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Application of Bipolar Theorem in $L^{\mathrm{P}}(\Omega, F, \mu)$ Space ,$0 < \mathrm{P} \le 1$

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Abstract :

in this paper we applied the bipoler theorem on $L^{\mathrm{P}}(\Omega, F, \mu)$ space, $0 < \mathrm{P} \le 1$. *if* we have $C \subseteq L_{+}^{\mathrm{P}}(\Omega, F, \mu)$ we define the *p*-polar of *C* $p - C^{\circ} = \{ g \in L_{+}^{p} : E[f.g]^{p} \le 1 \; \forall f \in C \}$ and we proved it closed, *p*-solid, *p*-convex set in L_{+}^{P}. Also we define the *p*-bipolar of *C* $p - C^{\circ\circ} = \{ f \in L_{+}^{p} : E[f.g]^{p} \le 1 \forall g \in p - c^{\circ} \}$and we proved it the smallest closed, *p*-solid, *p*-convex *set in* $L_{+}^{\mathrm{P}}(\Omega, F, \mu)$ containing *C* ,$0 < \mathrm{P} \le 1$.

1. Introduction :

let (Ω, F, μ) be a probability space and denote by $L^{\mathrm{P}}(\Omega, F, \mu)$ or $L^{\mathrm{P}}(\Omega)$,$0 < \mathrm{P} \le 1$ the set of all real – valued F measurable functions $\mathrm{X}(\Omega)$ defined $\mu - a.e$ on Ω such that $|\mathrm{X}(\Omega)|^{\mathrm{P}}$ is $\mu -$integrable over Ω [1] . By $L_{+}^{\mathrm{P}}(\Omega, F, \mu)$ we denote the positive part of $L^{\mathrm{P}}(\Omega, F, \mu)$ i.e

$L_{+}^{\mathrm{P}}(\Omega, F, \mu) = \{ f \in L^{\mathrm{P}}(\Omega, F, \mu), f \ge 0 \}$. In this paper *if* we have $C \subseteq L_{+}^{\mathrm{P}}(\Omega, F, \mu)$,$0 < \mathrm{P} \le 1$ we define the *p*- polar of *C* denoted by $(\mathrm{P} - C^{\circ})$ and define the p-bipolar of *C* denoted by $(\mathrm{P} - C^{\circ\circ})$. then we applied bipoler theorem for *C* and named it *p*–bipoler theorem.

2. Elementary definitions and concepts :

2.1 Definition [2]. A collection *F* of subsets of a non–empty set Ω is called $\sigma-$field or $\sigma-$ algebra on Ω if

1. $\Omega \in F$.
2. if $A \in F$ then $A^c \in F$.
3. if $\{A_n\}$ is a sequence of sets in *F* then $\overset{\infty}{\underset{n=1}{U}} A_n \in F$.

A measurable space is a pair (Ω, F) where Ω is a non-empty set and *F* is a $\sigma-$field on Ω .

2.2 Definition [2]. any member of $\sigma-$field *F* is called a measurable set or (measurable with respect to the $\sigma-$ field *F*).

2.3 Definition [3] . A measure on a $\sigma-$field *F* is a non-negative extended real valued function μ on *F* such that whenever A1, A2 , A3 , … Form a finite or countably infinite collection of disjoint sets in *F* , We have $\mu(\bigcup_n A_n) = \sum \mu(A_n)$.
If $\mu(\Omega)=1$, μ is called a probability measure .a measure space is a triple (Ω, F, μ) Where Ω is a set , *F* is a $\sigma-$ field of subsets of Ω, and μ is a measure on *F*. if μ is a probability measure on *F* then the triple (Ω, F, μ) is called probability space.

2.4 Definition [3]. let (Ω_1, F_1) and (Ω_2, F_2) be two measureable spaces a function $f : \Omega_1 \to \Omega_2$ is said to be measureable function (relative to F_1 and F_2) if $f^{-1}(B) \in F_1 \ \forall B \in F_2$.
we say *f* is Borel measureable function on (Ω_1, F_1) if F_2 is the set of all open set on Ω_2.

2.5 Definition [4]. A random variable X on a probability space (Ω, F, P) is a Borel measurable function from Ω to *IR* . i.e. $X : \Omega \to IR$ is random variable iff $\forall a \in IR, \{X \le a\} \in F$.

2.6 Definition [4] . A sequence $\{x_n\}$ of random variables is said to be converge Almost every where(surely) to a random variable x , written $x_n \xrightarrow{a.s} x$ or $x_n \xrightarrow{a.e} x$ if $P\{\lim_{n\to\infty} x_n = x\} = 1$.

