The Analytical Method To Solve Time Fractional Navier-Stokes Equations

الطريقه التحليلية لحل معادلات نافيير- ستوكس الزمنية الكسرية

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Abstract

In this study, we present a new method for solving time-fractional Navier-Stokes equations analytically. For this purpose, we use Sumudu Transform depending on Homotopy perturbation method to find the solution of fractional Navier-Stokes equations. The derivatives of non-integer order are described in the Caputo sense. We also refer to applied examples of its proving the efficiency and validity of the proposed idea.

Keyword: Sumudu transform; Homotopy perturbation method; Navier-Stokes equations.

الخلاصة

في هذه الدراسة، نقدم طريقة جديدة لحل معادلات نافيير-ستوكس الزمنية الكسرية تحليليا. لهذا الغرض، نستخدم تحويل سومودو (Sumudu Transform) اعتمادا على طريقة اضطراب هوموتوبي (HPTM) لإيجاد حل معادلات نافيير- ستوكس الكسرية. باستخدام تعريف (Caputo) للتفاضل الكسري. ونشير أيضا إلى أمثلة تطبيقية تثبت مدى كفاءة وصلاحية الفكرة المقترحة.

الكلمات المفتاحية: تحويل سومودو(Sumudu Transform). طريقة اضطراب هوموتوبي (HPTM). معادلات نافيير-ستوكس.

1. Introduction

We are known that the last years was characterized by the development of many phenomena and computational problem in different sciences such as physics, mathematics and engineering, especially when linked to the fractional calculus [1-8].

The fractional calculus [3,5,9,10,11,12] plays a fundamental role in mathematical applications as it is considered a precise definition of derivatives and integration to non-integer order.

The solutions for the fractional equations such as Navier-Stokes equations have been studied by many authors using powerful methods in obtaining the exact and approximate solution [13,14,15,16].

The classical Navier-Stokes equations [17] are given by

$$D_t \underline{u} + (\underline{u}.\nabla)\underline{u} = -\frac{1}{\rho}\nabla p + \upsilon \nabla^2 \underline{u}, \quad (1)$$

$$\nabla.\underline{u} = 0, \quad (2)$$

where ρ is the density, p is the pressure, υ is the kinematics viscosity, $\underline{u}$ is the velocity and t is the time. In recent studies, El-Shahed and Salem [18] and Momani and Odibat [16] have develop the integer derivatives of time Navier-Stokes equations above to non integer order $\alpha,\ 0<\alpha\le 1$.

The solutions of fractional Navier-Stokes equations are studied by many researchers using effective techniques in getting exact and approximate solutions, such as Momani et al. [16] have solved time fractional Navier-Stokes equation while Wang et al. [19] they found the approximate solution of the fractional Navier-Stokes equation.

The key objective of this paper is to present approximate solutions of time-fractional Navier-Stokes equations using Sumudu transform based on Homotopy perturbation method. In this paper we consider the time-fractional Navier-Stokes model as

$$D_t^{\alpha}u(x,t)+(u(x,t).\nabla)u(x,t)=-\frac{1}{\rho}\nabla p+\upsilon\nabla^2 u(x,t) \qquad (3)$$

$$u(x,0)=u_0 \qquad (4)$$

2. Some basic preliminaries

Some of the basic concepts and definitions which shall be used in this article regarding the fractional derivative, Sumudu transform and homotopy perturbation method will be reviewed in a simple way.

2.1 Fractional derivatives

We will discuss some definitions and basic notes that are linked to fractional calculus [10,11].

Definition 1: A real function $u(t)$, t > 0 is said to be in space $C_\mu, \mu\in R$ if it exists a real number $p>\mu,$ such that $u(t)=t^p u_1(t)$, where $u_1(t)\in C(0,\infty)$, and it is said to be in the space C_μ^n if and only if $u^{(n)}\in C_\mu, n\in \mathrm{N}$.

