A Variational Approach to the Two-endpoint Boundary Value Problem of Route Location in Cost Heterogeneous Terrain

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Abstract

Presented in this paper is a variational solution of the two-endpoint boundary value problem of route location in cost heterogeneous terrain. The terrain consists of two cost subregions separated by a single straight cost boundary line. The integral concerning the total transportation cost is minimized to obtain the optimum route. The analytical solution developed on the basis of calculus of variations was applied to different cases of cost ratios using a computer program, which was written in MCAD language. A comparison is made between the developed analytical solution based on variational technique with that based on Snell's law of light refraction and excellent agreement was obtained.

مقترب تغايري لمسألة القيم الحدية ذات النهايتين المتعلقة بايجاد موقع الطريق في المناطق المختلطة الكلف

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الخلاصة

يقدم هذا البحث حلاً يعتمد على حساب التغاير لمسألة قيم الحدود ذي نقطتي النهاية الخاصة بتحديد مسار منظومات المواصلات في مناطق غير متجانسة الكلفة. أن هذه المنطقة غير المتجانسة تحتوي على منطقتين مختلفتين في الكلفة يفصلها خط كلفة حدودي مستقيم. أن تكامل دالة الكلفة الكلية للمواصلات يجب أن تقلل إلى حدها الأدنى للحصول على المسار الأمثل. أن الحل التحليلي المستحدث في هذا البحث الذي يعتمد عل حساب التغاير طبق على حالات مختلفة من نسب الكلفة باستخدام برنامج بالحاسوب كتب من قبل الباحثين بلغة MCAD. ولقد تم مقارنة هذا الحل التحليلي المستحدث الذي استخدم تقنية حساب التغاير مع حل مبني على استخدام قانون سنيل لانكسار الضوء، وكانت النتائج متطابقة تماماً.

1. Introduction

Transportation routes are commonly constructed to facilitate the movement of people and commodities between spatially distinct points. An objective of planning transportation route is to minimize the cost of movement of people and commodities between terminals. For the homogeneous plane, a straight line identifies the most economical route between two terminals. If the terminals are located within a region that is heterogeneous with respect to land use, the identification of the route that minimizes total costs is most difficult. A method for identifying the optimal route when the region containing the two terminals is cost heterogeneous, is to subdivide this region into two or more homogeneous subregions separated by straight cost boundary lines.

From mathematical point of view, such problems belong to the two endpoint boundary value problems (Boas, 1966 and Howard *et al.*, 1968). The solution to such endpoints boundary-value problem can be achieved using the calculus of variations (Wylie and Barrett, 1985).

The total transportation cost of a route between the terminals P_1 and P_2 can be written as follows:

$$C = \int_{P_1}^{P_2} c(x, y)\, ds \tag{1}$$

where:

$c(x,y)$ = unit cost per unit length over a small element of length ds.

C = total cost

$$ds = \sqrt{(dx)^2 + (dy)^2} \tag{2}$$

To find the optimum route, the integral of eq. (1) representing the total cost should be minimum. This variational problem is rather difficult in practice, the terrain surface is not simple in a mathematical sense.

The calculus of variations, invented by John Bernoulli (1667-1748) involves problems like that of finding a path $y = f(x)$ from a point $P_1(x_1,y_1)$ to a point $P_2(x_2,y_2)$ that minimizes the value of an integral of eq. (1). Finney and Thomas (1990) reported that Newton, Leibniz, and Euler all worked on variational problems as part of calculus, but the contributions of Joseph Louis Lagrange (1736-1813), beginning about 1760, forged the calculus of variations into a self standing branch of mathematics. While working on the calculus of variations, Lagrange developed the elegant procedure that is common today as the method of Lagrange multipliers (Finney and Thomas, 1990).

2. Total Transportation Costs

A mathematical approach to the problem of best route location was introduced by Boukidis and Werner (1963). They showed that the total transportation cost between the terminals P_1 and P_2 could be written as in eq. (1).

In practice, the terrain surface is difficult in a mathematical sense so that a typical route may be taken as an assemblage of continuous segments as shown in Fig. 1 so that:

