The Detour Polynomials of the Corona of Graphs

Ali Aziz Ali
aliazizali1933@yahoo.com
Academic Professor
University of Mosul, Mosul, Iraq

Gashaw A. Muhammed-Saleh
gashaw.mohammed@su.edu.krd
College of Science
University of Salahaddin, Erbil, Iraq

Received on: 05/04/2012 **Accepted on: 28/06/2012**

ABSTRACT

A new graph distance based polynomial, called detour polynomial, is introduced. The detour polynomial and the detour index of the corona $G_1 \circ G_2$ of two connected disjoint graphs G_1 and G_2 are obtained in this paper.

Keywords: distance, detour distance, detour polynomial, detour index, corona.

متعددات حدود الالتفاف لبيانات كورونا

علي عزيز علي
أستاذ أكاديمي
جامعة الموصل، الموصل، العراق

كه شاو محمد صالح
كلية العلوم
جامعة صلاح الدين، أربيل، العراق

تاريخ استلام البحث: 2012/04/05 **تاريخ قبول البحث: 2012/06/28**

الملخص

في هذا البحث استعرضنا متعددات حدود جديدة معتمدة على مسافة الالتفاف في البيانات. وقد تضمن البحث الحصول على متعددة حدود الالتفاف ودليل الالتفاف لإكليل $G_1 \circ G_2$ لبيانين متصلين ومنفصلين عن بعضهما G_1 و G_2.

الكلمات المفتاحية: المسافة، مسافة الالتفاف، متعددة حدود الالتفاف، دليل الالتفاف، الاكليل.

1. Introduction :

The concept of Hosoya polynomial was first put forward in 1988 by Hosoya [**14**]. Several authors, such as [**1**]-[**4**], [**12**], and [**14**]-[**17**] had obtained Hosoya polynomials for special graphs, graphs having some kind of regularity and for compound graphs obtained by using some well-known binary operations in graph theory.

In this paper, we consider finite connected graphs without loops or multiple edges. For undefined concepts and notations see [**6**] and [**13**].

The standard distance $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. This is not the only way, however, that distance has been defined on the vertex set of a connected graph. The length of a longest $u-v$ path between two vertices u and v in a connected graph is called the **detour distance** $D(u,v)$ between u and v. As with standard distance, detour distance is a metric on the vertex set of any connected graph [7]. A $u-v$ path of length $D(u,v)$ is $u-v$ **detour**. It is clear that $D(u,v)=1$ if and only if uv is a bridge of G, and $D(u,v)=p(G)-1$ if and only if G contains a Hamiltonian $u-v$ path. Furthermore,

$d(u,v)=D(u,v)$ for every two vertices u and v of G if and only if G is a tree. It is possible, however, that $d(u,v)=D(u,v)$ for some pairs u, v of distinct vertices in a graph that contains no bridges. The **detour eccentricity** $e_D(v)$ of a vertex v is the maximum detour distances from v to all other vertices in G. The **detour radius** $rad_D(G)$ of a connected graph G is the minimum detour eccentricity among the vertices of G, and the **detour diameter** $diam_D(G)$ is the maximum detour eccentricity among the vertices of G. Since $d(x,y)\le D(x,y)$ for every two vertices x and y in G, it follows that $e(v)\le e_D(v)$ for every vertex v.Therefore, $rad(G)\le rad_D(G)$ and $diam(G)\le diam_D(G)$ for every connected graph G (see [9], [10] and [18]). In fact $rad_D(G)\le diam_D(G)\le 2rad_D(G)$, see [9].

Let G be a connected graph of order p and size q. The **detour polynomial** of G, denoted by $D(G;x)$, is defined by

$$D(G;x)=\sum_{u,\,v\in V} x^{D(u,v)}\ ,\ u\ne v\,;\ V=V(G)\,.$$

It is clear that $D(G;x)=\sum_{k\ge 1} C_D(G,k)\,x^k$, in which $C_D(G,k)$ is the number of unordered pairs u , v such that $D(u,v)=k$.