Definition 2: The Riemann-Liouville fractional integral operator of order $\alpha>0$, of a function $u\in C_\mu, \mu\ge -1$, is defined as

$$I^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds,\ \alpha>0$$

$$I^0 u(t)=u(t).$$

Definition 3: The fractional derivative with Caputo is written as

$$D^{\alpha}u(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-x)^{m-\alpha-1}u^{(m)}(x)\,dx, \qquad m-1<\alpha\le m, m\in N, t>0$$

Also, the important properties for the fractional derivative with Caputo as follows

i) $$D^{\alpha}t^{\gamma}=\begin{cases}\dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}t^{\gamma-\alpha}, & for\ \gamma\in\{0,1,2,\ldots\}\ and\ \gamma\ge\lceil\alpha\rceil\ or\ \gamma\notin N\ and\ \gamma>\lfloor\alpha\rfloor \\ 0, & for\ \gamma\in\{0,1,2,\ldots\}\end{cases} \qquad (5)$$

ii) $$I^{\alpha}D^{\alpha}u(t)=u(t)-\sum_{i=0}^{m-1}\frac{u^{(i)}(0)}{i!}t^i. \qquad (6)$$

2.2 Sumudu transform

The Sumudu transform is a new integral transform. This transform has many interesting properties and defined over the set of functions

$$A=\{f(t):\exists M,\tau_1,\tau_2>0,|f(t)|<Me^{\frac{|t|}{\tau_j}},\ if\ \ t\in(-1)^j\times[0,\infty)\}. \quad (7)$$

The sumudu transform is defined as follow:

$$G(u)=S(f(t))=\int_0^{\infty} f(ut)e^{-t}dt, \qquad -\tau_1<u<\tau_2 \quad (8)$$

Definition 4: The Sumudu transform of fractional order derivative introduced by Caputo is given by

$$S(D_x^{\alpha} f(t))=\frac{1}{u^{\alpha}}S[f(t)]-\sum_{k=0}^{n-1}\frac{1}{u^{\alpha-k}}[f^{(k)}(t)]_{x=0},\ n-1<\alpha\le n\ ,n\in N \quad (9)$$

Definition 5: The inverse Sumudu transform define as

$$S^{-1}[G(s)]=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}G(\frac{1}{s})\frac{ds}{s} \quad (10)$$

In [20] has been discuss properties of Sumudu transform, and was displayed Sumudu transform for many functions, we present some of it:

1. $if\ \ f(t)=1\quad then\quad S[f(t)]=S[1]=1$
2. $if\ \ f(t)=\frac{t^n}{\Gamma(n+1)}\qquad then\qquad S[f(t)]=S[\frac{t^n}{\Gamma(n+1)}]=u^n\ ,\ n>0$
3. $S[f(t)\pm g(t)]=S[f(t)]\pm S[g(t)]$

2.3 Homotopy perturbation method [23]

We will present in this sub-section a brief overview of the Homotopy perturbation technique. Consider differential equation

$$L(u)+N(u)=f(r),\quad r\in\Omega, \quad (11)$$

with the boundary conditions:

$$B(u,\frac{\partial u}{\partial n})=0,\ r\in\Gamma, \quad (12)$$

where L is a linear operator, while N is nonlinear operator, Γ is the boundary of the domain Ω and $f(r)$ is a source function. The He's homotopy perturbation technique [21-24] defines the homotopy $v(r,p):\Omega\times[0,1]\to R$ which satisfies

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[L(v)+N(v)-f(r)]=0, \quad (13)$$

where $0\le p\le 1$ is an impending parameter, u_0 is an initial approximation which satisfies the boundary conditions. We can express the solution for Eq.(13) by default in a series form and as follows:

$$v=v_0+pv_1+p^2v_2+... \quad (14)$$

Series solution of Eq.(11), therefore can be readily obtained:

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots \tag{15}$$

3. The proposed Sumudu Transform Homotopy Perturbation Method (STHPM)

To study solution technique of fractional Navier-Stokes equation by using Sumudu transform Homotopy Perturbation method, we consider fractional Navier-Stokes model and initial condition as follows

$$D_t^\alpha f = P + \upsilon(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}), \quad 0 < \alpha \le 1 \tag{16}$$

$$f(r,0) = f_0 \tag{17}$$

Applying the Sumudu transform of Eq. (16), we get

$$S(D_t^\alpha f) = S(P) + S(\upsilon(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r})) . \tag{18}$$

From the definition (4), we have

$$\frac{1}{u^\alpha} S(f(r,t)) = \sum_{k=0}^{n-1} \frac{1}{u^{\alpha-k}} (f^{(k)}(r,0)) + S(P) + S(\upsilon(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r})) \tag{19}$$

then,

$$S(f(r,t)) = \sum_{k=0}^{n-1} u^k f^{(k)}(r) + u^\alpha P + u^\alpha (S(\upsilon(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}))). \tag{20}$$

