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Simultaneous Approximation by Szàsz-Kantorovich Operators

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Abstract:
In this paper, we study the simultaneous approximation by Szàsz-Kantorovich type operators. We prove that these operators are converged to the function being approximated. Also, we discuss a Voronovaskaja-type asymptotic formula in simultaneous approximation for these operators.

Keywords: Simultaneous approximation, Szàsz-Kantorovich type operators, Voronovaskaja-type asymptotic formula.

1. *Introduction*

For a function $f \in C_\gamma[0,\infty) := \{g \in C[0,\infty): |g(t)| < M(1+t)^\gamma$ for some $M > 0, \gamma > 0\}$, and $n \in N := \{1,2,\dots\}$. The classical Szàsz operators are known as [1]:

$$S_n(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x)\, f(\frac{k}{n}),$$

where $q_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!},\ x \in [0,\infty)$.

In [2], Sun studied the simultaneous approximation of functions and their derivatives by the classical Szàsz operators.

Gupta and et al. defined a new sequence of Szàsz -Beta operators as follows [3]:

$$B_n(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x) \int_0^\infty b_{n,k}(t) f(t)dt,$$

where $b_{n,k}(t) = \frac{(n+k)!}{k!(n-1)!} t^k (1+t)^{-(n+k+1)}$. In addition, they studied a Voronovaskaja type asymptotic formula for these operators in simultaneous approximation.

In [4], Dumanand et al. studied the Szàsz-Kantorovich operators preserving linear functions, which are defined as:

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$$S_n(f;x) = n\sum_{k=0}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt. \qquad (1,1)$$

In 2016 [6], Atakut and Büyükyazıcı introduced a generalization of the Kantorovich Szász type operators defined by means of the Brenke type polynomials and obtained convergence properties of these operators by using Korovkin's theorem.

This paper is a continuation of the reference [4]. In [4], the authors studied ordinary approximation and they did not state a Voronovaskaja-formula for the operators (1.1).

In this paper, we study the Szàsz-Kantorovich operators in simultaneous approximation. We did not find any study of these types (Kantorovich operators) in simultaneous approximation and this is because of the difficulty of derive a recurrence relation of m-th order moment. We solve this problem by using a general formula of the term $S_n(t^m;x)$.

Lemma 1.1 [4]

For the operators $S_n(f;x)$ and $m \in N^0 := \{0,1,2,\dots\}$, we have:

(a) $S_n(1;x) = 1,$

(b) $S_n(t;x) = x + \frac{1}{2n},$

(c) $S_n(t^2;x) = x^2 + \frac{2x}{n} + \frac{1}{3n^2},$

(d) $S_n(t^m;x) = x^m + \frac{m^2}{2n}x^{m-1} + T.L.P.(x) + \frac{1}{n^{m(m+1)}},$

where $T.L.P.(x)$ means Terms in Lower Powers of x.

Our next lemma shows that the nature terms which are used in the description of the m-th order moment for the operators $S_n(f:x)$.

The m-th order moment for the operators $S_n(f:x)$ is defined by: [4]

$$T_{n,m}(x) = M_n((t-x)^m;x) = n\sum_{k=0}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt.$$

Theorem 1.1

For the function $T_{n,m}(x)$, we have:

$$T_{n,m}(x) = \sum_{i=0}^{m} \binom{m}{i} (-x)^{m-i} \left\{ x^m + \frac{m^2}{2n}x^{m-1} + TLP(x) + \frac{1}{n^m(m+1)} \right\},$$

where $m = 0, 1, 2, 3, \dots$.

Proof: By direct computation, we have

$$T_{n,m}(x) = n\sum_{k=0}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^m\, dt$$

$$= n\sum_{k=0}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \sum_{i=0}^{m} \binom{m}{i} t^i (-x)^{m-i}\, dt$$

$$= \sum_{i=0}^{m} \binom{m}{i} (-x)^{m-i} \left\{ n\sum_{k=0}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} t^i\, dt \right\}$$

$$= \sum_{i=0}^{m}\binom{m}{i}(-x)^{m-i}\left\{x^m + \frac{m^2}{2n}x^{m-1} + T.L.P.(x) + \frac{1}{n^{m(m+1)}}\right\}.$$

2. Main Results

Our first result is a Voronovaskaja-type asymptotic formula for the operators $S_n(f(t);x)$ in ordinary approximation.