Amić and Trinajstić [5] were first to consider the **detour index** $dd(G)$ defined as the sum of the detour matrix elements above the main diagonal. The detour index is used in quantitative structure-activity relationship (QSAR) studies. Lukovits [15] tested the detour index on the correlation of the boiling points of alkanes of cycloalkanes. The detour index can also be obtained from the detour polynomial, because

$$dd(G)=\sum_{u,\,v} D(u,v)=\left.\frac{d\ D(G;x)}{dx}\right|_{x=1}=\sum_{k\ge 1} kC_D(G,k)\,.$$

Let $C_D(G;v,k)$ be the number of vertices $u\ (\ne v)$ such that $D(u,v)=k$. Then, we define for each vertex v of G:

$$D(G;v,x)=\sum_{k\ge m(v)}^{e_D(v)} C_D(G;v,k)x^k=\sum_{\substack{u\in V\\ u\ne v}} x^{D(v,u)}\,,$$

where $e_D(v)$ is the detour eccentricity of vertex v, and $m(v)$ is the minimum detour distance from v. This polynomial is called the **detour polynomial of vertex** v. It is clear that

$$D(G;x)=\frac{1}{2}\sum_{v\in V} D(G;v,x)\,.$$

2. The Corona $G_1\circ G_2$:

The **corona** of two disjoint graphs G_1 and G_2 of orders p_1 and p_2, respectively , is the graph $G=G_1\circ G_2$ defined by taking one copy of G_1 and p_1 copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in the i^{th} copy of G_2, as illustrated in Fig.2.1, where the copies of G_2 are denoted by $G_1', G_2', \ldots, G_{p_1}'$,

$V_1=V(G_1)=\{v_1,\ v_2,\ldots,v_{p_1}\}$,

$U^{(i)}=V(G_i')=\{u_1^{(i)},\ u_2^{(i)},\ldots,u_{p_2}^{(i)}\}$, for $i=1,\ 2,\ldots,\ p_1$,

$V_2 = V(G_2) = \{u_1, u_2, ..., u_{p_2}\}$,

and

$U = \bigcup_{i=1}^{p_1} U^{(i)}$.

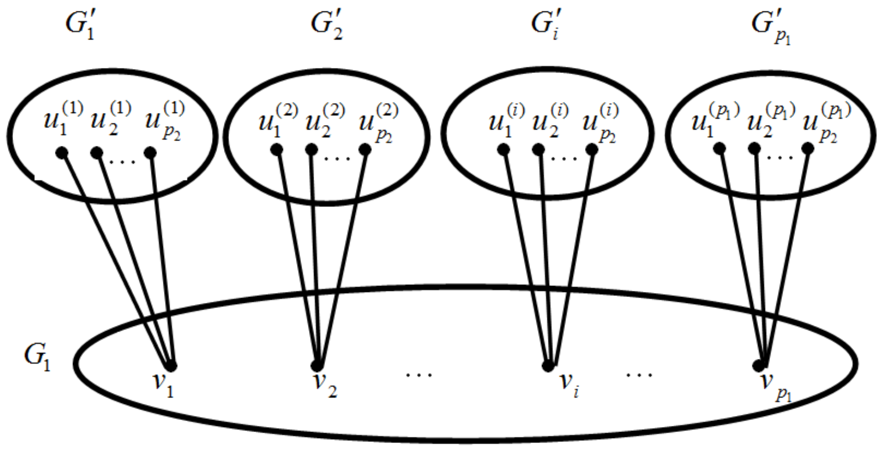

Fig.2.1 The corona $G_1 \circ G_2$

It follows, from the definition of the corona $G_1 \circ G_2$, that $p(G_1 \circ G_2) = p_1(1+p_2)$, $q(G_1 \circ G_2) = q_1 + p_1 q_2 + p_1 p_2$ and diam($G_1 \circ G_2$)=diam(G_1)+2.
Note that $G_1 \circ G_2 \neq G_2 \circ G_1$ unless $G_1 \cong G_2$.
Thus, the corona is a binary graph operation, it was defined for the first time by Frucht and Harary [11] in 1970, and used in studying the automorphism group of graphs. Recently, in 2007, the Hosoya polynomial of the corona of two graphs , with respect to Steiner distance, was obtained [1]. So , we think that it is an interesting to obtain the Hosoya polynomial of the corona with respect to detour distance.

We begin discussing the detour distance and detour diameter of $G_1 \circ G_2$. Let $e_D^{(i)}(w)$ be the detour eccentricity of vertex w in the graph G_i , i=1,2 . Define the graph G_2^+ as $G_2 + K_1$.