2.7 Definition [1]. $L^{\mathrm{P}}(\Omega, F, \mu)$ or $L^{\mathrm{P}}(\Omega)$ $,0 < \mathrm{P} \le 1$

if (Ω, F, μ) is a measure space then $L^{\mathrm{P}}(\Omega, F, \mu)$ is the set of all Borel measurable functions $\mathrm{X}(\Omega)$ defined $\mu - a.e$ on Ω such that $|\mathrm{X}(\Omega)|^p$ is μ -

integrable over Ω .

$L^{\mathrm{P}}(\Omega)$ = $L^{\mathrm{P}}(\Omega, F, \mu)$ = $\{ f : \Omega \to IR, f$ measurable function } such that $\int\limits_{\Omega} |f|^p d\mu < \infty$

2.8 theorem [5].

1. $L^{\mathrm{P}}(\Omega, F, \mu)$, 0<P<1 is not locally convex space i.e it hasn't a neighborhood base at 0 consisting of convex sets.
2. $L^{\mathrm{P}}(\Omega, F, \mu)$, p=1 is locally convex space.

2.9 proposition [6].

1.the dual of $L^{\mathrm{P}}(\Omega, F, \mu)$ space, 0<P<1 is zero i.e $(L^{\mathrm{P}})^* = \{o\}$.

2. $(L^1)^* = L^\infty$

2.10 Definition [3]. Let (Ω, F, μ) be a probability space an *f-measurable* function *f* defined on Ω is said to be essentially bounded if there exists a constant α such that $| f | < \alpha$ μ -a.e now $L^\infty(\Omega, F, \mu)$, or in short , $L^\infty(\Omega)$ is the set of all *f* -measurable , essentially bounded functions defined μ -a.e on Ω .

3. main Result :

Bipolar theorem is applied on $L^{\mathrm{P}}(\Omega, F, \mu)$ space, when p=1 because there is not problem in this manner,also its applied on $L^{\mathrm{P}}(\Omega, F, \mu)$ space ,when 0<P<1

3.1 Definition let $C \subseteq L_+^p$ we define the *p*- polar denoted by $(p - C^\circ)$ of C by $p - C^\circ = \{ g \in L_+^p : E[f.g]^p \le 1 \, \forall f \in C \}$ and the *p*-bipolar denoed by $p - C^{\circ\circ}$ of *C* by

$$p-C^{\circ\circ}=\{\ f\in L_{+}^{p}:E[f.g]^{p}\leq 1\forall g\in p-c^{\circ}\ \}$$

3.2 Definition we call a subset $C\subseteq L_{+}^{p}$ *p*-solid , if $f\in C$ and $0\leq g\leq f$ implies that $g\in C$. the set *C* is said to be closed in measure or simply closed , if it is closed with respect to the topology of convergence in measure .

3.3 Definition A set $D\subset L_{+}^{p}$ is *p*-convex if $\lambda\, f_1+(1-\lambda)\, f_2\in D\forall\ f_1, f_2\in D\, and\, 0\leq\lambda\leq 1.$ or *D* is convex if $\lambda D+(1-\lambda)D\subset D\ forall\, 0\leq\lambda\leq 1$. for all $f_1,f_2\in D$.

3.4 Definition [5]. for $A\in F$, where *F* is a $\sigma-$field , we denote by *C\A* the restriction of *C* to *A* , i.e $\{\gamma\,\chi_A,\gamma\in C\}$ with $\chi_A=1$ on *A* and 0 other wise . We denote similarly *P\A* the restriction of *P* to *A*.

3.5 Definition A subset $C\subseteq L^{p}(\Omega,F,P)$ is bounded in measure if , for all $\in>0$, there is $M>0$ such that $\mu[\|f\|>M]<\in$ for $f\in C$.

3.6 Definition we say that $C\subseteq L_{+}^{p}\ (\Omega,F,P)$ is hereditarily unbounded on a set $B\in F$ if , for every $A\subset B$ with $p(A)>0$, the restriction of *C* to A fails to be bounded subset of $L_{+}^{p}\ (\Omega,F,P)$.

3.7 Lemma Let C be *p*-convex and *p*-solid subset of $L_{+}^{\mathrm{P}}(\Omega,F,\mu')$. There exists a partition of Ω in to disjoint sets Ω_a, $\Omega_b\in F$ such that The partition $\{\Omega_a,\Omega_b\}$ is the unique partition of Ω satisfying (1) and (2) .

proof :

Denote by β the family of sets $\mathrm{B}\in F$, $p(B)>0$,verifying for $\in>0$ there is $f\in C$, s.t. $p[\mathrm{B}\bigcap\{f<\in^{-1}\}]<\in$

Note that β is closed under countable unions : for $(B_n)^{\infty}_{n=1}$ is β and $\in > 0$ find elements $(f_n)^{\infty}_{n=1}$ in C such that p$[B_n \bigcap \{f_n < 2^n \in^{-1}\}] < 2^{-n} \in$ then since C convex and solid F$_N = \sum_{n=1}^{N} 2^{-n} f_n$ is in C and for N large enough $P[B \bigcap \{F_N < \in^{-1}\}] < \in$ hence there is a set of maximal measure in β, which we denote by Ω_a and which is unique up to null- sets . let $\Omega_b = \Omega \setminus \Omega_a$. From this we conclude

1.C is hereditarily unbounded in probability on Ω_a.

