Numerical Solution for Non-linear Boussinesq System Using the Haar Wavelet Method

Ekhlass S. Al-Rawi **Ahmed F. Qasim**
drekhlass-alrawi@uomosul.edu.iq *ahmednumerical@yahoo.com*
College of Computer Sciences and Mathematics
University of Mosul, Mosul, Iraq

Received on: 02/09/2012 **Accepted on: 30/01/2013**

ABSTRACT

In this paper, an operational matrix of integrations based on the Haar wavelet method is applied for finding the numerical solution of non-linear third-order boussinesq system and the numerical results were compared with the exact solution. The accuracy of the obtained solutions is quite high even if the number of calculation points is small, by increasing the number of collocation points the error of the solution rapidly decreases as shown by solving an example. We have been reduced the boundary conditions in the solution by using the finite differences method with respect to time. Also we have reduced the order of boundary conditions used in the numerical solution by using the boundary condition at x=L instead of the derivatives of order two with respect to space.

Keywords: Boussinesq System, Haar Wavelets Method, Operational Matrix.

الحل العددي لنظام Boussinesq اللاخطي باستخدام طريقة مويجة Haar

اخلاص سعدالله الراوي **احمد فاروق قاسم**
كلية علوم الحاسوب والرياضيات
جامعة الموصل، الموصل، العراق

تاريخ استلام البحث: 2012/09/02 **تاريخ قبول البحث: 2013/01/30**

الملخص

في هذا البحث، تم تطبيق مصفوفة العوامل للتكاملات التي تعتمد على مويجة Haar لإيجاد الحل العددي لنظام Boussinesq غير الخطي من الرتبة الثالثة وقد تم مقارنة النتائج مع الحل المضبوط. إن دقة الحلول التي تم الحصول عليها عالية وان كانت عدد نقاط الشبكة المحسوبة صغيرة وكلما زادت عدد نقاط الشبكة المحسوبة فان الدقة تزداد والخطأ يتناقص وقد تم توضيح ذلك بمثال. لقد تم أيضا تخفيض رتبة الشروط الحدودية المطلوبة في الحل العددي وذلك باستخدام طريقة الفروقات المنتهية بالنسبة للزمن وكذلك تم تخفيض رتبة الشروط الحدودية بالنسبة للبعد وذلك باستخدام الشروط الحدودية عند نهاية الفترة x=L بدلا من استخدام الشروط الحدودية للمشتقة الثانية.

الكلمات المفتاحية: نظام Boussinesq ، طريقة مويجة Haar ، مصفوفة العوامل.

1. Introduction.

As a powerful mathematical tool, Wavelet analysis has been widely used in image digital processing, quantum field theory, numerical analysis and many other fields in the recent years.

Haar wavelets have been applied extensively for signal processing in communications and physics research, and more mathematically focused on differential equations and even non-linear problems. After discrediting the differential equation in a conventional way like the finite difference approximation, wavelets which can be used for algebraic manipulations in the system of equations obtained which may lead to better condition number of the resulting system [10].

Using the operational matrix of an orthogonal function to perform integration for solving, identifying and optimizing a linear dynamic system has several advantages: (i) the method is computer oriented, thus solving higher order differential equation becomes a matter of dimension increasing; (ii) the solution is a multi-resolution type and (iii) the answer is convergent, even the size of increment is very large .

The main characteristic of the operational method is to convert a differential equation into an algebraic one, and the core is the operational matrix for integration. We start with the integral property of the basic orthonormal matrix, $\phi(t)$ by writing the following approximation:

$$\underbrace{\int_0^t\int_0^t\int_0^t \ldots\ldots \int_0^t}_{k} \phi(t)\, \underbrace{dt\ dt\ dt \ldots dt}_{k} \cong Q_\phi^k \phi(t) \quad , \qquad \ldots(1)$$

where $\phi(t)=[\vec{\varphi}_0(t) \quad \vec{\varphi}_1(t) \quad \ldots \quad \vec{\varphi}_{m-1}(t)]^T$ in which the elements $\vec{\varphi}_0(t), \vec{\varphi}_1(t), \ldots, \vec{\varphi}_{m-1}(t)$ are the discrete representation of the basic functions which are orthogonal on the interval [0,1) and Q_ϕ is the operational matrix for integration of $\phi(t)$ [9].

Lepik [6] studied the application of the Haar wavelet transform to solve integral and differential equations, he demonstrated that the Haar wavelet method is a powerful tool for solving different types of integral equations and partial differential equations. AL-Rawi and Qasem [1] found the numerical solution for nonlinear Murray equation by the operational matrices of Haar wavelet method and compared the results of this method with the exact solution, they transformed the nonlinear Murray equation into a linear algebraic equations that can be solved by Gauss-Jordan method. Hariharan and Kannan [5] developed an accurate and efficient Haar transform or Haar wavelet method for some of the well-known non-linear parabolic partial differential equations. The equations include the Nowell-whitehead equation, Cahn-Allen equation, FitzHugh-Nagumo equation, and other equations.

Many authors have studied the solution for non-linear boussinesq systems.

Bona et al [3] derived a four-parameter family of boussinesq systems from the two-dimensional Euler equations for free-surface flow and formulated criteria to help decide which of these equations one might choose in a given modeling situation. The analysis of the systems according to these criteria is initiated. Bona et al. [4] are investigating numerically generalized solitary wave solutions of two coupled Kdv systems of boussinesq type. We present numerical experiments describing the generation and evolution of such waves, their interactions, the resolution of general initial profiles into sequences of such waves, and their stability under small perturbations. Anotonopoulos et al. [2] are studing three initial-boundary-value problems for Bona-Smith family of Boussinesq systems corresponding, respectively, to nonhomogeneous Dirichlet, reflection, and periodic boundary conditions posed at the endpoints of a finite spatial interval, and establish existence and uniqueness of their solutions. He proved that the initial-boundary-value problem with Dirichlet boundary conditions is well posed in appropriate spaces locally in time, while the analogous problems with reflection and periodic boundary conditions are globally well posed under mild restrictions on the initial data. Micu et al. [8] are studing the internal controllability and stabilizability of a family of boussinesq systems. The space of the controllable data for the associated linear system is determined for all values of the four parameters. As an application of this newly established exact controllability, some simple feedback controls are constructed such that the resulting closed-loop systems are exponentially stable.

In this paper, we study the numerical solution for non-linear third-order boussinesq system by the operational matrices of Haar wavelet method and we compare the results of this method with the exact solution.

We organized our paper as follows. In section 2, the Haar wavelet is introduced and an operational matrix is established. Section 3 function approximation is presented. In section 4 Haar wavelets used to solve nonlinear boussinesq system. Section 5 deals with the reducing of the order of boundary conditions used in the numerical solution. Numerical results are presented in section 6. Concluding remarks are given in section 7.

