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Smarandache Rings and Smarandache Elements

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ABSTRACT

In this paper, we study some Smarandache (S) notions in some types of rings. Conditions are given under which $\mathbb{Z}_n$ is a Smarandache ring. We study Smarandache ideals, Smarandache subrings and Smarandache weakly Boolean rings. We discuss some types of Smarandache elements in rings. Moreover, we get some other results.

Keywords: Smarandache ring, Smarandache ideal, Smarandache subring, Smarandache weakly Boolean ring, Smarandache SS-element, Smarandache super idempotent and Smarandache semi idempotent.

حلقات السمرنداشية وعناصر السمرنداشية

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الملخص

في هذا البحث ندرس بعض المفاهيم السمرنداشية فى بعض انواع الحلقات. لقد أعطينا الشروط التى تجعل $\mathbb{Z}_n$ حلقة سمرنداشية. و ندرس المثاليات السمرنداشية و الحلقات الجزئية السمرنداشية و الحلقات البوليانية الضعيفة السمرنداشية. كما نناقش بعض العناصر السمرنداشية فى االحلقات. وقد حصلنا على بعض النتائج.

الكلمات المفتاحية: حلقات سمرنداشية، مثاليات سمرنداشية، حلقات جزئية سمرنداشية، حلقات بوليانية ضعيفة سمرنداشية، عناصر -SS سمرنداشية، عناصر القوى سمرنداشية، شبه القوى سمرنداشية.

1. Introduction

Smarandache ring (S-ring) is introduced by F. Smarandache [1], it is defined to be a ring $\mathcal{R}$ (not necessary commutative), such that a proper subset of $\mathcal{R}$ is a field with respect to the operations induced. In [2] Vasantha Kandasamy introduced many Smarandache concepts such as; S-ideal (A Smarandache ideal is defined as an ideal $\mathcal{J}$, such that a proper subset of $\mathcal{J}$ is a field with respect with the same operations induced), S- subring(Let $\mathcal{R}$ be a ring. A proper subset $\mathcal{S}$ of $\mathcal{R}$ is said to be a Smarandache subring of $\mathcal{R}$ if $\mathcal{S}$ has a proper subset F which is a field and $\mathcal{S}$ is a subring of $\mathcal{R}$), weakly Boolean rings (Let $\mathcal{R}$ be a ring. We say $\mathcal{R}$ is a weakly Boolean ring if $\mathrm{x}^{\mathrm{n}(x)} = \mathrm{x}$ for all $x \in \mathcal{R}$ and some natural number $\mathrm{n}(x) > 1$), Smarandache weakly Boolean rings (Let $\mathcal{R}$ be a ring. We say $\mathcal{R}$ is Smarandache weakly Boolean ring, if we have a S-subring $\mathcal{S}$ of $\mathcal{R}$ such that $\mathcal{S}$ is a weakly Boolean ring). In section one of this paper we give conditions under which $\mathbb{Z}_\mathrm{n}$, is a Smarandache ring and we study Smarandache ideals, Smarandache subrings and Smarandache weakly Boolean rings. In section two, we discuss some types of Smarandache elements in rings such as Smarandache SS-element, S-super idempotent and S-semi idempotent elements.

2. Substructures in Smarandache rings

In this section we give conditions under which $\mathbb{Z}_n$ is a Smarandache ring and which answer an open problem given by W. B. Vasantha Kandasamy [2] and we get some other results.

Theorem 2.1

If n has the prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m} q$ (p_i, q are distinct primes), $\alpha_i \geq 1 (1 \leq i \leq m)$, then $\mathbb{Z}_n$ is an S-ring.