Proposition 2.1: Let w_1 and w_2 be any two distinct vertices of $G = G_1 \circ G_2$. Then, the detour distance $D_G(w_1, w_2)$ equals:

1) $D_{G_1}(w_1, w_2)$, if $w_1, w_2 \in V(G_1)$;
2) $D_{G_2^+}(w_1, w_2)$, if $w_1, w_2 \in V(G'_i)$ for $1 \le i \le p_1$;
3) $1 + e_D^{(2)}(w_2)$, if $w_1 = v_i$ and $w_2 \in V(G'_i)$ for some i, $1 \le i \le p_1$;
4) $D_{G_1}(v_i, v_j) + e_D^{(2)}(w_2) + 1$, if $w_1 = v_i$ and $w_2 \in V(G'_j)$, where $i \neq j$ and $1 \le i, j \le p_1$;
5) $D_{G_1}(v_i, v_j) + e_D^{(2)}(w_1) + e_D^{(2)}(w_2) + 2$, if $w_1 \in V(G'_i)$, $w_2 \in V(G'_j)$, where $i \neq j$ and $1 \le i, j \le p_1$.

Proof: 1) Obvious .
2) Any detour between two vertices of G'_i , $1 \le i \le p_1$ must contain vertex v_i and does not contain another vertex of G_1. Therefore $D_G(w_1, w_2) = D_{G_2^+}(w_1, w_2)$.

3) If w_2' is the detour eccentric vertex of w_2 in G_i', that is $D_{G_2}(w_2, w_2') = e_D^{(2)}(w_2)$, then a detour from v_i to w_2 consists of the edge $v_i w_2'$ followed by the detour in G_i' between w_2' to w_2. Therefore $D_G(w_1, w_2) = 1 + e_D^{(2)}(w_2)$.

4) In this case a detour between w_1 and w_2 in G consists of a detour between v_i and v_j in G_1 followed by the edge $v_i w_2'$ then the detour between w_2' and w_2 in G_j', here w_2' is the detour eccentric vertex of w_2 in G_j'. Thus

$$D_G(w_1, w_2) = D_{G_1}(v_i, v_j) + 1 + e_D^{(2)}(w_2).$$

5) A detour between a vertex w_1 of G_i' and a vertex w_2 of G_j', $i \neq j$, is constructed from a detour from w_1 to w_1' (the detour eccentric vertex of w_1), followed by the edge $w_2' v_j$, then a detour from v_i to v_j in G_1, then followed by the edge $v_j w_2'$ and finally followed by a detour $w_2' - w_2$ in G_j'. The length of this detour is $D_{G_1}(v_i, v_j) + e_D^{(2)}(w_1) + e_D^{(2)}(w_2) + 2$. Hence , the proof is complete. ■

Proposition 2.2: For a connected nontrivial graph G_1 and any connected graph G_2, $diam_D(\ G_1 \circ G_2) = diam_D(G_1) + 2diam_D(G_2) + 2$.

Proof: Let v_i, v_j be two vertices of G_1 such that $D_{G_1}(v_i, v_j) = diam_D(G_1)$; and let u_l, u_k be two vertices of G_2 such that $D_{G_2}(u_l, u_k) = diam_D(G_2)$.

One may easily check from Fig.2.1 and Proposition 2.1(5) , that

$$D_G(u_l^{(i)}, u_k^{(j)}) = D_{G_i'}(u_l^{(i)}, u_k^{(i)}) + 1 + D_{G_1}(v_i, v_j) + 1 + D_{G_j'}(u_l^{(j)}, u_k^{(j)})$$
$$= \mathrm{diam_D}(G_1) + 2\,diam_{\mathrm{D}}(G_2) + 2.$$

Now, let w_1, w_2 be any two vertices of $G_1 \circ G_2$, then:

1- If w_1, $w_2 \in V(G_1)$, then $D_G(w_1, w_2) \leq diam_D(G_1)$;

2- If w_1, $w_2 \in V(G_r')$ for some r, $1 \leq r \leq p_1$, then, let Q be a $w_1 - w_2$ detour. If $v_r \notin V(Q)$, then $D_G(w_1, w_2) \leq diam_D(G_2)$.