2. Haar wavelet

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval [0,1] by ([5] , [6] and [7]):

$$h_i(x)=\begin{cases} 1 & \dfrac{k}{m}\le x<\dfrac{k+1/2}{m} \\ -1 & \dfrac{k+1/2}{m}\le x<\dfrac{k+1}{m} \\ 0 & otherwise \quad in \quad [0,1) \end{cases} \qquad \ldots(2)$$

Integer $m=2^j$, $(j=0,1,2,....,J)$ indicates the level of the wavelet; k=0,1,2,..,m-1 is the translation parameter. Maximal level of resolution is J. The indix i is calculated according to the formula i=m+k+1; in the case of minimal values. m=1,k=0 we have i=2, the maximal value of i is $i=2M=2^{j+1}$. It is assumed that the value i=1 corresponds to the scaling function for which $h_1=1 \;\; in\ [0,1]$. Let us define the collocation points $x_l=(l-0.5)/2M$, $(l=1,2,...,2M)$ and discredit the Haar function $h_i(x)$; in this way we get the coefficient matrix $H(i,l)=(h_i(x_l))$, which has the dimension 2M*2M.

The operational matrix of integration P, which is a 2M square matrix, is defined by the equation: [6]

$$P_{i,1}(x)=\int_0^{x_l} h_i(x)\,dx \qquad \ldots(3)$$

$$P_{i,v+1}(x)=\int_0^{x} P_{i,v}(x)\,dx \quad , \qquad v=1,2,... \qquad \ldots(4)$$

These integrals can be evaluated using equation (2) are given below:

$$P_{i,1}(x)=\begin{cases} x-\alpha & for\ x\in[\alpha,\beta) \\ \gamma-x & for\ x\in[\beta,\gamma) \\ 0 & elsewhere \end{cases} \qquad \ldots(5)$$

$$P_{i,2}(x)=\begin{cases}\frac{1}{2}(x-\alpha)^2 & for\ x\in[\alpha,\beta)\\ \frac{1}{4m^2}-\frac{1}{2}(\gamma-x)^2 & for\ x\in[\beta,\gamma)\\ \frac{1}{4m^2} & for\ x\in[\gamma,1)\\ 0 & elsewhere\end{cases} \quad \ldots(6)$$

$$P_{i,3}(x)=\begin{cases}\frac{1}{6}(x-\alpha)^3 & for\ x\in[\alpha,\beta)\\ \frac{1}{4m^2}(x-\beta)+\frac{1}{6}(\gamma-x)^3 & for\ x\in[\beta,\gamma)\\ \frac{1}{4m^2}(x-\beta) & for\ x\in[\gamma,1)\\ 0 & elsewhere\end{cases} \quad \ldots(7)$$

We also introduce the following notations:

$$R_{i,\nu}=\int_0^1 P_{i,\nu}(x)\,dx \quad \ldots(8)$$

$$R_{i,1}(x)=P_{i,2}(1)=\begin{cases}\frac{1}{2}(1-\alpha)^2 & for\ x\in[\alpha,\beta)\\ \frac{1}{4m^2}-\frac{1}{2}(\gamma-1)^2 & for\ x\in[\beta,\gamma)\\ \frac{1}{4m^2} & for\ x\in[\gamma,1)\\ 0 & elsewhere\end{cases}$$

3. Function Approximation

Any square integrable function u(x) in the interval [0,1] can be expanded by a Haar series of infinite terms :

$$u(x)=\sum_{i=0}^{\infty}c_i\,h_i(x) \quad , \quad i\in\{0\}\cup N \quad \ldots(9)$$

where the Haar coefficients c_i are determined as follows:

$$c_0=\int_0^1 u(x)h_0(x)dx \qquad , \quad c_n=2^j\int_0^1 u(x)h_i(x)dx$$

$$i=2^j+k,\quad j\ge 0,\quad 0\le k<2^j,\quad x\in[0,1)$$

Such that the following integral square error ε is minimized:

$$\varepsilon=\int_0^1\left[u(x)-\sum_{i=0}^{m-1}c_ih_i(x)\right]^2dx,\quad m=2^j,\quad j\in\{0\}\cup N$$

Usually, the series expansion of (10) contains infinite terms for smooth u(x). If u(x) is piecewise constant by itself, or may be approximation as piecewise constant during each subinterval, then u(x) will be terminated at finite m terms, that is:

$$u(x)=\sum_{i=0}^{m-1} c_i h_i(x)=c_{(m)}^T\, h_{(m)}(x) \qquad \ldots(10)$$

where the coefficients $c_{(m)}^T$ and the Haar function vector $h_{(m)}(x)$ are defined as:

$c_{(m)}^T=[c_0,\quad c_1,\quad \ldots.,\quad c_{m-1}]$, and $h_{(m)}(x)=[h_0(x), h_1(x),\ldots\ldots, h_{m-1}(x)]^T$

where T means transpose. (see [6])

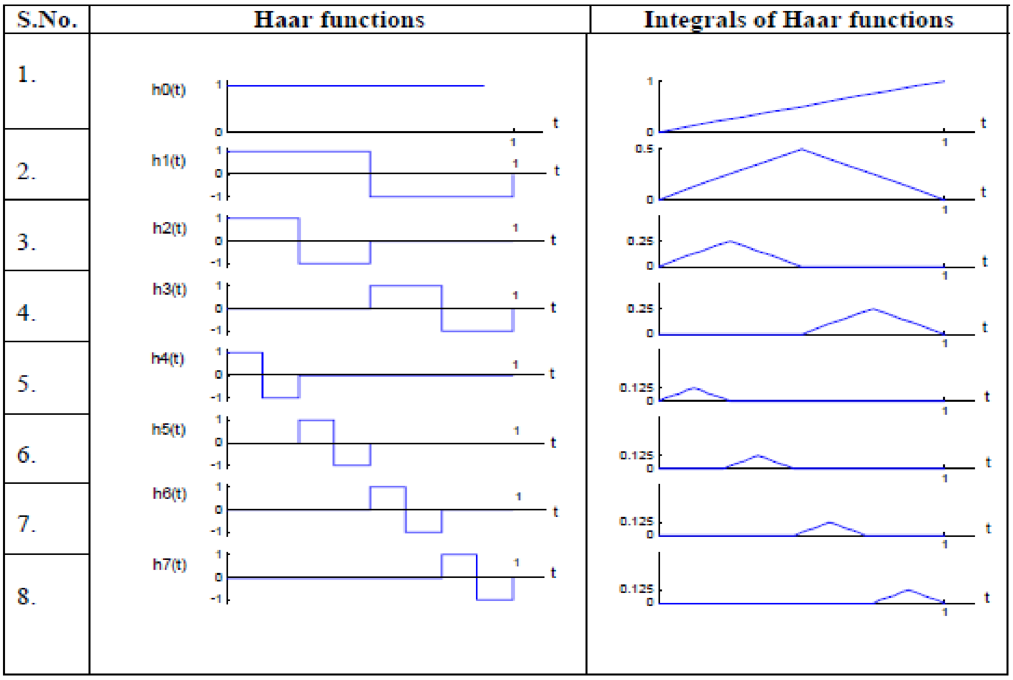

Fig. 1. First eight Haar functions [6]