2.the restriction $C \setminus \Omega_a$ of *C* to Ω_a is bounded in probability

3.8 theorem let C be a *p*-convex set in $L^{P}_{+}(\Omega, F, \mu')$ such that $\mu' = \mu \setminus \Omega_b$.

1. If $p(\Omega_b) > 0$ then there exists probability measure *Q* equivalent to μ' such that *C* is bounded in $L'(\Omega, F, Q)$.
2. Let *D* be a smallest closed, *p*-convex and *p*-solid set containing *C* then $D = D \setminus \Omega_b \oplus L^{P}_{+}(\Omega, F, \mu) \setminus \Omega_a$.

Proof :

1. let C be solid convex set in $L^{P}_{+}(\Omega, F, \mu')$ and let $k = C \bigcap L^{1}(\Omega, F, \mu')$ then k convex, solid set in $L^{1}_{+}(\Omega, F, \mu')$ then there exists probability measure $Q \approx \mu'$ such that k is bounded in $L^{1}_{+}(\Omega, F, \mu')$ k is dense in C then C$\subset \bar{k}$ and since k bounded then $\bar{k}$ is bounded hence C is bounded in $L^{1}(\Omega, F, Q)$.
2. obviously $D \subset D \setminus \Omega_b \oplus L^{P}_{+}(\Omega, F, \mu) \setminus \Omega_a$ to show the reverse inclusion let *f* = v+w with v$\in D \setminus \Omega_b$ and w $\in$ $L^{P}_{+} \setminus \Omega_a$ to show *f*$\in D$ .for every $n \in \mathrm{N}$,we find an $f_n \in C$ such that $\mu\, [\{f_n \le n^2\} \bigcap \Omega_a\}] \le 1 \setminus n$ since $h_n = (1-(1 \setminus n))v + (1 \setminus n)(f_n \wedge (nw)) \in D$
and $h_n \longrightarrow v + w = f$ in measure, it follows that $f \in D$.

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3.9. *P*-Bipolar theorem

for a set $C \subseteq L^{p}_{+}(\Omega, F, \mu)$ then

1. The p-polar of C $p-C^{\circ}=\{ g\in L_{+}^{p} : E[f.g]^{p}\le 1\ \forall f\in C \}$ is a closed, p-solid, p-convex subset of $L_{+}^{p}(\Omega,F,\mu)$.
2. The p-bipolar of C $p-C^{\circ\circ}=\{ f\in L_{+}^{p} : E[f.g]^{p}\le 1\ \forall g\in p-c^{\circ} \}$ is the smallest closed , p-convex ,p-solid set in $L_{+}^{p}(\Omega,F,\mu)$ containing C .

proof :

1. To prove $p-C^{\circ}$ is closed subset of L_{+}^{p} let $\{h_n\}$ be a sequence in $p-C^{\circ}$ such that $h_n\to h$, $h\in L_{+}^{p}$ then there exists subsequence $\{h_{nm}\}$ in $p-C^{\circ}$ such that $h_{nm}\to h\ a.e$.

Hence $f\,h_{nm}\to fh\ \ a.e$ then $E[f\,h_{nm}]^{p}\to E[fh]^{p}\ \ a.e$ since $\{h_{nm}\}$ in $p-C^{\circ}$ then $E[fh_{nm}]^{p}\le 1\ \ \forall f\in C$

hence $E[f\,h]^{p}\le 1\ \ \forall f\in C$ then we get $h\in p-C^{\circ}$. hence $p-C^{\circ}$ is p-closed .

Now to prove $p-C^{\circ}$ is p-convex subset of L_{+}^{p} let $f,g\in p-C^{\circ}$ and $0\le\lambda\le 1$.

Since $f\in p-C^{\circ}$ then $E[f\,h]^{p}\le 1\ \ \forall h\in C$ and $g\in p-C^{\circ}$ then $E[g\,h]^{p}\le 1\ \ \forall h\in C$

$$E[h(\lambda f+(1-\lambda)g)]^{p}=E[\lambda hf+(1-\lambda)hg]^{p}\le\lambda^{p}E[hf]^{p}+(1-\lambda)^{p}E[hg]^{p}$$

$$\therefore\ \le\lambda^{p}+(1-\lambda)^{p}\le 1\ \ \forall h\in C, 0<p\le 1. \qquad \therefore$$

$E[h(\lambda f+(1-\lambda)g]^{p}\le 1\ \ \forall h\in C\longrightarrow\lambda f+(1-\lambda)g\in p-C^{\circ}$ hence $p-C^{\circ}$ is p-convex set.