Proof: Let $\mathcal{H}$ be the principal ideal of $\mathbb{Z}_n$ generated by $= p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$. This means that $\mathcal{H} = \{0, \gamma, 2\gamma, \dots, (q-1)\gamma\}$. We claim that $\mathcal{H}$ is a field. Suppose there exist $a, b \in \mathcal{H}$ such that $a \neq 0$, $b \neq 0$ and $ab \equiv 0 \pmod{n}$. Since $a, b \in \mathcal{H}$ then $a = r_1 p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ and $b = r_2\ p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ for some $r_1, r_2 \in \{1, 2, \dots, q-1\}$. Thus

$r_1 r_2\ p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_m^{2\alpha_m} \equiv 0 \pmod{n}$, so

$r_1 r_2\ p_1^{2\alpha_1} p_2^{2\alpha_2} \dots p_m^{2\alpha_m} = k\ \ p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m} q$,

for some integer k, thus

$r_1 r_2\ p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m} = k q$, so

$q \mid r_1 r_2\ p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$. But $q \nmid p_i$ $(1 \leq i \leq m)$, since each p_i is a prime number. Hence either $q \mid r_1$ or $q \mid r_2$, which is a contradiction. Therefore $\mathcal{H}$ has no divisors of zero. Since $(\gamma, q) = 1$, the linear congruence $\gamma x \equiv 1 \pmod{q}$, has a unique solution modulo q say x_0 [3]. Now, $\gamma x_0 \in \mathcal{H}$ and $\gamma x_0 = 1 + k q$ for some integer k. Hence for each $\gamma \in \mathcal{H}$ $(1 \leq i \leq q-1)$,

$i\gamma\gamma x_0 = i\gamma\,(1+qk) = i\gamma + ik\gamma q \equiv i\gamma \pmod{\gamma q}$.

Therefore γx_0 is an identity element of $\mathcal{H}$, therefore $\mathcal{H}$ is an integral domain. Consequently it is a field

Example 2.1

Consider $\mathbb{Z}_n$, $n = 180 = 2^2\, 3^2 5$. Using the notations in Theorem 2.1, $\mathcal{H} = \langle 2^2\, 3^2 \rangle$, q = 5, thus $\mathcal{H} = \{ 0, 36, 72, 108, 144 \}$ is a field. Since $5 \mid (36-1)$, hence 36 acts as the identity element. This implies that $\mathbb{Z}_{180}$ is an S-ring.

We remark that $\mathbb{Z}_p$ is not an S-ring where p is prime, since it has no non-trivial ideals.

Theorem 2.2

If $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$, $\alpha_i \geq 2$, p_i are distinct prime numbers $(1 \leq i \leq m)$, then $\mathbb{Z}_n$ is not an S-ring.

Proof: Since $\mathbb{Z}_n$ is a principal ideal ring, every ideal of $\mathbb{Z}_n$ is of the form $\langle p_1^{\beta_1} p_2^{\beta_2} \dots p_m^{\beta_m} \rangle$ where $(0 \leq \beta_i \leq \alpha_i)$ and $(1 \leq i \leq m)$. If $\mathcal{J}$ is an ideal of $\mathbb{Z}_n$, then $\mathcal{J}$ contains an element x of the form $x = p_1\, p_2 \dots p_n$. Put $r = \max\{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Hence $x^r = (p_1\, p_2 \dots p_n)^r = k\, p_1^{\alpha_1} \dots p_m^{\alpha_m}$, for some positive integer k, which implies that $x^r \equiv 0 \pmod{n}$, hence each ideal $\mathcal{J}$ contains a non zero nilpotent element, therefore no ideal of $\mathbb{Z}_n$ is a field

Proposition 2.3

If $\mathcal{R}$ is a Boolean ring different from $\mathbb{Z}_2$, then $\mathcal{R}$ is an S-ring.

Proof: Suppose $\mathcal{R}$ is finite. Since $\mathcal{R} \ncong \mathbb{Z}_2$, then $\mathcal{R} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \dots \oplus \mathbb{Z}_2$ [4]. Consider $\mathcal{F} = \mathbb{Z}_2 \oplus \{0\} \oplus \dots \oplus \{0\}$. $\mathcal{F}$ is a subring of $\mathcal{R}$ which is isomorphic to $\mathbb{Z}_2$. Thus $\mathcal{F}$ is a subfield of $\mathcal{R}$, hence $\mathcal{R}$ is an S-ring.