4 . Mathematical Model

In this section, we will consider the following nonhomogeneous system [8]:

$$\begin{aligned}&\frac{\partial u}{\partial t_*}+\frac{\partial v}{\partial x_*}+\frac{\partial}{\partial x_*}(u\,v)+a\frac{\partial^3 v}{\partial x_*^3}-b\frac{\partial^3 u}{\partial x_*^2\,\partial t_*}=f(x_*,t_*)\\ &\frac{\partial v}{\partial t_*}+\frac{\partial u}{\partial x_*}+v\frac{\partial v}{\partial x_*}+c\frac{\partial^3 u}{\partial x_*^3}-d\,\frac{\partial^3 v}{\partial x_*^2\,\partial t_*}=g(x_*,t_*)\end{aligned} \qquad \ldots(11)$$

where f and g are given functions. For $x_*\in(0,2\pi), t_*\in(0,\mathrm{T})$. With initial conditions [8]:

$$u(x_*,0)=u_o(x_*)\quad v(x_*,0)=v_o(x_*)\quad x_*\in[0,2\pi] \qquad \ldots(12)$$

and periodic boundary conditions :

$$\begin{aligned}&\frac{\partial^r v}{\partial x_*^r}(0,t_*)=\frac{\partial^r v}{\partial x_*^r}(2\pi,t_*)\qquad for\qquad t_*\in(0,T)\quad ,\quad 0\le r\le r_o\\ &\frac{\partial^q u}{\partial x_*^q}(0,t_*)=\frac{\partial^q u}{\partial x_*^q}(2\pi,t_*)\qquad for\qquad t_*\in(0,T)\quad ,\quad 0\le q\le q_o\end{aligned} \qquad \ldots(13)$$

where the parameters a, b, c, and d are constant, but not completely independent . they are in fact required to fulfill the relations :

$$a+b=\frac{1}{2}(\Phi 2-1/3) \qquad c+d=\frac{1}{2}\;(1-\Phi\, 2)\;\ge 0$$

where Φ € [0 ,1]. The number of boundary conditions depends on the values of the parameters . for instance, if a=b=0 then $r_o = 0$, if a=0 and b≠0 then $r_o = 1$ and if a≠0 then $r_o = 2$ the values of q0 depend on the parameters c and d in a similar way .

Since the Haar wavelets are defined for $x \in [0,1]$, we must first normalize system (11) and initial-boundary condition (12) and (13) in regard to x .

We change the variables [7]:

$$x = \frac{1}{2\pi}(x_* - x_{\min}) \quad , \quad t = t_* - t_0 \quad , \quad L = b - a = 2\pi - 0$$

then the system (11) and the initial –boundary conditions (12) become:

$$\begin{aligned} &u_t + \frac{1}{L}v_x + \frac{1}{L}(uv)_x + \frac{a}{L^3}v_{xxx} - \frac{b}{L^2}u_{xxt} = f(x,t) \\ &v_t + \frac{1}{L}u_x + \frac{1}{L}vv_x + \frac{c}{L^3}u_{xxx} - \frac{d}{L^2}v_{xxt} = g(x,t) \end{aligned} \qquad \ldots(14)$$

with initial –boundary conditions:

u(x,0) = u0 (x) , v (x,0) =v0 =(x) …(15)

$$\begin{aligned} &\frac{\partial^r v}{\partial x^r}(0,t) = \frac{\partial^r v}{\partial x^r}(1,t) \qquad for \qquad t \in (0,T) \quad , \quad 0 \le r \le r_o \\ &\frac{\partial^q u}{\partial x^q}(0,t) = \frac{\partial^q u}{\partial x^q}(1,t) \qquad for \qquad t \in (0,T) \quad , \quad 0 \le q \le q_o \end{aligned} \qquad \ldots(16)$$

Now, let us divide the interval (0,T] into N equal parts of length Δt = T /n and denote to ts = (s-1) Δt , s = 1,2,…..n.

We assume that $\dot{u}'''(x,t)$ can be expanded in terms of Haar wavelets as follows:

$$\dot{u}'''(x,t) = \sum_{i=0}^{m-1} c_s(i)\, h_i(x) = C_m^T\, h_m(x) \qquad t \in (t_s, t_{s+1}] \qquad \ldots(17)$$

where the row vector $C_{(m)}^T$ is constant in the subinterval $t \in (t_s, t_{s+1}]$. Integrating (15) with respect to t from t_s to t and third with respect to x from 0 to x , we obtain:

$$u'''(x,t) = (t - t_s)C_m^T\, h_m(x) + u'''(x,t_s) \qquad \ldots(18)$$

$$\begin{aligned} u''(x,t) &= (t - t_s)C_m^T\, P_{i,1}(x) + [u''(x,t_s) - u''(0,t_s)] \\ &\quad + u''(0,t\ ) \end{aligned} \qquad \ldots(19)$$

$$\begin{aligned} u'(x,t) &= (t - t_s)\,C_m^T\, P_{i,2}(x) + [u'(x,t_s) - u'(0,t_s)] \\ &\quad + x[u''(0,t\ ) - u''(0,t_s)] + u'(0,t\ ) \end{aligned} \qquad \ldots(20)$$

$$\begin{aligned} u(x,t) &= (t - t_s)\,C_m^T\, P_{i,3}(x) + [u\ \ (x,t_s) - u\ \ (0,t_s) - x\,u'(0,t_s)] \\ &\quad + \frac{x^2}{2}[u''(0,t\ ) - u''(0,t_s)] + x\,u'(0,t\ ) + u(0,t) \end{aligned} \qquad \ldots(21)$$

Now, the differential of equation (21) with respect t , we get:

$$\dot{u}(x,t) = C_m^T\, P_{i,3}(x) + \frac{x^2}{2}\dot{u}''(0,t\ ) + x\dot{u}''(0,t\ ) + \dot{u}(0,t) \qquad \ldots(22)$$

We can use the backward finite different formula in equation (22) , we get :

$$\dot{u}(x,t) = C_m^T\, p_{i,3}(x) + \frac{x^2}{2}\left[\frac{u''(0,t) - u''(0,t_s)}{\Delta t}\right] + \ x\left[\frac{u'(0,t) - u'(0,t_s)}{\Delta t}\right] + \left[\frac{u(0,t) - u(0,t_s)}{\Delta t}\right] \qquad \ldots(23)$$