Using the definition of the inverse Sumudu transform, we have

$$f(r,t) = \sum_{k=0}^{n-1} \frac{t^k}{\Gamma(k+1)} f^{(k)}(r) + \frac{t^\alpha}{\Gamma(\alpha+1)} P + S^{-1}(u^\alpha (S(\upsilon(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r})))), \tag{21}$$

then

$$f(r,t) = G(r,t) + S^{-1}(u^\alpha (S(\upsilon(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r})))), \tag{22}$$

the term G forms the initial conditions and in addition to source term. Now application of the homotopy perturbation technique

$$f(r,t)=\sum_{n=0}^{\infty} p^n f_n(r,t) \tag{23}$$

substituting (23) in (22), we get

$$\sum_{n=0}^{\infty} p^n f_n(r,t)=G(r,t)+P(S^{-1}(u^{\alpha}(S(\upsilon(\frac{\partial^2}{\partial r^2}(\sum_{n=0}^{\infty} p^n f_n(r,t))+\frac{1}{r}\frac{\partial}{\partial r}(\sum_{n=0}^{\infty} p^n f_n(r,t)))))),$$

which is the hybrid of the Sumudu transform and the homotopy perturbation method utilizing He's polynomials. Looking at the coefficients of like powers of p, the following approximate are acquired

$$p^0 : f_0(r,t)=G(r,t)$$
$$p^1 : f_1(r,t)=S^{-1}(u^{\alpha}(S(\upsilon(\frac{\partial^2 f_0}{\partial r^2}+\frac{1}{r}\frac{\partial f_0}{\partial r}))))$$
$$p^2 : f_2(r,t)=S^{-1}(u^{\alpha}(S(\upsilon(\frac{\partial^2 f_1}{\partial r^2}+\frac{1}{r}\frac{\partial f_1}{\partial r}))))$$
$$p^3 : f_3(r,t)=S^{-1}(u^{\alpha}(S(\upsilon(\frac{\partial^2 f_2}{\partial r^2}+\frac{1}{r}\frac{\partial f_2}{\partial r}))))$$
$$\vdots$$

then the solution of (16) is given by

$$f(r,t)=f_0(r,t)+f_1(r,t)+f_2(r,t)+\ldots=\sum_{n=0}^{\infty} f_n(r,t) \tag{24}$$

4. Application

In order to apply this technique we will take two examples with analytical solution to show the efficiency of proposed technique for solve time fractional Navier-Stokes equation described in the section above.

Example (1): We consider the following time fractional Navier-Stokes model,

$$D_t^{\alpha} f=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r} \quad , \quad 0<\alpha\le 1 \tag{25}$$

with the initial condition

$$f(r,0)=r \tag{26}$$

Using the property of Sumudu transform on both sides of equation (25) with initial condition (26), we have

$$S(f(r,t)) = r + u^{\alpha}(S(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r})) \, . \tag{27}$$

By applying the inverse Sumudu transform

$$f(r,t) = r + S^{-1}(u^{\alpha}(S(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}))) \, . \tag{28}$$

Now, by applying the Homotopy perturbation method, we get

$$\sum_{n=0}^{\infty} p^n f_n(r,t) = r + P(S^{-1}(u^{\alpha}(S(\frac{\partial^2}{\partial r^2}(\sum_{n=0}^{\infty} p^n f_n(r,t)) + \frac{1}{r}\frac{\partial}{\partial r}(\sum_{n=0}^{\infty} p^n f_n(r,t)))))),$$

The coefficients of forces p by the Comparing, we obtain

$$p^0 : f_0(r,t) = r$$

$$p^1 : f_1(r,t) = S^{-1}(u^{\alpha}(S(\upsilon(\frac{\partial^2 f_0}{\partial r^2} + \frac{1}{r}\frac{\partial f_0}{\partial r})))) = \frac{1}{r}\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$p^2 : f_2(r,t) = S^{-1}(u^{\alpha}(S(\upsilon(\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r})))) = \frac{1}{r^3}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$p^3 : f_3(r,t) = S^{-1}(u^{\alpha}(S(\upsilon(\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r})))) = \frac{9}{r^5}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\vdots$$

then the solution of Eq. (25) is given by

$$f(r,t) = f_0(r,t) + f_1(r,t) + f_2(r,t) + \ldots = r + \sum_{n=1}^{\infty} \frac{1^1 \times 3^2 \times \ldots \times (2n-3)^2}{r^{2n-1}} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}$$

when $\alpha = 1$, the solution of the problem become

$$f(r,t) = r + \sum_{n=1}^{\infty} \frac{1^1 \times 3^2 \times \ldots \times (2n-3)^2}{r^{2n-1}} \frac{t^n}{n!} \tag{29}$$

we will see the same result when compare this solution with the solution by Baizar et al. [14] and Momoni and Odibat [16].