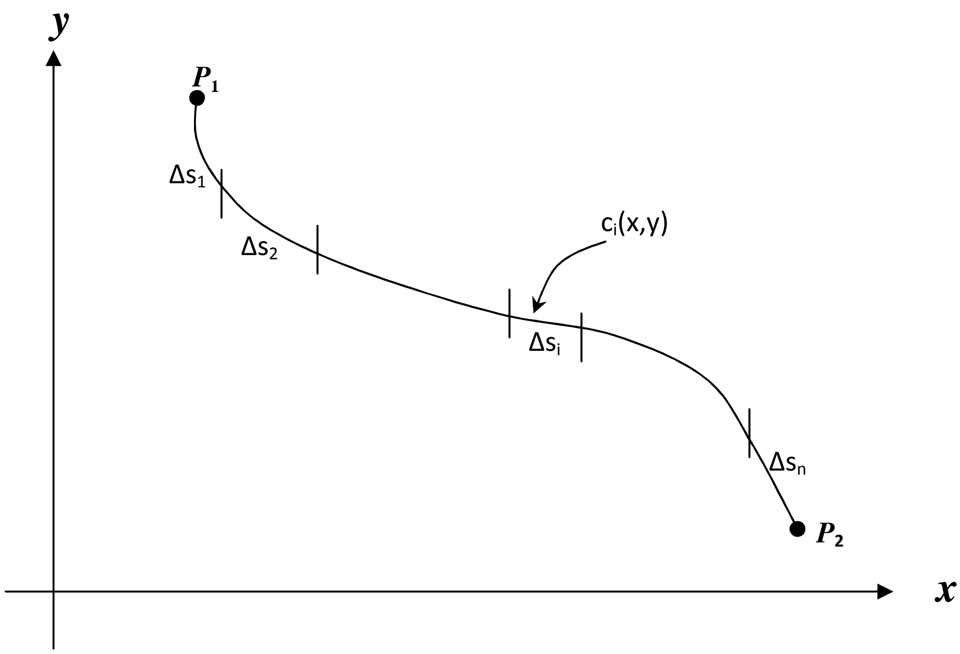

Fig. 1 . Dividing a route into segments.

$$C_T = \sum_{i=1}^{m} c_i(x,y)\,\Delta s_i \tag{3}$$

where:

C_T = total costs

c_i = unit cost per unit length, over a small segment of length Δs_i.

Δs_i= finite length of the route for the i^{th} segment.

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} \tag{4}$$

m = number of segments.

Boukidis and Werner (1963) expressed the cost function as follows:

$$c(x,y) = a(s) + b(s)f \tag{5}$$

where:

s = the path between point P_1 and P_2 measured positive from P_1 towards P_2.

$a(s)$ = unit construction cost which is a function of the length s of the route starting from P_1 .

$b(s)$ = weight factor converting the flow dimension to equivalent construction cost.

f = flow factor.

Thus, eq. (1) becomes:

$$C_T = \int_{P_1}^{P_2} [a(s) + b(s)f]\,ds \tag{6}$$

Boukidis and Werner (1963) also stated that for the case of a heterogeneous terrain in which the unit transportation cost is constant within each subregion bounded between two adjacent cost boundary lines ϕ_i, ϕ_{i+1} , eq. (3) can be written as follows [see Fig. 2] :

$$C_T = \sum_{i=1}^{m} (a_i + b_i f_i)\left| p_i - p_{i-1} \right| \tag{7}$$

where:

$$\left| p_i - p_{i-1} \right| = \sqrt{(x_{c_i} - x_{c_{i-1}})^2 + (y_{c_i} - y_{c_{i-1}})^2} \tag{8}$$

$m = n+1$ = number of cost subregions.

n = number of cost boundary lines.

$x_{c_{i-1}}$, y_{c_i} = coordinate of the i^{th} corner point.

When a terrain could be approximated by a finite number of cost areas (each having constant unit transportation cost), it would become necessary to find the points on the cost boundary lines at which the route crosses. Boukidis and Werner (1963) reported that using the regression technique (Ott,1984; McCuen, 1985; Kennedy and Neville, 1986; Akai, 1994), these contour lines could be described mathematically. The equation of the cost boundary curve to fit a set of points can be written as follows:

$$\phi_i(x_i, y_i) = 0 \tag{9}$$

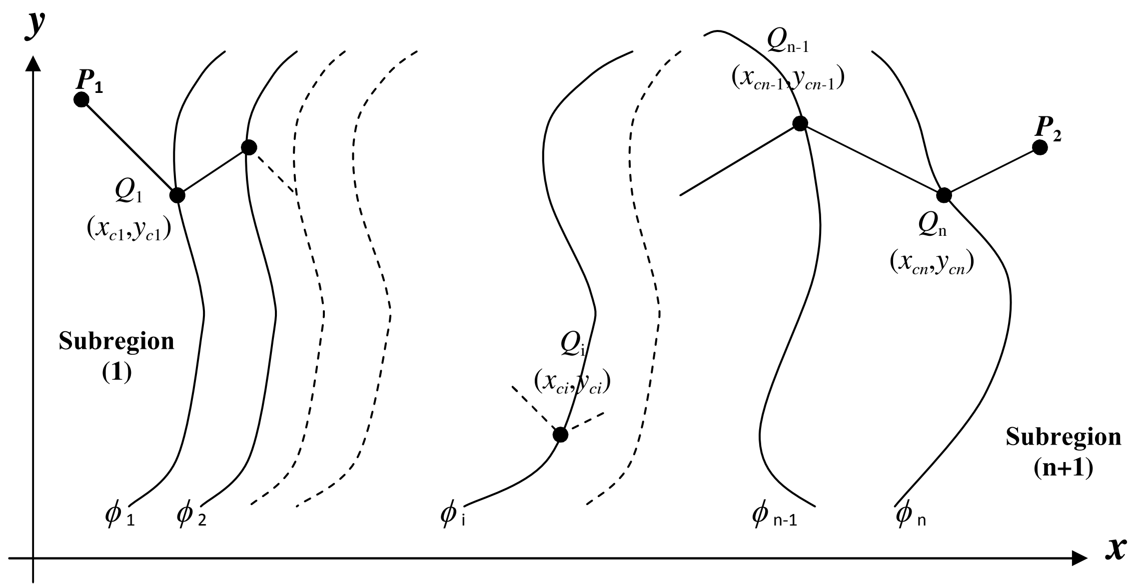

Fig. 2. Approximating a cost heterogeneous terrain by various cost subregions.