If $v_r \in V(Q)$, then $Q - v_r$ consists of two paths in G_r' each of length not more than $diam_D(G_2)$. Thus $D_G(w_1, w_2) \leq 2diam_D(G_2) + 2$;

3- If $w_1 \in V(G_t')$ and $w_2 \in V(G_s')$, $t \neq s$, then
$D_G(w_1, w_2) \leq diam_D(G_1) + 2diam_D(G_2) + 2$;

4- If $w_1 \in V(G_1)$ and $w_2 \in V(G_m')$, then
$D_G(w_1, w_2) \leq diam_D(G_1) + 1 + diam_D(G_2)$.

Therefore, for all cases of w_1 and w_2, we have

$$D_G(w_1, w_2) \leq diam_D(G_1) + 2diam_D(G_2) + 2.$$

Hence , the proof is complete. ■

The **minimum detour** of a connected graph G is denoted by $D_{\min}(G)$, and defined as

$$D_{\min}(G) = \min\{D(u, v) : u \neq v, u, v \in V(G)\}.$$

Let H_1 and H_2 be disjoint connected graphs, and let u_1 and u_2 be vertices of H_1 and H_2, respectively. Then , the **vertex identified graph** $H_1 \bullet H_2$ is obtained from H_1 and H_2 by identifying the vertices u_1 and u_2. We notice, that

$$D_{\min}(H_1 \bullet H_2) = \min\{D_{\min}(H_1)\ ,\ D_{\min}(H_2)\}\,.$$

Applying this fact , we can easily prove the following proposition, which determines $D_{\min}(G_1 \circ G_2)$.

Proposition 2.3 : For disjoint connected nontrivial graphs G_1 and G_2, we have

$$D_{\min}(G_1 \circ G_2) = \min\{D_{\min}(G_1)\ ,\ D_{\min}(G_2^+)\}\ .$$

3. The Detour Polynomial of $G_1 \circ G_2$:

To determine the detour polynomial of the corona $G_1 \circ G_2$, we introduce the **detour eccentric polynomial** of a connected graph G, defined as follows:

$$D_{ecc}(G;x) = \sum_{v \in V(G)} x^{e_D(v)}\ .$$

For example:

$$D_{ecc}(C_p;x) = px^{p-1}$$

$$D_{ecc}(P_p;x) = \begin{cases} 2\sum\limits_{i=1}^{\frac{p}{2}} x^{p-i}, & \text{for even } p \\ x^{\frac{p-1}{2}} + 2\sum\limits_{i=1}^{\frac{p-1}{2}} x^{p-i}, & \text{for odd } p. \end{cases}$$

We shall obtain the detour polynomial of the corona $G_1 \circ G_2$ in terms of $D(G_i;x)$ and $D_{ecc}(G_i;x)$, $i = 1, 2$.

We set the following definitions:

$$D(G,V_1;x) = \sum_{i=1}^{p_1} D(G,v_i;x)\,,$$

$$D(G,U^{(i)};x) = \sum_{j=1}^{p_2} D(G,u_j^{(i)};x)\,,$$

$$D(G,U;x) = \sum_{i=1}^{p_1} D(G,U^{(i)};x)\,.$$

Then

$$D(G;x) = \frac{1}{2}\left[D(G,V_1;x) + D(G,U;x)\right].$$

Proposition 3.1:

$$D(G,V_1;x) = 2D(G_1;x) + xp_1 D_{ecc}(G_2;x) + 2xD_{ecc}(G_2;x)D(G_1;x)\,.$$

Proof: From Proposition 2.1, for each $v_i \in V(G_1)$, we have

$$D(G,v_i;x) = \sum_{\substack{j=1 \\ j \neq i}}^{p_1} x^{D(v_i,v_j)} + \sum_{w \in U^{(i)}} x^{D(v_i,w)} + \sum_{\substack{w \in U^{(j)} \\ j \neq i}}^{p_1} x^{D(v_i,w)}\ .$$

Summing over all i=1, 2, …, p_1, we get

$$D(G,V_1;x)=\sum_{\substack{i,j\in V_1\\ i\neq j}} x^{D_{G_1}(v_i,v_j)}+p_1\sum_{w\in V_2} x^{1+e_{G2}(w)}+\sum_{i=1}^{p_1}\sum_{\substack{w\in U^{(j)}\\ i\neq j}} x^{1+D_{G_1}(v_i,v_j)+e_{G2}(u)}$$

$$=2\,D(G_1;x)+p_1xD_{ecc}(G_2;x)+xD_{ecc}(G_2;x).2D(G_1;x)\,.\quad \blacksquare$$