Now, suppose $\mathcal{R}$ is an infinite Boolean ring and let $0 \neq a \in \mathcal{R}$. Consider $\mathcal{F}= \{0, a\}$. Then $a + a = 0$ and $a^2 = a$, hence a acts as the identity element, this implies that $\mathcal{F}$ is a subfield of $\mathcal{R}$.

Lemma 2.4

Let m and n be positive integers. If m divides n, then there exists a ring homomorphism from $\mathbb{Z}_n$ onto $\mathbb{Z}_m$.

Proof: Define $\phi: (\mathbb{Z}_n, +,\cdot) \longrightarrow (\mathbb{Z}_m, +,\cdot)$ by $\phi(x) = x(\text{mod } m)$. It is easy to show that ϕ is an onto ring homomorphism [5].

Proposition 2.5

If $n = pq$ or $n = p^m$ (p *and* q are distinct primes), $m \geq 2$, then there exists an ideal I of $\mathbb{Z}_n$ such that $\mathbb{Z}_n/I$ is not an S-ring.

Proof: If $n = pq$, then $\mathbb{Z}_n \cong \langle p\rangle \oplus \langle q\rangle$.Consequently, $\mathbb{Z}_n/\langle p\rangle \cong \langle q\rangle$ and $\mathbb{Z}_n/\langle q\rangle \cong \langle p\rangle$. Clearly $|\langle p\rangle| = q$ and $|\langle q\rangle| = p$, which implies that $\langle p\rangle \cong \mathbb{Z}_q$ and $\langle q\rangle \cong \mathbb{Z}_p$. Therefore, Z_n/I is not an S-ring. If $n = p^m$, let $r = p^i$ $(1 \leq i \leq m)$. Then by Lemma 2.4 there exists an onto ring homomorphism $\phi: (\mathbb{Z}_n, +,\cdot) \longrightarrow (\mathbb{Z}_r, +,\cdot)$. Hence $\mathbb{Z}_n/\ker\phi \simeq \mathbb{Z}_{p^i}$ (by fundamental theorem on ring homomorphism) . Therefore $\mathbb{Z}_n/\ker\phi$ is not an S-ring.

Proposition 2.6

If $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_m}^{\alpha_m}$, $\alpha_i \geq 1, m > 1$ and p_i are distinct primes and n is not of the form pq, then Z_n/I is an S-ring for some ideal I of $\mathbb{Z}_n$.

Proof: By Lemma 2.4, there exists an onto ring homomorphism $\phi: (\mathbb{Z}_n, +,\cdot) \longrightarrow (\mathbb{Z}_{p_1 p_2 \dots p_m}, +,\cdot)$. Hence $\mathbb{Z}_n/\ker\phi \simeq \mathbb{Z}_{p_1 \dots p_m}$, consequently, is an S-ring.

Lemma 2.7

The number of ideals of the ring $\mathbb{Z}_n$, where $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \dots {p_m}^{\alpha_m}$ is the prime factorization of n, is equal to $(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_m + 1)$.

Proof: $(\mathbb{Z}_n, +)$ is a cyclic group of order n, hence by [5] for each divisor k of n, $\mathbb{Z}_n$ contains exactly one subgroup of order k and by [3] the number of positive divisors of n is $(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_m + 1)$. But each subgroup of $\mathbb{Z}_n$ is an ideal of the ring $\mathbb{Z}_n$, therefore the number of ideals is $(\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_m + 1)$.

Theorem 2.8

The number of S-ideals of the ring $\mathbb{Z}_n$, where $n = p_1 p_2 \dots p_r$, $r \geq 2$ and $(p_1 < p_2 < \cdots < p_r)$ equal to $\sum_{i=1}^{r}\binom{r}{i} - r$.