To simplify the problem, Boukidis and Werner (1963) suggested that a straight line could approximate every cost boundary curve, using linear regression analysis, without losing much accuracy. A set of functions can be built up as:

$$\phi_i = y_i - \alpha_i x_i - \beta_i = 0 \tag{10}$$

3. Minimum Cost as a Criterion for Optimum Route

Making use of the Lagrange multipliers (Boas, 1966; Finney and Thomas, 1990), the process of minimizing the total transportation cost to find the optimum route location can be simplified enormously by forming the function:

$$F = C_T + \sum_{i=1}^{n} \lambda_i \phi_i \tag{11}$$

where:

λ_i = the Lagrange multiplier for the i^{th} cost boundary line.

The values of x_{ci}, y_{ci} at which the function "*F*" becomes minimum can be determined by taking partial derivatives of "*F*" with respect to the variables x_{ci}, y_{ci}, λ_i and by setting each of these derivatives equal to zero as follows:

$$\left.\begin{aligned} \frac{\partial F}{\partial x_{ci}} &= 0 \\ \frac{\partial F}{\partial y_{ci}} &= 0 \\ \frac{\partial F}{\partial \lambda_i} &= 0 \end{aligned}\right\} \quad for \;\; i = 1,2,\ldots,n \tag{12}$$

For the case of two cost subregions the optimum route connecting two stations in different cost subregions as shown in Fig. 3 can be found by solving eqs. (7) and (8) as follows:

$$C_T = (a_1 + b_1 f)|Q_1 - P_1| + (a_2 + b_2 f)|P_2 - Q_1| \tag{13}$$

where:

$$|Q_1 - P_1| = \sqrt{(x_c - x_1)^2 + (y_c - y_1)^2} \quad (13.\ a)$$

$$|P_2 - Q_1| = \sqrt{(x_2 - x_c)^2 + (y_2 - y_c)^2} \quad (13.\ b)$$

$a_{1,} a_2$ = unit construction cost in subregions 1 and 2 respectively.
$b_{1,} b_2$= weight factor in subregions 1 and 2 respectively.
f = flow factor (assumed the same for both regions).
x_c, y_c= coordinates of the point at which the optimum route meets the cost boundary line (i.e. corner point), subjected to the constraint:

$$\phi_1 = y_c - \alpha_1 x_c - \beta_1 = 0 \quad (14)$$

For this special case of two cost subregions, eq. (11) becomes:

$$F = C_T + \lambda_1 \phi_1 \quad (15)$$

The substitution of eqs. (13) and (14) into eq. (15) yields:

$$F = (a_1 + b_1 f)|Q_1 - P_1| + (a_2 + b_2 f)|P_2 - Q_1| + \lambda_1 [y_c - \alpha_1 x_c - \beta] \quad (16)$$

Applying eq. (12), three equations with three unknowns x_c, y_c and λ_1 can be obtained. The solution of these equations yields the required corner point and hence the optimum route required.

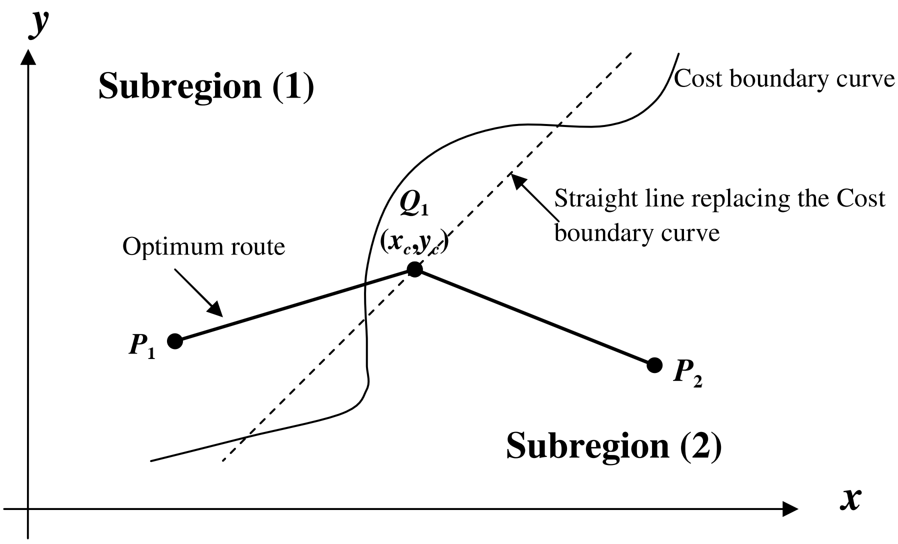

Fig. 3. Straight-line approximation for a single cost boundary curve, after Boukidis and Werner (1963).