Proposition 3.2:

$$D(G,U;x)=2p_1D(G_2^+;x)-p_1xD_{ecc}(G_2;x)+xD_{ecc}(G_2;x).2D(G_1;x)$$
$$+x^2\left[D_{ecc}(G_2;x)\right]^2.2D(G_1;x)\,.$$

Proof: From Proposition 2.1, for each $u_i^{(j)}$, $i=1,2,...,p_2$ and $j=1,2,...,p_1$, we have

$$D(G,u_i^{(j)};x)=D(G_2^+;u_i^{(j)};x)+\sum_{\substack{k=1\\ k\neq j}}^{p_1} x^{D_{G_1}(v_j,v_k)+1+ecc_D(u_i)}+\sum_{\substack{n=1,\,k=1\\ n\neq j}}^{n=p_1,\,p_2=k} x^{2+ecc_D(u_i)+ecc_D(u_k)+D_{G_1}(v_j,v_n)}\,.$$

Summing over both $j=1,2,...,p_1$ and $i=1,2,...,p_2$, we get , using the above notations and definitions:

$$D(G,U;x)=p_1[2D(G_2^+;x)-xD_{ecc}(G_2;x)]+xD_{ecc}(G_2;x).2D(G_1;x)$$
$$+x^2D_{ecc}(G_2;x).D_{ecc}(G_2;x).2D(G_1;x)\,.$$

Hence, the proof is complete. $\blacksquare$

Theorem 3.3: For connected nontrivial graphs G_1 and G_2, the detour polynomial of the corona graph $G_1\circ G_2$ is

$$D(G_1\circ G_2;x)=[1+xD_{ecc}(G_2;x)]^2.D(G_1;x)+p_1D(G_2^+;x)\,.$$

Proof: From Propositions 3.1 and 3.2, we obtain

$$D(G_1\circ G_2;x)=\frac{1}{2}[2D(G_1;x)+p_1xD_{ecc}(G_2;x)+2xD_{ecc}(G_2;x)D(G_1;x)$$
$$+2p_1D(G_2^+;x)-p_1xD_{ecc}(G_2;x)+2xD_{ecc}(G_2;x).D(G_1;x)$$
$$+2x^2D(G_1;x)[D_{ecc}(G_2;x)]^2]$$
$$=D(G_1;x)+2xD_{ecc}(G_2;x).D(G_1;x)+p_1D(G_2^+;x)$$
$$+x^2[D_{ecc}(G_2;x)]^2.D(G_1;x)$$
$$=[1+xD_{ecc}(G_2;x)]^2.D(G_1;x)+p_1D(G_2^+;x)\,.\quad\blacksquare$$

To obtain the detour index of the corona $G_1\circ G_2$, we give the definition of the **eccentricity sum of a graph**, denoted by $S_{ecc}(G)$:

$$S_{ecc}(G)=\sum_{v\in V(G)} ecc_D(v)\,.$$

Corollary 3.4: The detour index of $G_1\circ G_2$ is given as

$$dd(G_1\circ G_2)=(1+p_2)^2.dd(G_1)+p_1(p_1-1)(p_2+1)[p_2+S_{ecc}(G_2)]+p_1.dd(G_2^+)\,,$$

Proof: Taking the derivative of $D(G_1\circ G_2;x)$ given in Theorem 3.3, with respect to *x*, we get

$$D'(G_1\circ G_2;x)=[1+xD_{ecc}(G_2;x)]^2.D'(G_1;x)\,.$$
$$+2[1+xD_{ecc}(G_2;x)].[D_{ecc}(G_2;x)+xD'_{ecc}(G_2;x)].D(G_1;x)$$
$$+p_1D'(G_2^+;x)$$

Then , putting x = 1 , we get

$$dd(G_1 \circ G_2) = D'(G_1 \circ G_2;1)$$

$$= (1+p_2)^2 dd(G_1) + 2(1+p_2)[p_2 + D'_{ecc}(G_2;1)].\frac{1}{2}p_1(p_1-1) + p_1 dd(G_2^+)$$