Example (2): We consider the following time fractional Navier-Stokes model,

$$D_t^{\alpha} f = P + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} \quad , \quad 0 < \alpha \le 1 \tag{30}$$

with the initial condition

$$f(r,0)=1-r^2 \tag{31}$$

By applying the technique in section three, we get

$$\sum_{n=0}^{\infty} p^n f_n(r,t) = 1-r^2 + P\frac{t^\alpha}{\Gamma(\alpha+1)} + P(S^{-1}(u^\alpha(S(\frac{\partial^2}{\partial r^2}(\sum_{n=0}^{\infty} p^n f_n(r,t)) + \frac{1}{r}\frac{\partial}{\partial r}(\sum_{n=0}^{\infty} p^n f_n(r,t))))))$$

The coefficients of forces p by the Comparing, we obtain

$$p^0 : f_0(r,t) = 1-r^2 + P\frac{t^\alpha}{\Gamma(\alpha+1)}$$

$$p^1 : f_1(r,t) = S^{-1}(u^\alpha(S(\upsilon(\frac{\partial^2 f_0}{\partial r^2} + \frac{1}{r}\frac{\partial f_0}{\partial r})))) = -\frac{4t^\alpha}{\Gamma(\alpha+1)}$$

$$p^2 : f_2(r,t) = S^{-1}(u^\alpha(S(\upsilon(\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r})))) = 0$$

$$p^3 : f_3(r,t) = S^{-1}(u^\alpha(S(\upsilon(\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r})))) = 0$$

$$\vdots$$

the result of Eq. (30) as follows

$$f(r,t) = f_0(r,t) + f_1(r,t) + f_2(r,t) + \ldots = 1-r^2 + (P-4)\frac{t^\alpha}{\Gamma(\alpha+1)}$$

when $\alpha = 1$, the solution of the problem become

$$f(r,t) = 1-r^2 + (P-4)t \tag{32}$$

The numerical results for time fractional Navier-Stokes equation with t=1, P=1 and different values of r and α , shown in Figure 1, where it is the same as ADM by Momani and Odibat [16].

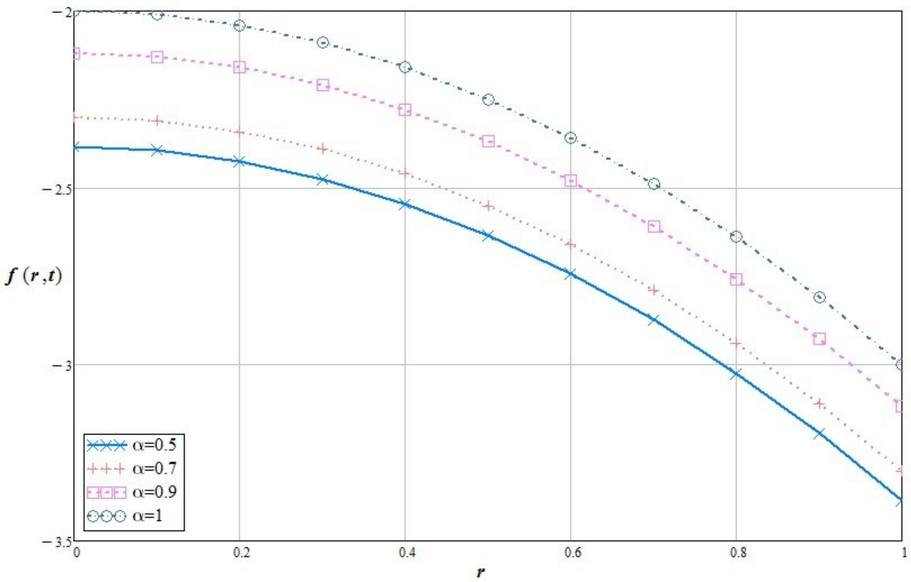

Fig. 1 Graph of $f(r,t)$ with t=P=1 for different values of α .

5. Conclusion

From this work we can conclude that the proposed Sumudu transform homotopy perturbation method is effective for solving this model of equation because the method is simple and smooth. However, what is the ideal solution and the appropriate ways to solve a system of these equations and this can be studied in future research.

6. References

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