4. The Location of Optimum Route

The function F given by eq. (16) can be written in the following form:

$$F = (a_1 + b_1 f)\sqrt{(x_c - x_{P_1})^2 + (y_c - y_{P_1})^2} + (a_2 + b_2 f)\sqrt{(x_{P_2} - x_c)^2 + (y_{P_2} - y_c)^2} + \lambda_1 [y_c - \alpha_1 x_c - \beta] \quad (17)$$

However, as the unit transportation cost is constant within each cost subregion, eq. (17) can be written as follows:

$$F = c_1\sqrt{(x_c - x_{P_1})^2 + (y_c - y_{P_1})^2} + c_2\sqrt{(x_{P_2} - x_c)^2 + (y_{P_2} - y_c)^2} + \lambda_1 [y_c - \alpha_1 x_c - \beta] \quad (18)$$

For the optimum route, eq. (12) yields:

1) $$\frac{\partial F}{\partial x_c}=0=\frac{c_1(2x_c-2x_{P_1})}{2\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2}}+\frac{c_2(2x_{P_2}-2x_c)}{2\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2}}-\alpha_1\lambda_1$$

or

$$\begin{aligned}0=c_1(x_c-x_{P_1})&\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2}\\&+c_2(x_{P_2}-x_c)\sqrt{(y_c-y_{P_1})^2+(x_c-x_{P_1})^2}\\&+\lambda_1\sqrt{(y_{P_2}-y_c)^2+(x_{P_2}-x_c)^2}\cdot\sqrt{(y_c-y_{P_1})^2+(x_c-x_{P_1})^2}\end{aligned}\qquad(19)$$

2) $$\frac{\partial F}{\partial y_c}=0=\frac{c_1(2y_c-2y_{P_1})}{2\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2}}+\frac{c_2(2y_{P_2}-2y_c)}{2\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2}}+\lambda_1$$

or

$$\begin{aligned}0=c_1(y_c-y_{P_1})\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2}&+c_2(y_{P_2}-y_c)\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2}\\+\lambda_1\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2}&\cdot\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2}\end{aligned}\qquad(20)$$

3) $$\frac{\partial F}{\partial \lambda_1}=0=y_c-\alpha_1x_c-\beta_1\qquad(21)$$

It is shown in Appendix (A) that the solution of the three simultaneous eqs. (19), (20), and (21) can be reduced to the solution of the following fourth order algebraic equation:

$$A+Bx_c+Cx_c^2+Dx_c^3+Ex_c^4=0\qquad(22)$$

where:

$$\left.\begin{aligned}&A=I^2LH^2-F^2k\\&B=2I^2mHL-2I^2GH^2-2NF^2-2mFk\\&C=mI^2H^2-4mI^2HG+LI^2m^2-mF^2+4FN-km^2\\&D=2I^2m^2H-2GI^2m^2+2m^2F+2Nm^2\\&E=I^2m^3-m^3\\&m=1+\alpha_1^2\\&H=\alpha_1\beta_1-\alpha_1y_{P_1}-x_{P_1}\\&F=\alpha_1y_{P_2}-\alpha_1\beta_1-x_{p_2}\\&L=y_{P_2}^2-2\beta_1y_{P_2}+\beta_1^2+x_{P_2}^2\\&k=y_{P_1}^2-2\beta_1y_{P_1}+\beta_1^2+x_{P_1}^2\\&G=\alpha_1y_{P_2}+\alpha_1\beta_1+x_{P_2}\\&N=x_{P_1}-\beta_1\alpha_1^2y_{P_1}+\alpha_1y_{P_1}\\&I=c_1/c_2\end{aligned}\right\}\qquad(23)$$

By solving eq. (22), the *x*-value of the corner point will be found. The corresponding *y*-coordinate of the corner point will be obtained from eq. (21). Once the coordinates x_c, y_c are determined, the corner point and hence, the optimum route are determined. It is worth mentioning that eq. (22) possesses generally four different roots. These roots can include both complex and real roots. For more than one real root, the governing root will be that yielding the minimum cost and hence the maximum saving in cost. The saving in cost is defined as follows:

$$\text{Saving in cost} = \frac{\text{Cost of direct route} - \text{Cost of optimum route}}{\text{Cost of direct route}} \quad (24)$$

For the case shown in Fig. 4, use can be made of the following data:

$$c_1 = 10000$$
$$c_2 = 10$$
$$\alpha_1 = 1$$
$$\beta_1 = -20$$
$$x_{P_1} = 0 \quad , \quad y_{P_1} = 0$$
$$x_{P_2} = 50 \quad , \quad y_{P_2} = 0$$

Therefore, eq. (22) becomes:

$$463992*10^6 - 1151998*10^6 x_c + 943999*10^5 x_c^2 - 287998*10^4 x_c^3 + 31999900\, x_c^4 = 0 \quad (25.\ a)$$

or

$$14499.795 - 36000.050\, x_c + 2950.006\, x_c^2 - 90\, x_c^3 + x_c^4 = 0 \quad (25.\ b)$$

The solution of eq. (25) revealed one pair of complex roots and the following two real roots (Kreyszig, 1999):

$$r_1 = x_{c_1} = 10.054$$
$$r_2 = x_{c_2} = 9.945$$

Depending on the maximum saving in cost [see eq. (24)], the first root ($r_1 = x_{c_1} = 10.054$) is the governing root. Accordingly, eq. (21) yields (y_c = -9.946). Making use of eqs. (13. a) and (13. b), the total length of the first and second segments of the optimum route can be obtained as (55.473) length unit. The total cost of the optimum route can be obtained from eq. (13) as (141835.074) cost units. This route yields (29.19)% saving in cost relative to the direct route.