Proof: By Theorem 1.1 the number of ideals of $\mathbb{Z}_n$ which are fields is r, namely $\langle p_1 p_2 \dots p_{r-1}\rangle, \langle p_1 p_2 \dots p_{r-2} p_r\rangle, \dots, \langle p_2 p_3 \dots p_r\rangle$. Clearly no ideal of them is S-ideal. The ideal $\langle p_1\rangle$ generated by p_1 is of order $p_2 p_3 \dots p_r$, so it contains the element $p_1 p_2 \dots p_{r-1}$. Hence $\langle p_1 p_2 \dots p_{r-1}\rangle \subset \langle p_1\rangle$, but $\langle p_1 p_2 \dots p_{r-1}\rangle$ is a field, therefore $\langle p_1\rangle$ is a S-ideal. A similar argument shows that all $\langle p_i\rangle$, $(2 \leq i \leq r)$ are S-ideals.

Now, consider $\langle p_1p_2\rangle$, the ideal generated by p_1p_2, its order is $p_3 p_4 \dots p_r$, so it also contains $p_1 p_2 \dots p_{r-1}$, hence $\langle p_1 p_2 \dots p_{r-1}\rangle \subset \langle p_1p_2\rangle$. Therefore $\langle p_1p_2\rangle$ is S-ideal. By similar way if we choose any two distinct primes p_i, p_j from $\{p_1, p_2, \dots, p_r\}$, then $\langle p_ip_j\rangle$ is a S-ideal for $(1 \leq i, j \leq r, i \neq j)$. Continuing in this manner the ideal generated by multiple of $r - 2$ primes from $\{p_1, p_2, \dots, p_r\}$ is S-ideal. Consequently we get that the number of S-ideals is $\sum_{i=1}^{r}\binom{r}{i} - r$.

Note that, by Lemma 2.7, the number of non zero ideals of $\mathbb{Z}_n$ is $2^r - 1 = \sum_{i=1}^{r}\binom{r}{i}$.

Theorem 2.9

Let $\mathcal{R} = \mathrm{F}_1 \oplus \mathrm{F}_2 \oplus \ldots \oplus \mathrm{F}_n$ where $\mathrm{n} \geq 2$ and F_i is a non prime field. Then every non-trivial ideal of $\mathcal{R}$ is an S-ideal.

Proof: Let $\mathcal{J}$ be a non zero proper ideal of the ring $\mathcal{R}$. Then by [6] $\mathcal{J}$ is of the form $\mathcal{J} = \mathrm{I}_1 \oplus \mathrm{I}_2 \oplus \ldots \oplus \mathrm{I}_n$, where I_i is an ideal of F_i $(1 \leq \mathrm{i} \leq \mathrm{n})$. Hence at least one of I_i say $\mathrm{I}_k \neq \{0\}$, then $\mathrm{I}_k = \mathrm{F}_k$, but F_k is not a prime field. If F_k is of characteristic 0 then by [7] F_k contains a proper subfield which is isomorphic to $\mathbb{Q}$, hence $\{0\} \oplus \ldots \oplus \{0\} \oplus \mathbb{Q} \oplus \{0\} \oplus \ldots \oplus \{0\}$ can be considered as a subfield of $\mathcal{J}$. If F_k has a prime characteristic p, then by [7] F_k contains a proper subfield which is isomorphic to $\mathbb{Z}_p$, hence $\{0\} \oplus \ldots \oplus \{0\} \oplus \mathbb{Z}_p \oplus \{0\} \oplus \ldots \oplus \{0\}$ is a subfield of $\mathcal{J}$. Consequently, $\mathcal{J}$ is an S-ideal.

Theorem 2.10

Let $\mathcal{R} = \mathrm{F}_1 \oplus \mathrm{F}_2$, where F_1 and F_2 are prime fields. Then no ideal of $\mathcal{R}$ is an S-ideal.

