On WJCP-Injective Rings

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ABSTRACT

As a generalization of right $JCP-$ injective rings, we introduce the nation of right $WJCP-$ injective rings, that is for any right nonsingular element a of R, there exists a positive integer n and $a^n \neq 0$ and any right R- homomorphism $f: a^nR \to R$, there exists $m \in R$ such that $f(a^nc) = m a^n c$ for all $c \in R$. In this paper, we first introduce and characterize a right $WJCP-$ injective rings . Next , connection between such ring and quasi $\pi-$regular rings and $S-$ weakly regular rings.

Keywords: right $JCP-$ injective rings, right $WJCP-$ injective rings, quasi $\pi-$regular rings , $S-$ weakly regular rings.

الحلقات الغامرة من النمط –WJCP

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الملخص

كتعميم للحلقات الغامرة اليمنى من النمط $-JCP$. سوف نعطي دراسة الحلقات الغامرة اليمنى من النمط $-WJCP$ ، بعبارة أخرى لأي عنصر غير منفرد أيمن a في R يوجد عدد صحيح موجب n و $a^n \neq 0$ وأي تشاكل أيمن f من a^nR إلى R يوجد $m \in R$ بحيث أن $f(a^nc) = m a^n c$ لكل $c \in R$. في هذا البحث سوف نقدم ونصف أولا دراسة تعميم الحلقات الغامرة اليمنى من النمط $-WJCP$. وبالتالي علاقة مثل هذه الحلقات مع الحلقات الأخرى مثل حلقات كوازي المنتظمة من النمط $-\pi$ و الحلقات المنتظمة الضعيفة من النمط $-S$.

الكلمات المفتاحية: الحلقات الغامرة اليمنى من النمط $-JCP$ ، الحلقات الغامرة اليمنى من النمط $-WJCP$، حلقات كوازي المنتظمة من النمط $-\pi$ ، الحلقات المنتظمة الضعيفة من النمط $-S$.

Introduction:

Throughout this paper R denotes an associative ring with identity, and R-modules are unital. For $a \in R$, r(a) and l(a) denote the right annihilator of a and left annihilator of a, respectively. We write J(R),Y(R),Z(R) for the Jacobson radical , the right singular ideal and the left singular ideal, respectively. An element $a \in R$ is called right (left) regular if r(a)=0 (l(a)=0).[10]

A ring R is reduced if $a^2 = 0$ implies $a = 0$ for all $a \in R$, and R is called right C_2- ring if every right ideal T which is isomomorphic to summand of R is a summand [4].

A right $R-$module M is said to be right $YJ-$injective [1], if for any $0 \neq a \in R$ there exists a positive integer n such that $a^n \neq 0$ and any right $R-$homomorphism of a^nR into M extended to one of R into M.

Call a right $R-$module M, $JCP-$injective, if for each $k \notin Y(R)$, any right $R-$homomorphism $kR \to M$ extended to R. Examples of these module include right $YJ-$injective modules. The concept of $JCP-$injective was first introduced by Wei, [7]. As a generalization of this concept [5], introduced $WJCP-$injective as a right $R-$module M , $WJCP-$injective, if for each $a \notin Y(R)$, then there exists a positive integer n such that $a^n \neq 0$ and every right $R-$ homomorphism from a^nR into M can be extended to one of R into M.

A ring R is called strongly $\pi-$regular ring, if for every $a \in R$ there exists a positive integer n such that $a^n = a^{n+1}b$. [2].

A ring R is called $S-$weakly regular ring if for all $a \in R$ then $a \in aRa^2R$ $(a \in Ra^2Ra)$. [3]

2- WJCP-Injective Rings:

In this section , some basic properties of $WJCP-$ injective rings are given ;

Definition 2.1:[5]

A right $R-$module M is said to be $WJCP-$injective if for each $a \notin Y(R)$ there exists a positive integer n such that $a^n \neq 0$ and every right $R-$homomorphism from a^nR into M can be extended to one of R into M. If R_R is $WJCP-$ injective ring, we call R is right $WJCP-$ injective ring.