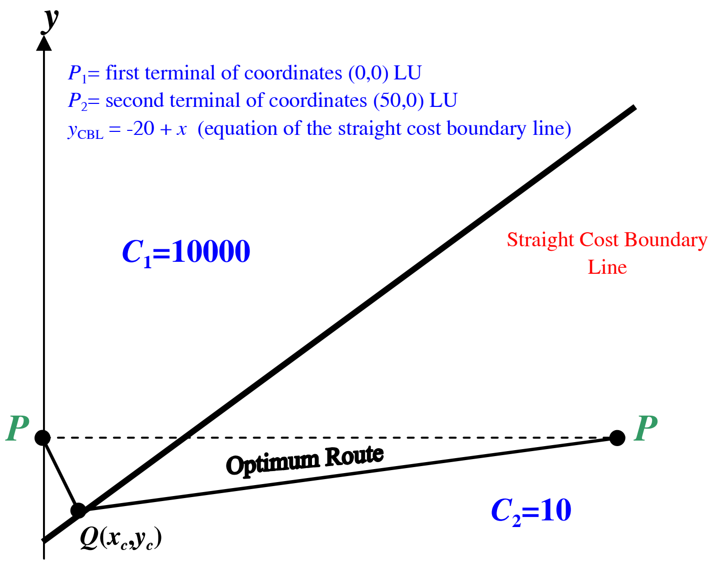

Fig. 4. Optimum route in a terrain separated by a single straight cost boundary line for a case of Al-Rubai'ee (2001).

5. Solution by Snell's Law of Light Refraction

To show the validity of the variational technique of the two endpoints boundary value problem of route location, the same example [Fig. 4] was solved by using the Snell's law of light refraction (Razouki and Al-Zubaidi, 1995).

According to Fig. 5, the coordinates of the estimated corner point $Q(x_c,y_c)$ can be obtained as follows:

$$\frac{s}{d} = \frac{x_b - x_a}{x_c - x_a} \tag{26}$$

$$\frac{s}{d} = \frac{y_b - y_a}{y_c - y_a} \tag{27}$$

where:

s = the distance between the two perpendicular projection points a and b on the cost boundary lines.

d = the distance between a and the corner point Q.

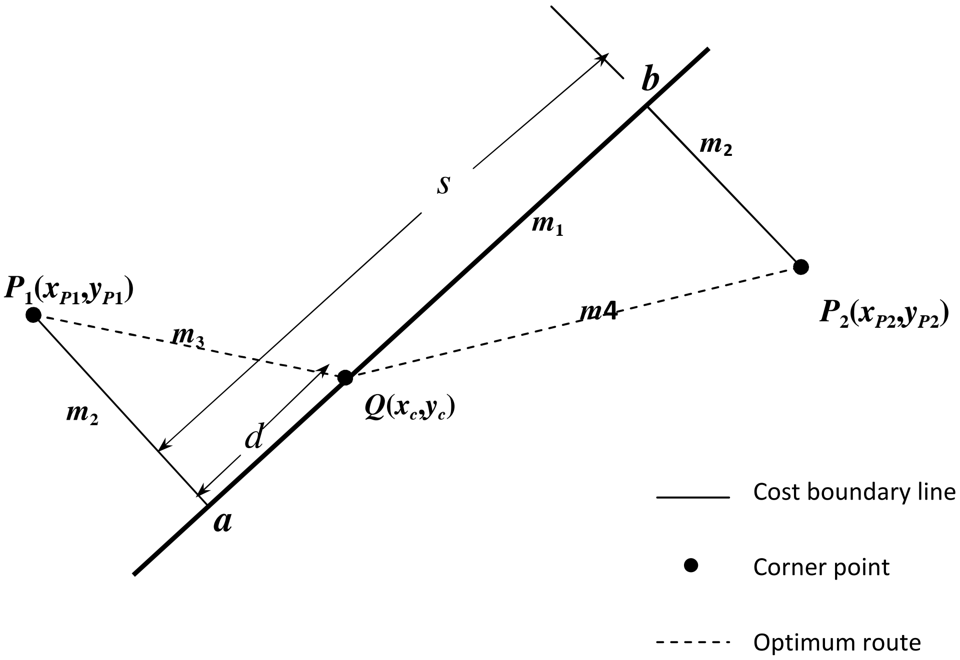

Fig. 5. Slopes of both projected and refracted lines.

For the case of Fig. 4, the coordinates of the first point a of the optimization interval are (10,-10) and of the second point b are (35, 15). The corner point should satisfy the cost boundary line equation as follows:

$$y_c = x_c - 20 \tag{28}$$

As shown in Fig. 5, the slope (m_3) of the first segment of the route $\overline{P_1Q}$, and the slope (m_4) of the second segment $\overline{QP_2}$ can be determined by following the simple relationships:

$$m_3 = \frac{y_{P_1} - y_c}{x_{P_1} - x_c} \tag{29}$$

$$m_4 = \frac{y_c - y_{P_2}}{x_c - x_{P_2}} \tag{30}$$

For this example, eq. (29) will be:

$$m_3 = \frac{y_c}{x_c} \tag{31}$$

and eq. (30) will be:

$$m_4 = \frac{y_c}{x_c - 50} \tag{31}$$

It remains to determine the objective function (Salman, 1991 and Al-Zubaidi, 1993), which is:

$$K = \frac{\sin e}{\sin r} - \frac{c_2}{c_1} \tag{33}$$

where:

e, r = the entry and refraction angles respectively.

c_1, c_2 = the total transportation costs within the first and second region respectively.