Now, the integrating equation (17) with respect to x from 0 to x , we get :

$$\dot{u}''(x,t) = C_m^T p_{i,1}(x) + \dot{u}''(0,t)$$

then, using the backward finite difference formula :

$$\dot{u}''(x,t) = C_m^T p_{i,1}(x) + [\frac{u''(0,t) - u''(0,t_s)}{\Delta t}] \quad \ldots(24)$$

also ,

$$\dot{u}'(x,t) = C_m^T p_{i,2}(x) + x\left[\frac{u''(0,t) - u''(0,t_s)}{\Delta t}\right] + \left[\frac{u'(0,t) - u'(0,t_s)}{\Delta t}\right] \quad \ldots(25)$$

Similarly, we assume that $\dot{v}'''(x,t)$ can be expanded in terms of Haar wavelets as follows:

$$\dot{v}'''(x,t) = \sum_{n=0}^{m-1} d_s(n) h_m(x) = D_m^T h_m(x) \quad ...(26)$$

$$v'''(x,t) = (t - t_s) D_m^T h_m(x) + v'''(x,t_s) \quad ...(27)$$

$$v''(x,t) = (t - t_s) D_m^T p_{i,1}(x) + [v''(x,t_s) - v''(0,t_s)] + v''(0,t) \quad ...(28)$$

$$v'(x,t) = (t - t_s) D_m^T p_{i,2}(x) + [v'(x,t_s) - v'(0,t_s)] + x[v''(0,t) - v''(0,t_s)] + v'(0,t) \quad \ldots(29)$$

$$v(x,t) = (t - t_s) D_m^T p_{i,3}(x) + [v(x,t) - v(0,t_s)] - x\, v'(0,t_s) + \frac{x^2}{2}[v''(0,t) - v''(0,t_s)] + + xv'(o,t) + v(0,t) \quad ...(30)$$

$$\dot{v}(x,t) = D_m^T p_{i,3}(x) + \frac{x^2}{2}[\frac{v''(0,t) - v''(o,t_s)}{\Delta t}] + x[\frac{v'(0,t_s) - v'(0,t_s)}{\Delta t}] + [\frac{v(0,t) - v(0,t_s)}{\Delta t}] \quad \ldots(31)$$

$$\dot{v}'(x,t) = D_m^T p_{i,2}(x) + x[\frac{v''(0,t) - v''(0,t_s)}{\Delta t}] + [\frac{v'(0,t) - v'(0,t_s)}{\Delta t}] \quad ...(32)$$

$$\dot{v}''(x,t) = D_m^T p_{i,1}(x) + [\frac{v''(0,t) - v''(o,t_s)}{\Delta t}] \quad \ldots(33)$$

Now, we can substitute the equations (18)–(25) respect to u(x,t) and the equations (27) – (33) respect to v(x,t) in the system (14) , we get :

$$\begin{aligned}
&C_m^T p_{i,3}(x) + \frac{x^2}{2\Delta t}[u''(0,t) - u''(0,t_s)] + \frac{x}{\Delta t}[u'(0,t) - u'(0,t_s)] + \frac{1}{\Delta t}[u(0,t) - u(0,t_s)] + \\
&\frac{1}{L}\Delta t D_m^T p_{i,2}(x) + \frac{1}{L}[v'(x,t_s)] + \frac{x}{L}[v''(0,t) - v''(0,t_s)] + \frac{1}{L}[v'(0,t) - v'(0,t_s)] + \\
&\frac{a}{L^3}\Delta t D_m^T h_m(x) + \frac{a}{L^3} v'''(x,t_s) - \frac{b}{L^2} C_m^T p_{i,1}(x) - \frac{b}{L^2 \Delta t}[u''(0,t) - u''(0,t_s))] = f(x,t\ ) \\
&-\frac{1}{L}[u(x,t_s)v'(x,t_s) + u'(x,t_s)v(x,t_s)] \quad \ldots(34a)
\end{aligned}$$

$$D_m^T p_{i,3}(x)+\frac{x^2}{2\Delta t}[v''(0,t)-v''(0,t_s)]+\frac{x}{\Delta t}[v'(0,t)-v'(0,t_s)]+\frac{1}{\Delta t}[v(0,t)-v(0,t_s)]+$$
$$\frac{1}{L}\Delta t C_m^T p_{I,2}(x)+\frac{1}{L}u'(x,t_s)+\frac{x}{L}[u''(0,t)-u''(0,t_s)]+\frac{1}{L}[u'(0,t)-u'(0,t_s)]+$$
$$\frac{c}{L^3}\Delta t C_m^T h_m(x)+\frac{c}{L^3}u'''(x,t_s)-\frac{d}{L^2}D_m^T p_{i,1}(x)-\frac{d}{L^2\Delta t}[v''(0,t)-v''(0,t_s)]$$
$$=g(x,t)-\frac{1}{L}[v(x,t_s)v'(x,t_s)] \qquad \ldots(34b)$$

Then

$$C_m^T p_{i,3}(x)-\frac{b}{L^2}C_m^T p_{i,1}(x)+\frac{\Delta t}{L}D_m^T p_{i,2}(x)+\frac{a\Delta t}{L^3}D_m^T h_m(x)$$
$$=f(x,t)-\frac{1}{L}[u(x,t_s)v'(x,t_s)+u'(x,t_s)v(x,t_s)]-\frac{x^2}{2\Delta t}[u''(0,t)-u''(0,t_s)]-\frac{x}{\Delta t}[u'(0,t)-u'(0,t_s)]$$
$$-\frac{1}{\Delta t}[u(0,t)-u(0,t_s)]-\frac{1}{L}v'(x,t_s)-\frac{x}{L}[v''(0,t)-v''(0,t_s)]-\frac{1}{L}[v'(0,t)-v'(0,t_s)]-\frac{a}{L^3}v'''(x,t_s)$$
$$+\frac{b}{L^2\Delta t}[u''(0,t)-u''(0,t_s)] \qquad \ldots(35a)$$

$$\frac{\Delta t}{L}C_m^T p_{i,2}(x)+\frac{c\Delta t}{L^3}C_m^T h_m(x)+D_m^T p_{i,3}(x)-\frac{d}{L^2}D_m^T p_{i,1}(x)$$
$$=g(x,t)-\frac{1}{L}[v(x,t_s)v'(x,t_s)]-\frac{x^2}{2\Delta t}[v''(0,t)-v''(0,t_s)]-\frac{x}{\Delta t}[v'(0,t)-v'(0,t_s)]$$
$$-\frac{1}{\Delta t}[v(0,t)-v(0,t_s)]-\frac{1}{L}u'(x,t_s)-\frac{x}{L}[u''(0,t)-u''(0,t_s)]-\frac{1}{L}[u'(0,t)-u'(0,t_s)]$$
$$-\frac{c}{L^3}u'''(x,t_s)+\frac{d}{L^2\Delta t}[v''(0,t)-v''(0,t_s)] \qquad \ldots(35b)$$

The function $u(0,t),u'(0,t),u''(0,t),v(0,t),v'(0,t)$ and $v''(0,t)$ are found by using the boundary condition (16) and the functions $u(x,t_s),u'(x,t_s),u'''(x,t_s),v(x,t_s),v'(x,t_s)$ and $v'''(x,t_s)$ are found by using initial conditions (15) in the first step .