Theorem 2.1

Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$. If f'' exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n\big(S_n(f(t);x) - f(x)\big) = \frac{1}{2}f'(x) + \frac{1}{2}xf''(x)$$

Proof: By Taylor's expansion of $f(t)$, we have

$$f(t) = f(x) + f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Hence

$$S_n(f(t);x) = f(x)S_n(1;x) + f'(x)S_n\big((t-x);x\big) + \frac{f''(x)}{2}S_n((t-x)^2;x)$$
$$+S_n(\varepsilon(t,x)(t-x)^2;x).$$

$$\lim_{n\to\infty} n\{S_n(f(t);x) - f(x)\} = \lim_{n\to\infty}\{f'(x)\frac{n}{2n} + \frac{f''(x)}{2}n\left(\frac{1}{n}x + \frac{1}{3n^2}\right)$$
$$+ nS_n(\varepsilon(t,x)(t-x)^2;x)\}$$
$$= \frac{1}{2}f'(x) + \frac{1}{2}xf''(x) + \lim_{n\to\infty} E,$$

where $E = nS_n(\varepsilon(t,x)(t-x)^2;x)$.

$$|E| = \left|n^2\sum_{k=0}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} \varepsilon(t,x)(t-x)^2\,dt\right|$$
$$\le n^2\sum_{k=0}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} |\varepsilon(t,x)|(t-x)^2\,dt$$
$$\le n^2\sum_{k=0}^{\infty} q_{n,k}(x)\int_{|t-x|<\delta} |\varepsilon(t,x)|(t-x)^2\,dt$$
$$\le n^2\varepsilon\sum_{k=0}^{\infty} q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2\,dt$$
$$= \varepsilon n\left\{\frac{x}{n} + \frac{1}{3n^2}\right\} = \varepsilon x = o(1).$$

Now, since $\varepsilon > 0$ is arbitrary, it follows that $E \to 0$ as $n \to \infty$.

The modulus of continuity of a continuous function $\omega: [a,b] \to R$ for $\delta > 0$ is defined as [5]:

$$\omega(\delta, f) \coloneqq max_{|x-y|<\delta}|f(x) - f(y)|.$$

Next, we give an error of approximation by the terms of modulus of continuity in ordinary approximation.

Theorem 2.2

Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$ and $0 \le q \le 2$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$, then for sufficiently large n, we have:

$$\|S_n(f(t);x)-f(x)\|_{C[a,b]} \le C_1 n^{-1}\sum_{i=0}^{q}\|f^{(i)}\|_{C[a,b]} + C_2\, n^{\frac{-1}{2}}\omega_{f^{(q)}}(n^{\frac{-1}{2}};(a-\eta\,,b+\eta)$$
$$+O(n^{-2}),$$

where C_1, C_2 are constants independent on f and n.

Proof: By our hypothesis

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)+h(t,x)\big(1-\chi(t)\big),$$

where ξ lies between t, x, and $\chi(t)$ is the characteristic function of the interval $(a-\eta\,,b+\eta)$.

For $t\in(a-\eta\,,b+\eta)$ and $x\in[a,b]$, we get

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)\,.$$

For $t\in[0,\infty)/(a-\eta\,,b+\eta)$ and $x\in[a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$S_n(f(t);x)-f(x)=\left\{\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}S_n((t-x)^i;x)-f(x)\right\}$$
$$+S_n(\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^i\chi(t);x)$$
$$+S_n(h(t,x)(1-\chi(t));x)$$

$:=\Sigma_1+\Sigma_2+\Sigma_3.$

By using Lemma 1.1, we get

$$\Sigma_1=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{m=0}^{j}\binom{j}{m}(-x)^{j-m}\{S_n(t^m;x)-f(x)\}$$
$$=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{m=0}^{j}\binom{j}{m}(-x)^{j-m}\{x^m+\frac{m^2}{2n}x^{m-1}+T.L.P.(x)+\frac{1}{(m+1)n^m}\}.$$