Clearly, right $YJ-$injective rings are right $WJCP-$ injective. The ring in Example (2.5,[5]) is a right $WJCP-$ injective which is not right $YJ-$injective.

Theorem 2.2:

A ring R is a right $WJCP-$injective if and only if for $a \notin Y(R)$ there exists a positive integer n such that $a^n \neq 0$ and $Ra^n = lr(a^n)$.

Proof:

Suppose that a ring R is right $WJCP-$injective. Then, for every $0 \neq a \notin Y(R)$, there exists a positive integer n such that $a^n \neq 0$ and every right $R-$homomorphism from a^nR into R can be extended to endomorphism of R. It is clear that $Ra^n \subseteq l(r(a^n))$. Let $x \in l(r(a^n))$, and $f : a^nR \to R$ are defined by $f(a^nr) = xr$, then, f is well defined right R- homomorphism because $xr(a^n) = 0$ So $r(a^n) \subseteq r(x)$. Since, R is right $WJCP-$injective, there exists $c \in R$ such that $f(a^n) = ca^n$. Then, $x = f(a^n) = ca^n \in Ra^n$ which implies that $lr(a^n) \subseteq Ra^n$. Consequently, $lr(a^n) = Ra^n$.

Conversely, let $a \notin Y(R)$ there exists a positive integer n such that $Ra^n = lr(a^n)$. Let $f : a^n R \to R$ be any right $R-$ homomorphism. Then, $r(a^n) \subseteq r(f(a^n))$, which implies $f(a^n) \in lr(f(a^n)) \subseteq lr(a^n) = Ra^n$ and therefore $f(a^n) = da^n$ for some $d \in R$, this shows that R is right $WJCP-$injective. #

Example: [7, Example 2.4]

Let V be a two- dimensional vector space over a field F, the trivial extension $R = T(F,V) = F \oplus V$ is commutative, local, artinian ring with $J^2 = 0$ and $J(R) = Y(R)$. Now , if $x \in R$ with $x \notin Y(R)$, then x is invertible. So, $l(r(x^n)) = R = Rx^n$. This implies that R is right $WJCP-$injective. #

Proposition 2.3:

Let R be a right $WJCP-$injective and $Ra^n \subseteq Ra$ for all $a \in R$ and a positive integer n. Then, any right regular element of R is left invertible.

Proof:

Let $a \in R$ and there exists a positive integer n such that $r(a^n) = 0$. Since, R is right $WJCP-$injective ring, then $R = lr(a^n) = Ra^n \subseteq Ra$ by Theorem 2.2 . In particular $ra = 1$ for some $r \in R$. Hence, a is left invertible. #

Wei and Chen [5] proved the following theorem:

Theorem 2.4:

Let R be right $WJCP-$injective ring .Then,

1- $Y(R) \subseteq J(R)$

2- R is a right C_2- ring.

Following [6], a right $R-$module is called $N-$flat if for each $a \in N(R)$ then, the mapping $I_M \otimes i : M \otimes_R Ra \to M \otimes_R R$ is monic,where $i : Ra \to R$ is the inclusion map .

Lemma 2.5:[6]

Let I be a right ideal of R. Then, R/I is $N-$flat right $R-$module if and only if $Ia = I \cap Ra$ for all $a \in N(R)$.

Theorem 2.6:

If R is a right $WJCP-$injective, $l(a) \subseteq r(a)$ for every $a \in R$ and every simple singular right $R-$module is $N-$flat, then $Z(R) = 0$.

Proof:

If $Z \neq 0$, then $0 \neq b \in Z$ such that $b^2 = 0$. We show that $Z + r(b) = R$. Otherwise there exists a maximal right ideal M such that $Z + r(b) \subseteq M$. If M is not an essential right ideal of R, then $M = r(e)$ where $e^2 = e \in R$. If $be \neq 0$, then $beR \cong eR$ as right R module by Theorem 2.4 (R is C_2- ring). $beR = gR$, where $g^2 = g \in R$ so, $g \in Z$ because $beR \subseteq Z$. This is a contradiction. So, $be = 0$. Then $e \in r(b) \subseteq M = r(e)$ which is impossible. Thus, M is an essential right ideal of R, so R/M is $N-$flat, by Lemma 2.5, $b=ab$ for some $a \in M$, so $1-a \in r(b) \subseteq M$ and then, $1 \in M$, which is a contradiction. Hence, $Z+r(b)=R$, let $1=x+y$, $x \in Z$, $y \in r(b)$ then, $b=bx$ and so $b(1-x)=0$ since $x \in Z$ and $l(x) \cap l(1-x)=0$, $l(1-x)=0$, hence $b=0$ which is a contradiction so $Z(R)=0$. #

3- The Connection between $WJCP-$ Injective Rings and other Rings.