The Haar coefficient vectors C_m^T and D_m^T are calculated from the solution of linear system (35).

5. Reducing of the Order of Boundary Conditions:

We can reduce the order of the boundary conditions used in equations (18)-(25) by using the boundary condition at x=1 and notation (8) instead of the derivatives $u''(0,t)$ and $u''(0,t_s)$.

The values of unknown term $u''(0,t)$ and $u''(0,t_s)$ can be calculated by integrating equation (19) from 0 to 1 and is given by:

$$\int_0^1 u''(x,t)=\int_0^1 (t-t_s)C_m^T P_{i,1}(x)\,dx+\int_0^1 u''(x,t_s)\,dx+\int_0^1 [u''(0,t)-u''(0,t_s)]\,dx$$

This implies that

$$u'(1,t)-u'(0,t)=(t-t_s)C_m^T R_{i,1}(x)+[u'(1,t_s)-u'(0,t_s)]+[u''(0,t)-u''(0,t_s)]$$

and

$$[u''(0,t)-u''(0,t_s)]=-(t-t_s)C_m^T R_{i,1}(x)+[u'(1,t)-u'(1,t_s)] -[u'(0,t)-u'(0,t_s)] \quad \ldots(36)$$

such that

$$R_{i,1}=P_{i,2}(1)=\begin{cases}\frac{1}{2}(1-\alpha)^2 & for\ x\in[\alpha,\beta)\\ \frac{1}{4m^2}-\frac{1}{2}(\gamma-1)^2 & for\ x\in[\beta,\gamma)\\ \frac{1}{4m^2} & for\ x\in[\gamma,1)\\ 0 & elsewhere\end{cases} \quad \ldots(37)$$

By substituting equation (36) in equations (18)-(25), we get:

$$u'''(x,t)=(t-t_s)C_m^T h_m(x)+u'''(x,t_s) \quad \ldots(38)$$

$$u''(x,t)=(t-t_s)C_m^T P_{i,1}(x)+u''(x,t_s) +\left[-(t-t_s)C_m^T R_{i,1}(x)+[u'(1,t)-u'(1,t_s)]-[u'(0,t)-u'(0,t_s)]\right] \quad ...(39)$$

$$u'(x,t)=(t-t_s)C_m^T P_{i,2}(x)+u'(x,t_s)+[u'(0,t)-u'(0,t_s)] +x\left[-(t-t_s)C_m^T R_{i,1}(x)+[u'(1,t)-u'(1,t_s)]-[u'(0,t)-u'(0,t_s)]\right] \quad ...(40)$$

$$u(x,t)=(t-t_s)C_m^T P_{i,3}(x)+u(x,t_s)+[u(0,t)-u(0,t_s)] +x[u'(0,t)-u'(0,t_s)] +\frac{x^2}{2}\left[-(t-t_s)C_m^T R_{i,1}(x)+[u'(1,t)-u'(1,t_s)]-[u'(0,t)-u'(0,t_s)]\right] \quad ...(41)$$

Now, the different equation (41) with respect t , we get:

$$\dot u(x,t)=C_m^T P_{i,3}(x)+\frac{x}{(t-t_s)}[u'(0,t)-u'(0,t_s)]+\frac{1}{(t-t_s)}[u(0,t)-u(0,t_s)] +\frac{x^2}{2(t-t_s)}\left[-(t-t_s)C_m^T R_{i,1}(x)+[u'(1,t)-u'(1,t_s)]-[u'(0,t)-u'(0,t_s)]\right] \quad ...(42)$$

$$\dot u'(x,t)=C_m^T p_{i,2}(x)+\frac{x}{\Delta t}\left[-\Delta t C_m^T R_{i,1}(x)+\left(u'(1,t)-u'(1,t_s)\right)-\left(u'(0,t)-u'(0,t_s)\right)\right]+ +\frac{1}{\Delta t}\left[u'(0,t)-u'(0,t_s)\right] \quad ...(43)$$

$$\dot u''(x,t)=C_m^T p_{i,1}(x)+\frac{1}{\Delta t}\left[-\Delta t C_m^T R_{i,1}(x)+\left(u'(1,t)-u'(1,t_s)\right)-\left(u'(0,t)-u'(0,t_s)\right)\right] \quad \ldots(44)$$

Similarly, we can reduce the order of the boundary conditions used in equations (27)-(33) by using the boundary condition at x=1 and notation (8) instead of the derivatives $v''(0,t)$ and $v''(0,t_s)$.

When x = 1 from equation (29), we get :

$$v''(0,t)-v''(0,t_s)=-(t-t_s)D_m^T R_{i,1}(x)+\left[v'(1,t)-v'(1,t_s)\right]-\left[v'(0,t)-v'(0,t_s)\right] \quad \ldots(45)$$

By substituting the equation (45) in equation (27) – (33), we get :-

$$v'''(x,t)=\Delta t D_m^T h_m(x)+v'''(x,t_s) \quad \ldots(46)$$

$$v''(x,t)=\Delta t D_m^T p_{i,1}(x)+v''(x,t_s)+\left[-\Delta t D_m^T R_{i,1}(x)+\left(v'(1,t)-v'(1,t_s)\right)-\left(v'(0,t)-v'(0,t_s)\right)\right] \quad ...(47)$$

$$v'(x,t)=\Delta t D_m^T p_{i,2}(x)+v'(x,t_s)+\left[v'(0,t)-v'(0,t_s)\right]+ \\ +x\left[-\Delta t D_m^T R_{i,1}(x)+\left(v'(1,t)-v'(1,t_s)\right)-\left(v'(0,t)-v'(0,t_s)\right)\right] \quad ...(48)$$

$$v(x,t)=\Delta t D_m^T p_{i,3}(x)+v(x,t_s)+\left[v(0,t)-v(0,t_s)\right]+x\left[v'(0,t)-v'(0,t_s)\right]+ \\ +\frac{x^2}{2}\left[-\Delta t D_m^T R_{i,1}(x)+\left(v'(1,t)-v'(1,t_s)\right)-\left(v'(0,t)-v'(0,t_s)\right)\right] \quad \dots(49)$$

$$\dot v(x,t)=D_m^T p_{i,3}(x)+\frac{x^2}{2\Delta t}\left[-\Delta t D_m^T R_{i,1}(x)+\left(v'(1,t)-v'(1,t_s)\right)-\left(v'(0,t)-v'(0,t_s)\right)\right]+ \\ +\frac{x}{\Delta t}\left[v'(0,t)-v'(0,t_s)\right]+\frac{1}{\Delta t}\left[v(0,t)-v(0,t_s)\right] \quad ...(50)$$

$$\dot v'(x,t)=D_m^T p_{i,2}(x)+\frac{x}{\Delta t}\left[-\Delta t D_m^T R_{i,1}(x)+\left(v'(1,t)-v'(1,t_s)\right)-\left(v'(0,t)-v'(0,t_s)\right)\right]+ \\ +\frac{1}{\Delta t}\left[v'(0,t)-v'(0,t_s)\right] \quad \dots(51)$$

$$\dot v''(x,t)=D_m^T p_{i,1}(x)+\frac{1}{\Delta t}\left[-\Delta t D_m^T R_{i,1}(x)+\left(v'(1,t)-v'(1,t_s)\right)-\left(v'(0,t)-v'(0,t_s)\right)\right] \quad \dots(52)$$