Based on the above, we get

$\|\Sigma_1\|_{C[a,b]}\le C_1\,n^{-1}(\sum_{i=0}^{q}\|f^{(i)}\|_{C[a,b]})+O(n^{-2})$, uniformly on [a,b], where

$\|f\|_{C[a,b]}=sup_{a\le x\le b}|f(x)|$

To estimate Σ_2 we proceed as follows:

$$|\Sigma_2|\le\left|S_n(\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^i\chi(t);x)\right|$$

$$\le\frac{\omega_{f^{(q)}}\big(\delta;(a-\eta,b+\eta)\big)}{q!}S_n(\{1+\frac{|t-x|}{\delta}\}|t-x|^q;x)$$

$$\leq \frac{\omega_{f^{(q)}}\left(\delta;(a-\eta,b+\eta)\right)}{q!}\{n\sum_{k=0}^{\infty}|q_{n,k}(x)|\int_{\frac{k}{n}}^{\frac{k+1}{n}}(|t-x|^{q}+\delta^{-1}|t-x|^{q+1})dt\}.$$

Now, s=0, 1, 2, ..., we have

$$n\sum_{k=0}^{\infty}q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|^{s}dt$$

$$\leq n\sum_{k=0}^{\infty}q_{n,k}(x)(\int_{\frac{k}{n}}^{\frac{k+1}{n}}dt)^{1/2}(\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)^{2s}\ dt)^{\frac{1}{2}}$$

$$\leq (\sum_{k=0}^{\infty}q_{n,k}(x))^{\frac{1}{2}}(n\sum_{k=0}^{\infty}q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}(t-x)^{2s}\ dt)^{\frac{1}{2}}$$

$$\leq (1)^{\frac{1}{2}}(T_{n,2s}(x))^{\frac{1}{2}}$$

$\leq O(n^{\frac{-s}{2}})$ uniformly on [a,b].

Choosing $\delta = n^{\frac{-1}{2}}$, then

$$\|\Sigma_2\|_{C[a,b]} \leq \frac{\omega_{f^{(q)}}\left(n^{\frac{-1}{2}};(a-\eta,b+\eta)\right)}{q!}[O\left(n^{\frac{-q}{2}}\right)+n^{\frac{1}{2}}\ O\left(n^{\frac{-(q+1)}{2}}\right)]$$

$$\leq C_2\, n^{\frac{-q}{2}}\omega_{f^{(q)}}\left(n^{\frac{-1}{2}};(a-\eta,b+\eta)\right).$$

Since $t \in [0,\infty)/(a-\eta\,,b+\eta)$, we can choose $\delta > 0$ in such a way that $|t-x| \geq \delta$ for all $x \in [a,b]$.

$$\|\Sigma_3\|_{C[a,b]} \leq n\sum_{k=0}^{\infty}q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}|h(t,x)|\,dt$$

For $|t-x| \geq \delta$, we can find $C > 0$ such that $|h(t,x)| \leq Ce^{\alpha t}$.

$$|\Sigma_3| \leq C\,n\sum_{k=0}^{\infty}q_{n,k}(x)\int_{\frac{k}{n}}^{\frac{k+1}{n}}e^{\alpha t}\,dt$$

$$\leq C\, n \sum_{k=0}^{\infty} q_{n,k}(x) \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} dt \right)^{\frac{1}{2}} \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} e^{2\alpha t}\, dt \right)^{\frac{1}{2}}$$

$$\leq C \left(\sum_{k=0}^{\infty} q_{n,k}(x) \right)^{1/2} \left(n \sum_{k=0}^{\infty} q_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} e^{2\alpha t}\, dt \right)^{1/2}$$

$$\leq C\, (1)^{1/2}\, O(n^{-s})$$

$\leq O(n^{-s})$ for any $s > 0$, uniformly on [a,b].
Combining the estimates of $\Sigma_1, \Sigma_2, \Sigma_3$, the required is immediate.

Our next theorem is a Voronovaskaja-type asymptotic formula for the operators $S_n(f;x)$ in simultaneous approximation. This is a generalization of Theorem 2.1, by put $r = 0$.