In this section, we give the relation between $WJCP-$injective, $S-$weakly regular rings , strongly $\pi-$ regular rings.

Following [7], a ring R is called right quasi regular if $a \in aRa$ for all $a \notin Y(R)$. Now, we give the generalized of quasi regular ring.

Definition 3.1:

A ring R is called right quasi $\pi-$regular rings if $a^n \in a^n R a^n$ for all $a \notin Y(R)$ and a positive integer n. Clearly R is $\pi-$regular ring if and only if R is right non singular and right quasi $\pi-$ regular.

Example:

Let Z_6 be a ring of integers modulo 6, then $Y(R)=\{0\}$ so for all $a \notin Y(R)$, there exists a positive integer n such that $a^n = a^n R a^n$.

Proposition 3.2:

The following conditions are equivalent for a ring R:

1- R is right quasi $\pi-$regular ring.
2- Every $R-$module is $WJCP-$injective.
3- Every cyclic $R-$module is $WJCP-$injective.

Proof:

$1 \to 2$:

Let M be an $R-$module, $a \in R$ with $a \notin Y(R)$ and $f: a^n R \to M$ any right R-homomorphism. Since, R is right quasi $\pi-$ regular rings $a^n = a^n b a^n$ for some $b \in R$. Let $a^n b = e$ and $f(e)=m$, where $m \in M$. Then, $g: R \to M$ is defined by $g(r)=mr$, $r \in R$ is a right R-homomorphism, and $g(a^n r) = m a^n r = f(e) a^n r = f(a^n b) a^n r = f(a^n b a^n) r = f(a^n) r = f(a^n r)$ which implies that M is $WJCP-$injective.

$2 \to 3$:

is trivial.

$3 \to 1$:

Let $a \notin Y(R)$. Since, $a^n R$ is $WJCP-$injective, then the identity map $a^n R \to a^n R$ can be extended to one of R into R. Hence, $a^n = a^n b a^n$ for some $b \in R$. Thus, R is right quasi $\pi-$ regular ring. #

A ring R right weakly principally small injective [5], if for any $0 \neq a \in J(R)$, there exists a positive integer n such that $a^n \neq 0$ and any $R-$homomorphism from $a^n R$ to R_R can be extended to R_R into R_R. Clearly, every right $YJ-$injective is right weakly principally small injective.

The following theorem is a generalization of [7,Theorem 2.9] .

Theorem 3.3:

R is right *YJ*-injective if and only if R is right $WJCP-$injective and right weakly principally small injective.

Proof:

Assume R is $YJ-$injective, then R is right $WJCP-$injective and weakly principally small injective.
Conversely, let R be a right $WJCP-$injective, then by Theorem 2.4, $Y(R)\subseteq J(R)$. Let $a\in R$. If $a\notin Y(R)$, then by Theorem 2.2.,, then $a\notin Y(R)$. If $l(r(a^n))=Ra^n$. Then, $x\in l(r(a^n))$ is clear. Let $Ra^n\subseteq l(r(a^n))$. $Ra^n=l(r(a^n))$ we claim that $a\in J(R)$ is a well defined f. Then, $f(a^n r)=xr$ be defined by $f:a^nR\to R$. Let $r(a^n)\subseteq r(x)$ right $R-$homomorphism. Since R is right weakly principally small injective, there exists a right $R-$homomorphism $g:R\to R$ such that $f(a^n)=g(a^n)$. Hence, $x=f(a^n)=g(a^n)=g(1)a^n\in Ra^n$ and so $l(r(a^n))\subseteq Ra^n$, hence $Ra^n=l(r(a^n))$ therefore, R is $YJ-$injective. #

Lemma 3.4:[8]

If R is $S-$ weakly regular ring if and only if R is reduced weakly regular ring.