The objective function K for the location of the optimal route connecting station P_1 with station P_2 across a straight cost boundary line at a corner point Q as shown in Fig. 6 should be minimized. The calculation of the entry angle (e) and refractive (r) is carried out by applying the following relationships (Salman, 1991):

$$90 - e = \alpha_1 = \tan^{-1}\left|\frac{m_3 - m_1}{1 + m_3 m_1}\right| \tag{34}$$

$$90 - r = \alpha_2 = \tan^{-1}\left|\frac{m_4 - m_1}{1 + m_4 m_1}\right| \tag{35}$$

$$e = 90 - \alpha_1 \tag{36}$$

$$r = 90 - \alpha_2 \tag{37}$$

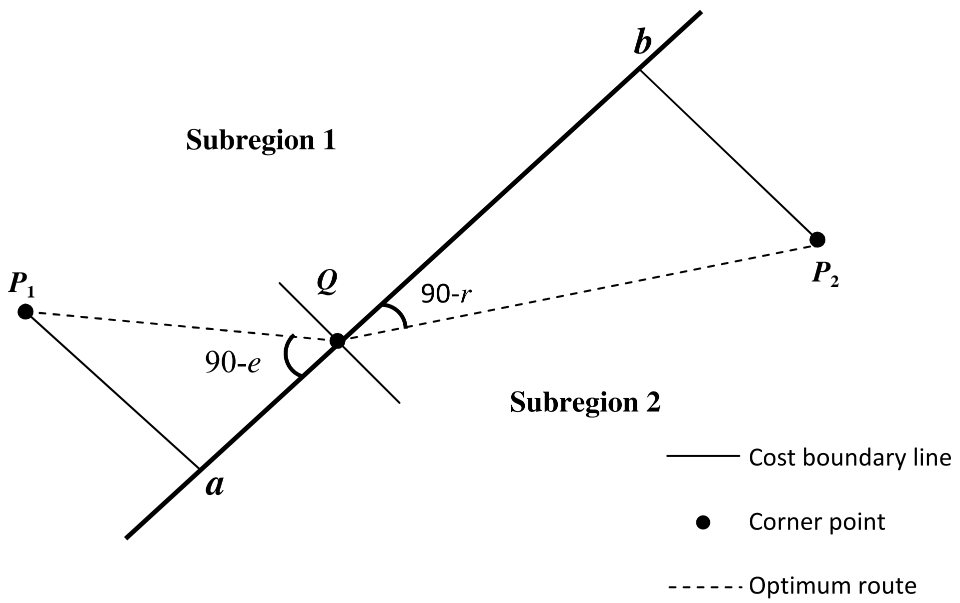

Fig. 6. The basic components of the objective function.

To find the minimum value for K, the optimization interval $[a,b]$ was subdivided into 100 uniform subintervals as recommended by Al-Zubaidi (1993). Then the objective function K was computed at the beginning and end of each subinterval by using eq. (33). The minimum value of K determines the location of the required corner point. Hence, the optimum route is determined. For the case of Fig. 4, the corner point is (10.054,-9.946). This result is the same as that obtained from the variational solution confirming the validity of the variational approach.

6. Effect of Cost Ratio on Optimum Route

To show the effect of cost ratio on optimum route location, the case of two cost subregions separated by a single straight cost boundary line shown in Fig. 4 is considered. Nine different values for the cost ratio c_1/c_2 namely (1000, 100, 10, 2, 1, 0.5, 0.1, 0.01, and 0.001) were adopted. Table (1) presents the summary of results of the effect of cost ratio on the saving in cost as well as the length ratio (i.e. ratio of length of the optimum route to the length of direct route).

Figure 7 shows the resulting optimal routes for the cost ratio (1000, 10, 2, 1, 0.5, 0.1, 0.01, and 0.001). It is quite obvious from this figure that the cost ratio has a significant effect on the optimum route. The route becomes as short as possible in the region with high cost and crosses a long distance in the cheap region.

Figure 8 shows that the saving in cost can be quite significant when the cost ratio is either two high (1000) or two low (0.001). Figure 9 shows the variation of length ratio with the cost ratio. It is quite obvious that the length ratio becomes unity for homogeneous terrain with $I = c_1/c_2 = 1.0$.

Table 1. Effect of cost ratio on saving in cost and on length ratio for the single straight cost boundary line of Fig. 4.

