

**Proposition(2-14)**: Let M and N are two R-modules. If M is P-ker-N-injective, then every R-monomorphism $f:M\rightarrow N$ is pointwise ker-split .

**Proof**: let f: $M\rightarrow N$ be any R-monomorphism and $a\in A$ . Define $h:f(M)\rightarrow M$ by $h(f(m))=m$ for all $m\in M$ . h is well-defined R-homomorphism , since M is P-ker-N-injective R-module and $f(a)\in f(M)$ . Thus by theorem (2-3) there exist an R-homomorphism $g:N\rightarrow M$ and an R-monomorphism $\alpha:M\rightarrow M$ such that $g(f(a))=(\alpha\circ h)(f(a))$ . put $g_a=g$ and since

$(\alpha \circ h)(f(a)) = \alpha(h(f(a))) = \alpha(a)$ , thus $(g_a \circ g)(a)=\alpha(a)$ . Therefore f is pointwise -ker- split R-homomorphism .□

**Corollary(2-15)**: If M is PQ-ker-injective R-module, then every R-monomorphism $\alpha:M\to M$ is pointwise ker-split .□

**Proposition (2-16**) : An R-module M is pointwise-ker-injective if and only if M is PQ-ker-E(M)-injective for each R-module M .

**Proposition (2-1)**: For each R-module M , the following statements are equivalent :

**(1)** M is pointwise-ker-injective.

**(2)** M is PQ-ker-N-injective , for every extended R-module N of M

**(3)**M is PQ-ker-E(M)-injective .□

By proposition(2-8) and [5,proposition(1-7)] we have the following corollary :

**corollary (2-18**): For a cyclic R-module M , the following statements are equivalent :

**(1)** M is an injective R-module

**(2)**M is pointwise injective R-module

**(3)**M is ker-injective R-module .

**(4)**M is PQ-ker-E(M)-injective R-module .□

Immediately from corollary(2-18) we have the following corollaries

**Corollary (2-19)**: the following statements are equivalent for a ring R :

(1) R is self- injective ring

(2) R is self-pointwise injective ring

(3) R is self-ker-injective ring .

(4) R is PQ-ker-E(M)-injective R-module .□

**Corollary (2-20)**:Every cyclic Z-module M is not PQ-ker-E(M)-injective.

**Proof** :Assume that acyclic Z-module M is PQ-ker-E(M)-injective. Thus by corollary(2-18) .M is injective Z-module and this a contradiction, since every finitely generated Z-module is not injective[7] .Therefore M is not PQ-ker-E(M)-injective Z-module .□

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# الموديولات شبه اغمارية النواة رئيسياً

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**الخلاصة:-**

في هذا البحث قدمنا مفهوم الموديولات شبه اغمارية النواة رئيسيا كتعميم فعلي لمفهومي الموديولات شبه الاغمارية رئيسيا والموديولات اغمارية النواة نقطيا . مجموعة من الخواص والتميزات للموديولات شبه اغمارية النواة رئيسيا قد اعطيت فمثلا برهنا ان الموديول M يكون شبه اغماري النواة رئيسيا اذا وفقط اذا كان لكل m,n عناصر في الموديول M بحيث ان $ann_R(m)\subseteq ann_R(n)$ فانه يوجد تشاكل متباين $\alpha$ من الموديول M الى نفسه وتشاكل g من الموديول M الى الموديول نفسه بحيث ان $g(n)=\alpha(m)$ . أخيرا درسنا بعض العلاقات بين الموديولات شبه اغمارية النواة رئيسيا واصناف اخرى من الموديولات .