Now, we have the following theorem:

Theorem 3.5:

Let R be a ring whose simple singular right $R-$modules are $WJCP-$injective. Then, R is reduced if and only if R is $S-$ weakly regular ring.

Proof:

If R is $S-$weakly regular ring then, R is reduced by Lemma 3.4.
Conversely , assume that R is reduced . For any $0\neq a\in R$, if $Ra^2R+r(a)\neq R$, then there exists a maximal right ideal M of R containing $Ra^2R+r(a)$. If M is not an essential right ideal in R, then $M=r(e)$, $e^2=e\in R$. Therefore, $ea=0$, since R is abelian, $ae=0$ hence, $e\in r(a)\subseteq M=r(e)$, which is a contradiction . So, M is an essential right ideal in R by hypothesis, R/M is $WJCP-$injective. Since, R is reduced, $Y(R)=0$. Hence, there exists a positive integer n such that $a^{2n}\neq 0$ and any right $R-$homomorphism $a^{2n}R\to R/M$ can be extended to $R\to M$. Set $f:a^{2n}R\to R/M$ is defined by $f(a^{2n}x)=x+M$, $x\in R$. Then, f is a well defined right $R-$homomorphism. Hence, there exists $g:R\to R/M$ such that
$1+M=f(a^{2n})=g(a^{2n})=g(1)a^{2n}=ca^{2n}+M$ where, $g(1)=c+M$, so $1-ca^{2n}\in M$. Since, $ca^{2n}\in Ra^2R\subseteq M$, $1\in M$ which is a contradiction. Hence, $Ra^2R+r(a)=R$. In particular, $ca^2d+x=1$, for some $c,d\in R$, $x\in r(a)$, then $a=aca^2d$. Therefore, R is $S-$weakly regular rings. #

Definition 3.6:[9]

R is called right $CAM-$ring, if for any maximal essential right ideal M of R (if it exists) and for any right sub ideal I of M which is either a complement right sub ideal of M or a right annihilator ideal in R, I is an ideal of M.

Show that semi prime right $CAM-$ring, R is either semi simple artinian or reduced

Lemma 3.7:[9]

If R is a semi prime right $CAM-$ring then, R is either semi simple artinian or reduced .

Theorem 3.8:

Let R be a semi prime right $CAM-$ring, quasi duo ring whose simple singular right $R-$modules are $WJCP-$injective. Then R is strongly $\pi-$regular ring .

Proof:

If R is not a semi simple artinian ring then, R is reduced so R is a right non singular ring. Let $0 \neq a \in R$. If $a^nR + r(a^n) \neq R$, then there exists a maximal right ideal M of R such that $a^nR + r(a^n) \subseteq M$. If M is not an essential right ideal of R, then $M = r(e)$ where $e = e^2 \in R$ because R is reduced $ea = ae = 0$ and $e \in r(a) \subseteq M = r(e)$ is contradiction. Hence, M is an essential right ideal of R and so R/M is a singular simple right $R-$module. By hypothesis R/M is right $WJCP-$injective. Then, there exists $c \in R$ such that $1 - ca^n \in M$. But, then $1 \in M$ because R is a quasi duo ring and M is an ideal. It is contradiction. Hence, $a^nR + r(a^n) = R$ and R is a strongly $\pi-$regular ring.#

We conclude the paper with a few characteristic properties of $WJCP-$injective ring.

Proposition 3.9:

Let R be a reduced ring and every left principle ideal is a left annihilator of an element in R. Then, the followings are

1- R is strongly regular.
2- R is right $YJ-$injective.
3- R is $WJCP-$injective.
4- R is simple $WJCP-$injective.
5- R is simple singular $WJCP-$injective.
6- R is $S-$weakly regular ring.

Proof:

$1 \to 2 \to 3 \to 4 \to 5$, $6 \to 1$ are trivial

$5 \to 6$: by Theorem 3.5. #

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