Cost Ratio	Corner point		Cost of optimum route CU (1)	Cost of direct route CU (2)	Saving in cost% = $\frac{(2)-(1)}{(2)}*100$	Length ratio
	x_c	y_c				
1000	10.054	-9.946	141835.074	200300.89	**29.189**	**1.067**
100	10.086	-9.915	1455.322	2030	**28.304**	**1.067**
10	11.853	-9.148	182.129	230	**20.803**	**1.067**
2	14.418	-5.583	66.938	70	**4.372**	**1.043**
1.0	20.000	0	50.000	50	**0**	**1.000**
0.5	27.742	7.741	75.935	80	**5.083**	**1.012**
0.1	33.613	13.613	249.300	320	**22.094**	**1.018**
0.01	35.14	15.140	2159.378	3020	**28.488**	**1.020**
0.001	34.983	14.983	21251.280	30020	**29.209**	**1.020**

CU = Cost Unit
LU = Length Unit

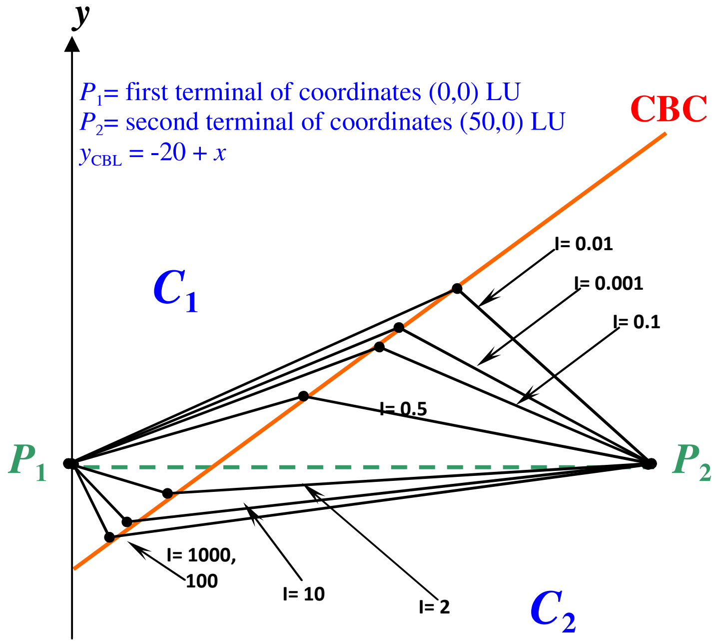

Fig. 7. Effect of cost ratio on optimum route location in a terrain separated by a straight cost boundary line.

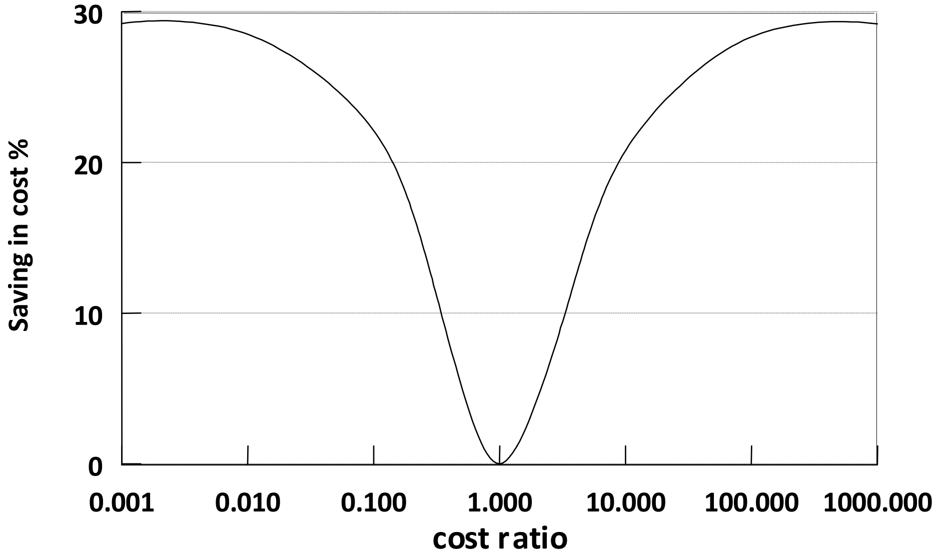

Fig. 8. Effect of cost ratio on saving in cost for the straight cost boundary line of Fig. 4.

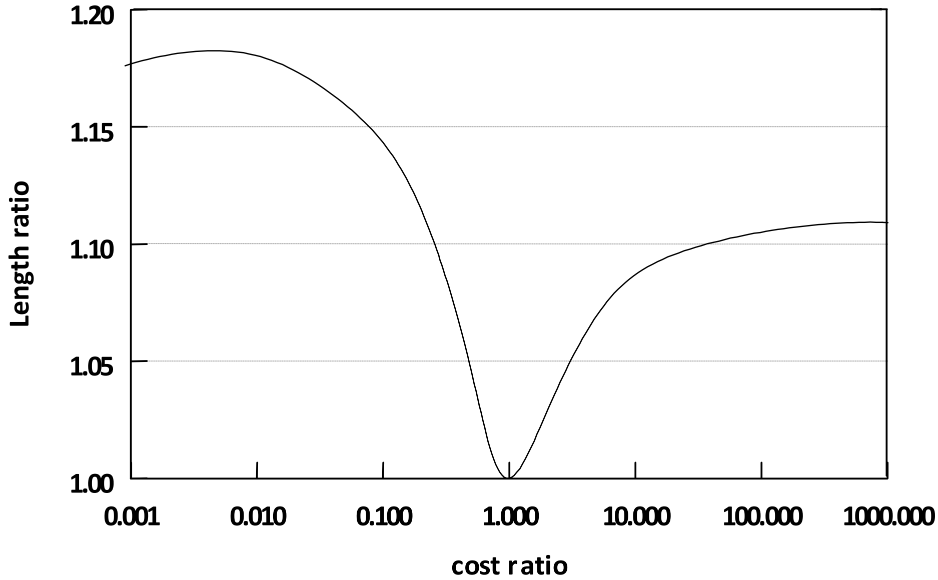

Fig. 9. Effect of cost ratio on length ratio for the straight cost boundary line of Fig. 4.