Now to prove $p-C^{\circ}$ is p-solid subset of L_{+}^{p} let $g\in p-C^{\circ}\Rightarrow E[f.g]^{p}\le 1\ \forall f\in C$ and $0<h\le g$ hence

$$0\le E[h.f]^{p}\le E[f.g]^{p}\le 1\ \ \forall f\in C$$

$$\therefore E[h.f]^{p}\le 1\forall f\in C\Rightarrow h\in p-C^{\circ}$$

Hence $p-C^{\circ}$ p- solid subset of $L_{+}^{p}(\Omega,F,\mu)$.

2. Let D be the smallest closed, p-convex, p-solid set in $L_{+}^{p}(\Omega,F,\mu)$ containing C then by theorem(3.8)

$D = D\setminus\Omega_b \oplus L_+^{\mathrm{P}}(\Omega,F,\mu)\setminus\Omega_a$ to prove $p-C^{\circ\circ}=D$,as $p-C^{\circ\circ}$ is closed, *p*-convex and *p*-solid and contains C one has $D\subset p-C^{\circ\circ}$.Suppose first that $p(\Omega_b)=0$ then we get

$p-C^{\circ\circ}\subset D=L^p(\Omega,F,\mu)$ since $D\subset p-C^{\circ\circ}$,one has the desired result.We now consider the case $p(\Omega_b)>0$,so that we can find aprobability measure μ' equivalent to μ such that D is bounded in $L'(\Omega,F,\mu')$ let us prove that $D\subset p-C^{\circ\circ}$ we can find some $f_\circ\in p-C^{\circ\circ}\setminus D$ and denote $f_b=f_\circ\setminus\Omega_b$.

define $D_b=\{f\setminus\Omega_b : f\in D\}$ we get $D_b\subset D$, D_b closed , bounded and *p*-convex set in $L_+^p(\Omega,F,\mu)$.

let $\tilde{D}_b=\{K\in L_+^1(\Omega,F,\mu):\exists g\in D_b\ \ s.t\ \ K\le g\ \ \mu-a.s\}$ we get $\tilde{D}_b\subset D,\tilde{D}_b$ closed, *p*-convex in $L_+^1(\Omega,F,\mu)$ to prove $f_b\in D$ let

$f_b\in L_+^1(\Omega,F,\mu)$ but $f_b\notin\tilde{D}_b$ since $\tilde{D}_b$ closed, *p*-convex in $L_+^1(\Omega,F,\mu)$ then we get $I\in L_+^\infty(\Omega,F,\mu)$ with $E[f_b.I]^p>1$ but $E[h.I]^p\le 1\ \ \forall h\in\tilde{D}_b$, $C\subseteq\tilde{D}_b$ Infact: let $f\in C$,since $C\subset D\Rightarrow f\in D$ than $f/\Omega_b\in D_b$, now since $f\le f/\Omega_b\Rightarrow f\in\tilde{D}_b$ then $E[h.I]^p\le 1\ \forall h\in C$, hence $I\in p-C^\circ$ and $E[f_b.I]^p>1$, this implies $f_b\notin p-C^{\circ\circ}$ then $f_0\notin p-C^{\circ\circ}$ and this contradiction
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تطبيق مبرهنة ثنائية القطب على الفضاء $L^{P}(\Omega, F, \mu)$
$0 < P \leq 1$

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الخلاصة:-

في هذا البحث طبقنا مبرهنة ثنائية القطب على الفضاء $L^{P}(\Omega, F, \mu)$ و $0 < P \leq 1$. لتكن المجموعة C مجموعة جزئية من الفضاء $L_{+}^{P}(\Omega, F, \mu)$ ، قمنا بتعريف القطب-p على المجموعة C بالشكل $p - C^{\circ} = \{\, g \in L_{+}^{p} : E[f.g]^{p} \leq 1 \ \forall f \in C \,\}$ ثم برهنا انه مجموعة مغلقة ، صلبه – p ، محدبة – p في $L_{+}^{P}(\Omega, F, \mu)$. ثم عرفنا ثنائي القطب - p على المجموعة C بالشكل $p - C^{\circ\circ} = \{\, f \in L_{+}^{p} : E[f.g]^{p} \leq 1 \forall g \in p - c^{\circ} \,\}$ وايضاً برهنا انه أصغر مجموعة مغلقة، صلبه – p ، محدبة – p في $L_{+}^{P}(\Omega, F, \mu)$ تحتوي C .