Now, we can substitute the equations (38) – (44) with respect to u(x,t) and the equations (46) – (52) with respect to v(x,t) in the system (14) , we get :

$$C_m^T p_{i,3}(x)-\frac{x^2}{2}C_m^T R_{i,1}(x)-\frac{b}{L^2}C_m^T p_{i,1}(x)+\frac{b}{L^2}C_m^T R_{i,1}(x) \\ +\frac{\Delta t}{L}D_m^T p_{i,2}(x)-\frac{\Delta t x}{L}D_m^T R_{i,1}(x)+\frac{a\Delta t}{L^3}D_m^T h_m(x) \\ = f(x,t)-\frac{1}{L}[u(x,t_s)v'(x,t_s)+u'(x,t_s)v(x,t_s)]-\frac{x^2}{2\Delta t}[u'(1,t)-u'(1,t_s)] \\ +\frac{x^2}{2\Delta t}[u'(0,t)-u'(0,t_s)]-\frac{x}{\Delta t}[u'(0,t)-u'(0,t_s)]-\frac{1}{\Delta t}[u(0,t)-u(0,t_s)] \\ -\frac{1}{L}v'(x,t_s)-\frac{1}{L}[v'(0,t)-v'(0,t_s)]-\frac{x}{L}[v'(1,t)-v'(1,t_s)]+\frac{x}{L}[v'(0,t)-v'(0,t_s)] \\ -\frac{a}{L^3}v'''(x,t_s)+\frac{b}{L^2\Delta t}[u'(1,t)-u'(1,t_s)]-\frac{b}{L^2\Delta t}[(u'(0,t)-u'(0,t_s)] \quad \dots(53a)$$

$$\frac{\Delta t}{L}C_m^T p_{i,2}(x)-\frac{\Delta t x}{L}C_m^T R_{i,1}(x)+\frac{c\Delta t}{L^3}C_m^T h_m(x)+D_m^T p_{i,3}(x) \\ -\frac{x^2}{2}D_m^T R_{i,1}(x)-\frac{d}{L^2}D_m^T p_{i,1}(x)+\frac{d}{L^2}D_m^T R_{i,1}(x) \\ = g(x,t)-\frac{1}{L}\left[v(x,t)v'(x,t_s)\right]-\frac{x^2}{2\Delta t}\left[v'(1,t)-v'(1,t_s)\right]+\frac{x^2}{2\Delta t}\left[v'(0,t)-v'(0,t_s)\right] \\ -\frac{x}{\Delta t}\left[v'(0,t)-v'(0,t_s)\right]-\frac{1}{\Delta t}\left[v(0,t)-v(0,t_s)\right]-\frac{1}{L}u'(x,t_s) \\ -\frac{1}{L}\left[u'(0,t)-u'(0,t_s)\right]-\frac{x}{L}\left[u'(1,t)-u'(1,t_s)\right]+\frac{x}{L}\left[u'(0,t)-u'(0,t_s)\right] \\ -\frac{c}{L^3}u'''(x,t_s)+\frac{d}{L^2\Delta t}\left[v'(1,t)-v'(1,t_s)\right]-\frac{d}{L^2\Delta t}\left[v'(0,t)-v'(0,t_s)\right] \quad \dots(53b)$$

The functions $u(0,t), u(1,t), u'(0,t), u'(1,t), v(0,t), v'(1,t) v'(0,t)$ are found by using the boundary conditions (69), and the functions $u(x,t_s), u'(x,t_s), u'''(0,t_s), v(x,t_s), v'(x,t_s)$ and $u'''(x,t_s)$ are found by using initial conditions (69) in the first step .

The Haar coefficient vectors C_m^T and D_m^T are calculated from the solution of the linear system (53a) and (53b)

6. Numerical Results

In this section, we have solved nonhomogenous Boussinesq system (14) with the initial-boundary conditions (15) and (16) by using two formula:

a-) We have solved system (14) with the initial-boundary conditions (15)-(16) by using the linear system (35a) and (35b) such that [8] :

$$\left.\begin{array}{lll} u(0,t\ ) = v(0,t\ ) = 0 & , & u(1,t\ ) = v(1,t\ ) = 0 \\ \dfrac{\partial u}{\partial x}(0,t\ ) = \dfrac{\partial v}{\partial x}(0,t\ ) = 2\pi\cos(t) & , & \dfrac{\partial u}{\partial x}(1,t\ ) = \dfrac{\partial v}{\partial x}(1,t\ ) = 2\pi\cos(2\pi + t) \\ \dfrac{\partial^2 u}{\partial x^2}(0,t\ ) = \dfrac{\partial^2 v}{\partial x^2}(0,t\ ) = 0 & , & \dfrac{\partial^2 u}{\partial x^2}(1,t\ ) = \dfrac{\partial^2 v}{\partial x^2}(1,t\ ) = 0 \end{array}\right\} \quad \ldots(54)$$

$$t\ \geq 0$$

$$u(x,0) = v(x,0) = \sin(2\pi x) \quad \ldots(55)$$

$$\begin{array}{l} f(x,t) = 2\sin(2\pi x)\cos(2\pi x)\cos^2(t) - (1+b)\sin(2\pi x)\sin(t) + (1-a)\cos(2\pi x)\cos(t) \\ g(x,t) = \sin(2\pi x)\cos(2\pi x)\cos^2(t) - (1+d)\sin(2\pi x)\sin(t) + (1-c)\cos(2\pi x)\cos(t) \end{array} \quad ...(56)$$

This process is started with the initial condition:

$$\begin{array}{ll} u(x,t_s) = v(x,t_s) = \sin(2\pi x) & 0 \leq x \leq 1 \\ u'(x,t_s) = v'(x,t_s) = 2\pi\cos(2\pi x) & 0 \leq x \leq 1 \\ u'''(x,t_s) = v'''(x,t_s) = -(2\pi)^3\cos(2\pi x) & 0 \leq x \leq 1 \end{array} \quad \ldots(57)$$

The exact solution of Boussinesq system (14) is given by:

$$u(x,t) = v(x,t) = \sin(2\pi x)\cos(t)$$

Results of the computer simulation are presented in Tables (1) and (2) where m=16 , where [4]:

$$L = 2\pi\ ,\ a = \frac{-7}{30}, b = \frac{7}{15}, c = \frac{-2}{5}, d = \frac{1}{2}, \Delta t = 10^{-5}, t = 0.001.$$

Table (1) Comparison of the numerical solution and the exact solution of u(x,t) when m=16.