7. Conclusions

The main conclusions of this work can be summarized as follows:

1. The variational approach yields an exact analytical solution for the optimum route, in a heterogeneous terrain with two cost subregions separated by a straight cost boundary line, without the need for any iterative approach.
2. The cost ratio has a significant effect on the optimum route location in heterogeneous terrain. The optimum route tends to be short in the subregion with the highest unit transportation cost and long in the cheap region.
3. The saving in cost for the optimum route is significantly affected by the cost ratio for any given geometry of the cost boundary line.

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9. Appendix A: Determination of Coordinates of Corner Point

It was shown under section (4) that the determination of the corner point for the optimum route requires the solution of the three simultaneous eqs. (19), (20) and (21). Equations (19) and (20) can be rewritten as follows:

$$c_1(x_c - x_{P_1})\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2} + c_2(x_{P_2}-x_c)\sqrt{(x_c - x_{P_1})^2+(y_c-y_{P_1})^2} = \alpha_1\lambda_1\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2}\cdot\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2} \quad \text{(A. 1)}$$

$$c_1(y_c - y_{P_1})\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2} - c_2(y_{P_2}-y_c)\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2} = -\lambda_1\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2}.\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2} \quad \text{(A. 2)}$$

The substitution of the right hand side of eq. (A. 2) in eq. (A. 1) yields:

$$c_1(x_c - x_{P_1})\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2} + c_2(x_{P_2}-x_c)\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2} = -\alpha_1 c_1(y_c-y_{P_1})\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2} + \alpha_1 c_1(y_{P_2}-y_c)\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2} \quad \text{(A. 3)}$$

or:

$$c_1(x_c - x_{P_1})\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2} + \alpha_1 c_1(y_c-y_{P_1})\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2} = \alpha_1 c_2(y_{P_2}-y_c)\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2} - c_2(x_{P_2}-x_c)\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2} \quad \text{(A. 4)}$$

or:

$$c_1\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-y_c)^2}\left[(x_c-x_{P_1})+\alpha_1(y_c-y_{P_1})\right] = c_2\sqrt{(x_c-x_{P_1})^2+(y_c-y_{P_1})^2}\left[\alpha_1(y_{P_2}-y_c)-(x_{P_2}-x_c)\right] \quad \text{(A. 5)}$$

or:

$$c_1\sqrt{(x_{P_2}-x_c)^2+(y_{P_2}-\beta_1-\alpha_1 x_c)^2}\left[x_c m + H\right] = c_2\sqrt{(x_c-x_{P_1})^2+(-y_{P_1}+\beta_1+\alpha_1 x_c)^2}\left[x_c m + F\right] \quad \text{(A. 6)}$$

or:

$$I\sqrt{mx_c^2 - 2x_c G + L}\left[x_c m + H\right] = \sqrt{mx_c^2 - 2x_c N + k}\left[x_c m + F\right] \quad \text{(A. 7)}$$

where:

$$\left.\begin{aligned} I &= c_1 / c_2 \\ m &= 1+\alpha_1^2 \\ G &= \alpha_1 y_{P_2} + \alpha_1\beta_1 + x_{P_2} \\ L &= y_{P_2}^2 - 2\beta_1 y_{P_2} + \beta_1^2 + x_{P_2}^2 \\ H &= \alpha_1\beta_1 - \alpha_1 y_{P_1} - x_{P_1} \\ N &= x_{P_1} - \beta_1\alpha_1^2 y_{P_1} + \alpha_1 y_{P_1} \\ k &= y_{P_1}^2 - 2\beta_1 y_{P_1} + \beta_1^2 + x_{P_1}^2 \\ F &= \alpha_1 y_{P_2} - \alpha_1\beta_1 - x_{P_2} \end{aligned}\right\} \quad \text{(A. 8)}$$

Squaring eq. (A. 7) yields:

$$I^2(mx_c^2 - 2x_c G + L)(x_c m + H)^2 = (mx_c^2 - 2x_c N + k)(x_c m + F)^2 \quad \text{(A. 9)}$$

Simplifying eq. (A. 9) yields the following fourth order algebraic equation:

$$A + Bx_c + Cx_c^2 + Dx_c^3 + Ex_c^4 = 0 \quad \text{(A. 10)}$$

where:

$$\left.\begin{aligned} A &= I^2 L H^2 - F^2 k \\ B &= 2I^2 mHL - 2I^2 GH^2 - 2NF^2 - 2mFk \\ C &= mI^2H^2 - 4mI^2HG + LI^2m^2 - mF^2 + 4FN - km^2 \\ D &= 2I^2m^2H - 2GI^2m^2 + 2m^2F + 2Nm^2 \\ E &= I^2m^3 - m^3 \end{aligned}\right\} \quad \text{(A. 11)}$$