Then , simplifying the above expression and noticing that

$$D'_{ecc}(G_2;1) = S_{ecc}(G_2) \ ,$$

we get the required formula of $dd(G_1 \circ G_2)$. ■

Example: Let $G_1 = C_n$ the cycle of order n, and $G_2 = K_m$, the complete graph of order *m*, then

$$D(C_n;x) = \begin{cases} n(x^{n-1} + x^{n-2} + \dots + x^{\frac{n+1}{2}}), & \text{for odd } n, \\ n(x^{n-1} + x^{n-2} + \dots + x^{1+\frac{n}{2}} + \frac{1}{2}x^{\frac{n}{2}}), & \text{for even } n, \end{cases}$$

$$D_{ecc}(K_m;x) = mx^{m-1},$$

and

$$D(G_2^+;x) = D(K_{m+1};x) = \frac{1}{2}m(m+1)x^m .$$

Thus, using Theorem 3.3, we get

$$D(C_n \circ K_m;x) = \frac{1}{2}nm(m+1)x^m$$

$$+(1+mx^m)^2.\begin{cases} n(x^{n-1} + x^{n-2} + \ \dots \ + x^{\frac{n+1}{2}}) \quad , & \textit{for odd } n \\ n(x^{n-1} + x^{n-2} + \ \dots \ + + x^{1+\frac{n}{2}} + \frac{1}{2}x^{\frac{n}{2}}) \quad , & \textit{for even } n \ . \end{cases}$$

REFERENCES

[1] Abdullah, H. O., **Hosoya polynomials of Steiner distance of Some graphs**, Ph.D. Thesis, University of Salahaddin, Erbil, Iraq, 2007.

[2] Ahmed, H. G., **On Wiener Polynomials of n-Distance in Graphs**, M.Sc. Thesis, Dohuk University, Dohuk, Iraq, 2007.

[3] Ali, A. A. and Ali, A. M.,Wiener polynomials of the generalized distance for some special graphs, **Rafiden. J. Comp. Sc. and Maths.**, **3**(2) (2006) 103-120.

[4] Ali, A. A. and Saeed, W. A., Wiener polynomials of the tensor product, **Rafiden. J. Sc., 17**(1) (2006) 67-77.

[5] Amić, D. and Trinajstić, N., On the Detour Matrix, **Croat. Chem. Acta**, **68** (1995) 53-62.

[6] Buckly, F. and Harary, F., **Distance in Graphs**, Addison-Wesley, Redwood, California. USA, 1990.

[7] Chartrand, G.; Escuadro, H. and Zhang, P., Detour distance in graphs, **J. Combin. Math. Combin. Comput.**, **53** (2005) 75-94.

[8] Chartrand, G.; Johns, G. L. and Tian S., Detour distance in graphs, **Annals of Discrete Mathematics**, **55** (1993) 127-136.

[9] Chartrand, G. and Zhang, P., Distance in graphs-taking the long view, **AKCE J. Graphs Combin.**, **1**(1) (2004) 1-13.

[10] Diudea, M.; Katona, G. and Lukovits, I., Detour and Cluj-Detour indices, **Croat. Chem. Acta. , 71**(3) (1998) 459-471.

[11] Frucht , R. and Harary , F. , On the corona of two graphs , **Aequationes Math.** , **4** (1970)322-324 .

[12] Gutman, I., Some properties of the Wiener polynomial, **Graph Theory Notes of New York**, **XXV** (1993), 13-18.

[13] Harary, F., **Graph Theory**, Addison-Wesley, Reading, Mass, 1969.

[14] Hosoya, H., On some counting polynomials in Chemistry, **Discrete Applied Mathematics**, **19** (1988) 239-257.

[15] Lukovits, I., The detour index, **Croat. Chem. Acta**, **69** (1996) 873-882.

[16] Saeed, W. A.,**Wiener polynomials of Graphs**, Ph.D. Thesis, Mosul University, Mosul, Iraq, 1999.

[17] Sagan, B. E., Yeh, Y. N. and Zhang, P., The Wiener polynomial of a graph, **International J. of Quantum Chemistry**, **60** (1999) 959-969.

[18] Santhakumaran, A. P. and Athisayanathan, S., On the connected detour number of a graph, **Journal of Prime Research in Mathematics**, **5** (2009) 149-170.