The value x	Wavelet solution of u(x,t)	Exact solution of u(x,t)	Absolute error
0.1963	0.19509022447441	0.19509022447098	3.4362e-012
0.5890	0.55556995549255	0.55556995523451	2.5804e-010
0.9817	0.83146919781950	0.83146919656777	1.2517e-009
1.3744	0.98078479348397	0.98078479001063	3.4733e-009
1.7671	0.98078479745969	0.98078479001063	7.4491e-009
2.1598	0.83146921041976	0.83146919656777	1.3852e-008

2.5525	0.55556997896256	0.55556995523451	2.3728e-008
2.9452	0.19509026338454	0.19509022447098	3.8914e-008
3.3379	-0.1950901616976	-0.1950902244709	6.2773e-008
3.7306	-0.5555698537465	-0.5555699552345	1.0149e-007
4.1233	-0.8314690302700	-0.8314691965677	1.6630e-007
4.5160	-0.9807845125815	-0.9807847900106	2.7743e-007
4.9087	-0.9807843190110	-0.9807847900106	4.7100e-007
5.3014	-0.8314683853672	-0.8314691965677	8.1120e-007
5.6941	-0.5555685433918	-0.5555699552345	1.4118e-006
6.0868	-0.1950877499729	-0.1950902244709	2.4745e-006

Table (2) Comparison of the numerical solution and the exact solution of v(x,t) when m=16.

The value x	Wavelet solution of v(x,t)	Exact solution of v(x,t)	Absolute error
0.1963	0.19509022447438	0.19509022447098	3.4011e-012
0.5890	0.55556995549153	0.55556995523451	2.5702e-010
0.9817	0.83146919781287	0.83146919656777	1.2451e-009
1.3744	0.98078479345550	0.98078479001063	3.4449e-009
1.7671	0.98078479736385	0.98078479001063	7.3532e-009
2.1598	0.83146921014530	0.83146919656777	1.3578e-008
2.5525	0.55556997826015	0.55556995523451	2.3026e-008
2.9452	0.19509026172483	0.19509022447098	3.7254e-008
3.3379	-0.1950901653992	-0.1950902244709	5.9072e-008
3.7306	-0.5555698616600	-0.5555699552345	9.3574e-008
4.1233	-0.8314690466654	-0.8314691965677	1.4990e-007
4.5160	-0.9807845457596	-0.9807847900106	2.4425e-007
4.9087	-0.9807843849577	-0.9807847900106	4.0505e-007
5.3014	-0.8314685146372	-0.8314691965677	6.8193e-007
5.6941	-0.5555687940181	-0.5555699552345	1.1612e-006
6.0868	-0.1950882315789	-0.1950902244709	1.9929e-006

b-) We have solved system (14) with the initial-boundary conditions (15) and (16) by using the linear system (53a) and (53b) such that [8]:

$$\left.\begin{array}{ll} u(0,t)=v(0,t)=0 \quad , & u(1,t)=v(1,t)=0 \\ \dfrac{\partial u}{\partial x}(0,t)=\dfrac{\partial v}{\partial x}(0,t)=2\pi\cos(t) \quad , & \dfrac{\partial u}{\partial x}(1,t)=\dfrac{\partial v}{\partial x}(1,t)=2\pi\cos(2\pi+t) \end{array}\right\} \quad t\geq 0 \qquad \ldots(58)$$

$$u(x,0)=v(x,0)=\sin(2\pi x) \qquad \ldots(59)$$

$$\begin{array}{l} f(x,t)=2\sin(2\pi x)\cos(2\pi x)\cos^2(t)-(1+b)\sin(2\pi x)\sin(t)+(1-a)\cos(2\pi x)\cos(t) \\ g(x,t)=\sin(2\pi x)\cos(2\pi x)\cos^2(t)-(1+d)\sin(2\pi x)\sin(t)+(1-c)\cos(2\pi x)\cos(t) \end{array} \qquad ...(60)$$

This process is started with initial condition:

$$u(x,t_s)=v(x,t_s)=\sin(2\pi x) \qquad 0\leq x\leq 1$$

$$u'(x,t_s)=v'(x,t_s)=2\pi\cos(2\pi x) \qquad 0\leq x\leq 1$$

$$u'''(x,t_s)=v'''(x,t_s)=-(2\pi)^3\cos(2\pi x) \qquad 0\leq x\leq 1$$

Results of the computer simulation are presented in tables (3) and (4) where m=16, where

$$L=2\pi\ ,\ a=\frac{-7}{30},b=\frac{7}{15},c=\frac{-2}{5},d=\frac{1}{2},\Delta t=10^{-5},t=0.001.$$

Table (3) Comparison of the Numerical Solution and the Exact Solution of u(x,t) when m=16.

The value x	Wavelet solution of u(x,t)	Exact solution of u(x,t)	Absolute error
0.1963	0.19509022454205	0.19509022447098	7.1074e-011
0.5890	0.55556995659652	0.55556995523451	1.3620e-009
0.9817	0.83146920004959	0.83146919656777	3.4818e-009
1.3744	0.98078479619490	0.98078479001063	6.1843e-009
1.7671	0.98078479984483	0.98078479001063	9.8342e-009
2.1598	0.83146921097669	0.83146919656777	1.4409e-008
2.5525	0.55556997511001	0.55556995523451	1.9876e-008
2.9452	0.19509025051388	0.19509022447098	2.6043e-008
3.3379	-0.1950901918556	-0.1950902244709	3.2615e-008
3.7306	-0.5555699160344	-0.5555699552345	3.9200e-008
4.1233	-0.8314691511077	-0.8314691965677	4.5460e-008
4.5160	-0.9807847389022	-0.9807847900106	5.1108e-008
4.9087	-0.9807847340725	-0.9807847900106	5.5938e-008
5.3014	-0.8314691367523	-0.8314691965677	5.9815e-008
5.6941	-0.5555698925947	-0.5555699552345	6.2640e-008
6.0868	-0.1950901602334	-0.1950902244709	6.4238e-008

Table (4) Comparison of the Numerical Solution and the Exact Solution of v(x,t) when m=16.