Theorem 2.3

Let $f \in C_\gamma[0,\infty)$ for some $\gamma > 0$. If $f^{(r+2)}(x)$ exists and it's continuous, where $r \in \mathbb{N} = \{0, 1, 2, \dots\}$, then

$$\lim_{n\to\infty} n\left(S_n^{(r)}(f(t);x) - f^{(r)}(x)\right) = \frac{r+1}{2} f^{(r+1)}(x) + \frac{x}{2} f^{(r+2)}(x).$$

Proof: By Taylor's expansion, we have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2},$$

where $\varepsilon(t,x) \to 0$ as $t \to x$.
Then,

$$S_n^{(r)}(f(t);x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} S_n^{(r)}((t-x)^i;x) + S_n^{(r)}(\varepsilon(t,x)(t-x)^{r+2};x)$$
$$\coloneqq E + E_1.$$

$$E = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} S_n^{(r)}((t-x)^i;x) = \sum_{i=r}^{r+2} \frac{f^{(i)}(x)}{i!} \left\{ \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} S_n^{(r)}(t^j;x) \right\}$$

$$= \frac{f^{(r)}(x)}{r!} S_n^{(r)}(t^r;x) + \frac{f^{(r+1)}(x)}{(r+1)!}\left\{(r+1)(-x)S_n^{(r)}(t^r;x) + S_n^{(r)}(t^{r+1};x)\right\}$$
$$+ \frac{f^{(r+2)}(x)}{(r+2)!}\left\{\frac{(r+2)(r+1)}{2}(-x)^2 S_n^{(r)}(t^r;x) + (r+2)(-x)S_n^{(r)}(t^{r+1};x) + S_n^{(r)}(t^{r+2};x)\right\}$$
$$= S_n^{(r)}(t^r;x)\left\{\frac{f^{(r)}(x)}{r!} + \frac{f^{(r+1)}(x)}{(r+1)!}(r+1)(-x) + \frac{f^{(r+2)}(x)}{(r+2)!}\frac{(r+2)(r+1)}{2}(-x)^2\right\}$$
$$+ S_n^{(r)}(t^{r+1};x)\left\{\frac{f^{(r+1)}(x)}{(r+1)!} + \frac{f^{(r+2)}(x)}{(r+2)!}(r+2)(-x)\right\}$$
$$+ S_n^{(r)}(t^{r+2};x)\left\{\frac{f^{(r+2)}(x)}{(r+2)!}\right\}.$$

Since,

$$S_n(t^r;x) = x^r + \frac{r^2}{2n}x^{r-1} + T.L.P.(x) + \frac{1}{n^r(r+1)}$$
$$S_n^{(r)}(t^r;x) = r! + zero$$

$$S_n^{(r)}(t^{r+1};x) = x(r+1)! + \frac{(r+1)^2}{2n} r!$$

$$S_n^{(r)}(t^{r+2};x) = x^2 \frac{(r+2)!}{2} + \frac{(r+2)^2}{2n}(r+1)!\, x$$

$$n\left(S_n^{(r)}(f(t);x) - f^{(r)}(x)\right) = \frac{nf^{(r)}(x)}{r!}\{r! - r!\}$$
$$+ \frac{nf^{(r+1)}(x)}{(r+1)!}\left\{(r+1)(-x)r! + (r+1)!\, x + \frac{(r+1)^2}{2n} r!\right\}$$
$$+ \frac{nf^{(r+2)}(x)}{(r+2)!}\left\{\frac{(r+2)(r+1)(-x)^2}{2} r! + (r+2)!\,(-x^2) + \frac{(r+2)(-x)(r+1)^2 r!}{2n} + x^2 \frac{(r+2)!}{2} + \frac{(r+2)^2}{2n}(r+1)!\, x\right\}.$$

From the above, we get the required result.

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التقريب المتعدد بمؤثرات *Szὰsz–Kantorovich*

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المستخلص:

في هذا البحث، قمنا بدراسة التقريب المتعدد لمؤثرات من نوع Szὰsz–Kantorovich. برهنا تقارب هذه المؤثرات للدالة المراد تقريبها. كذلك، ناقشنا الصيغة المشابهة لـ Voronovaskaja في التقريب المتعدد لهذه المؤثرات.