The value x	Wavelet solution of v(x,t)	Exact solution of v(x,t)	Absolute error
0.1963	0.19509022454146	0.19509022447098	7.0485e-011
0.5890	0.55556995659693	0.55556995523451	1.3624e-009
0.9817	0.83146920005372	0.83146919656777	3.4859e-009
1.3744	0.98078479620167	0.98078479001063	6.1910e-009
1.7671	0.98078479984803	0.98078479001063	9.8374e-009
2.1598	0.83146921096912	0.83146919656777	1.4401e-008
2.5525	0.55556997508411	0.55556995523451	1.9850e-008
2.9452	0.19509025046061	0.19509022447098	2.5990e-008
3.3379	-0.1950901919465	-0.1950902244709	3.2524e-008
3.7306	-0.5555699161759	-0.5555699552345	3.9059e-008
4.1233	-0.8314691513148	-0.8314691965677	4.5253e-008
4.5160	-0.9807847391926	-0.9807847900106	5.0818e-008
4.9087	-0.9807847344632	-0.9807847900106	5.5547e-008
5.3014	-0.8314691372570	-0.8314691965677	5.9311e-008
5.6941	-0.5555698932144	-0.5555699552345	6.2020e-008
6.0868	-0.1950901609407	-0.1950902244709	6.3530e-008

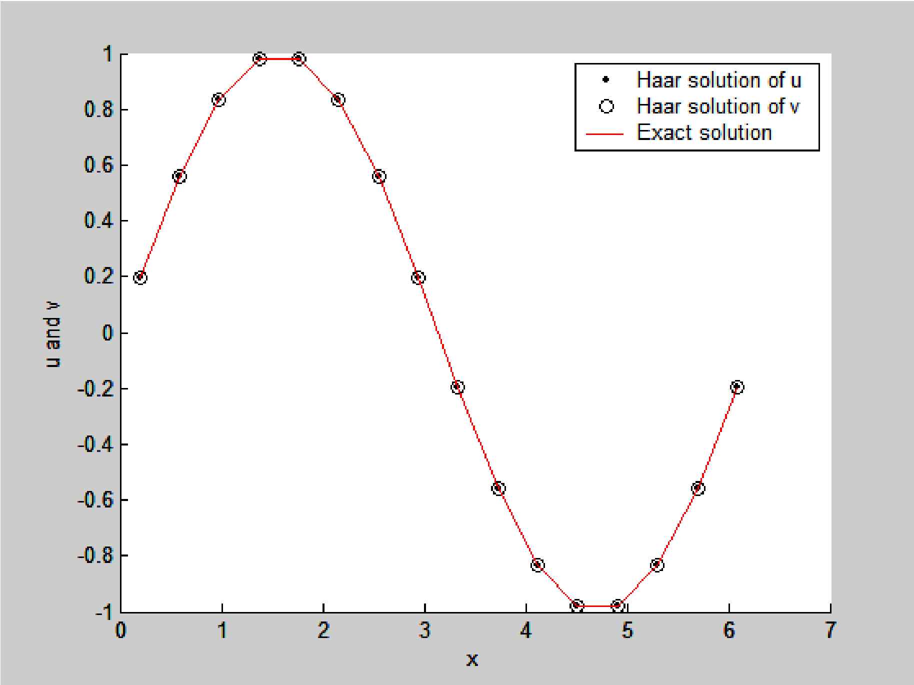

Fig. (2) Comparison of the Numerical Solutions and the Exact Solution When m=16.

Table (5) the mean square error of numerical solution when

$$L = 2\pi\ , a = \frac{-7}{30}, b = \frac{7}{15}, c = \frac{-2}{5}, d = \frac{1}{2},, t = 0.001$$

m	$\lvert u_{ex} - u \rvert$	$\lvert v_{ex} - v \rvert$
8	1.2400e-007	1.2304e-007
16	5.4617e-008	5.4492e-008
32	3.5526e-008	3.5503e-008
64	9.9082e-009	9.9058e-009
128	8.7514e-009	8.7489e-009

7. Conclusions

In this paper, solving the non-linear third-order Boussinesq system by using Haar wavelet method was discussed. The fundamental idea of Haar wavelet method is to convert the third-order Boussinesq system into a group of algebra equations which involves a finite number of variables.

We found that Haar wavelet had good approximation effect by comparing with exact solution of Boussinesq system at the same time. The bigger resolution J is obtained more accurate approximation in the solution, as note in tables (1) and (2) when m=16. Also, when m=32 , m=64 , m=128 , …, we can obtain the results closer to the exact values as noted in table (5).

We have also been reducing the boundary conditions used in the solution by using the finite different method with respect to time and by using the notation (8) when x=L respect to space and the results were of a high resolution as note in tables (3) and (4) and Figure (2). Matlab language is used in finding the results and figure draw, it's a characteristic at high accuracy and large speed.

REFERENCES

[1] Al-Rawi E. S. and Qasem A. F., (2010), “Numerical solution for nonlinear murray equation using the operational matrices of the Haar wavelets method”, Tikrit J. of pure science, Vol. 15, No. 2, Pp. 288-294.

[2] Antonopoulos D. C., Dougalis V. A. and Mitsotakis D. E. , (2009), “Initial-Boundary-Value problems for the Bona-Smith family of Boussinesq systems”, Adv. Differential Equations, Vol. (14), Pp. 27-53.

[3] Bona J. L. , Chen M., Saut J. C., (2002),”Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media I: derivation and linear theory”, J. Nonlinear Science, Vol. 12, Pp. 283-318.

[4] Bona J. L., Dougalis V. A., and Mitsotakis D. E., (2005), “Numerical solution of coupled-kdv systems of Boussinesq equations: II. Generation, interactions and stability of generalized solitary waves”, preprint submitted to Elsevier Science,

[5] Hariharan G. and Kannan K., (2010), “Haar wavelet method for solving some nonlinear parabolic equations”, J. Math. Chem., 48, Pp. 1044-1061.

[6] Lepik U., (2007), “Application of the Haar wavelet transform to solving integral and differential equations”, Proc. Estonian Acad. Sci. Phys. Math., Vol. 56, No. 1, pp.28-46.

[7] Lepik, U., (2007), “Numerical solution of evolution equations by the Haar wavelet method”, Applied Mathematics and Computation, 185, Pp. 695-704.

[8] Micu S., Ortega J. H., Rosier L., and Zhang B. Y., (2009), “Control and stabilization of a family of Boussinesq systems”, Discrete and Continuous Dynamical systems, Vol. 24, N0. 2, Pp.. 273-313.

[9] Wu J. L. and chen C. H., (2003), “A novel numerical method for solving partial differential equations via operational matrices”, proceeding of the 7th world multi conference on systemic, cybernetics and Informatics, Orlando, USA, Vol.5, pp.276-281.

[10] Zhi S., Li-Yuan D. and Qing J. C., (2007), “Numerical solution of differential equations by using Haar wavelets”, Proceeding of the international conference on wavelet analysis and pattern recognition, Beijing, China, Pp. 1039-1044.