Proof: Since each F_i is a prime field, $\mathcal{R}$ has only two nonzero proper ideals, namely $\mathcal{J}_1 = \mathrm{F}_1 \oplus \{0\}$ and $\mathcal{J}_2 = \{0\} \oplus \mathrm{F}_2$, and clearly $\mathcal{J}_i$, i = 1, 2 never contains a subfield, hence $\mathcal{J}_i$, i = 1, 2 is not an S-ideal

Theorem 2.11

Let $n = p^m\, q\, r$ $(m \geq 2)$, p, q and r are distinct primes $(p < q < r)$. If $q - 1 | r - 1$, then $\mathbb{Z}_n$ is an S-weakly Boolean ring.

Proof: By Theorem 2.1 $\mathbb{Z}_n$ is an S-ring and $F = \langle p^m\, q\rangle$ is a subfield of $\mathbb{Z}_n$. It is clear that the subring $\mathcal{H} = \langle p^m\rangle$ generated by p^m is a S-subring. We claim that

$(ip^m)^r \equiv ip^m \pmod{n}, (1 \leq i \leq qr)$,

Case1: $(i$, $qr) = 1$ consequently $(ip^m$, $qr) = 1$. Then by [3] $(ip^m)^{r-1} \equiv 1 \pmod{qr}$, which implies that $(ip^m)^{r-1} = 1 + kqr$, for some integer k , hence

$(ip^m)^r = ip^m + ikp^m\, qr$, therefore $(ip^m)^r \equiv ip^m \pmod{n}$.

Case 2: $(i\, , qr) \neq 1$, in this case either $(i\, , q) \neq 1$ or $(i\, , r) \neq 1$,

if $(i\, , r) \neq 1$, then $i = lr$, for some integer $l < q$.

Now, $\frac{i^r(p^m)^r - ip^m}{p^m qr} = \frac{i^r(p^m)^{r-1} - i}{qr} = \frac{(lr)^r(p^m)^{r-1} - lr}{qr} = \frac{l((lrp^m)^{r-1} - 1)}{q}$ … (1).

Since $q \nmid lrp^m$ hence by [3]

$(lrp^m)^{q-1} \equiv 1 \pmod{q}$, then $(lrp^m)^{q-1} - 1 = k_1\, q$,

for some integer k_1, substituting in equation (1) we get $\frac{i^r(p^m)^r - ip^m}{p^m qr} = \frac{k_1 l((lrp^m)^{r-1} - 1)}{((lrp^m)^{q-1} - 1)}$.

But by hypothesis $q - 1 | r - 1$, we obtain $\frac{i^r(p^m)^r - ip^m}{p^m qr} = k_2$,

for some integer k_2 , therefore $(ip^m)^r \equiv ip^m \pmod{n}$.

If $(i\, , q) \neq 1$, then $i = tq$, for some positive $\mathrm{t} < r$. By the same way, we get

$\frac{i^r(p^m)^r - ip^m}{p^m qr} = \frac{t\, k_3((tqp^m)^{r-1} - 1)}{((tqp^m)^{r-1} - 1)} = t\, k_3$ .for some integer k_3, which implies that

$(ip^m)^r \equiv ip^m \pmod{n}$. This completes the proof.

Theorem 2.12

The group ring $\mathbb{Z}_2\mathrm{G}$, where $G = \langle \mathrm{g} \mid \mathrm{g}^m = 1 \rangle$ is a cyclic group of an odd order $m > 1$ is an S-weakly Boolean ring .

Proof: Let $\mathcal{S} = \{0\, , 1\, , \mathrm{g} + \mathrm{g}^2 + \cdots + \mathrm{g}^{m-1}\, , 1 + \mathrm{g} + \mathrm{g}^2 + \cdots + \mathrm{g}^{m-1}\}$. It is clear that $(\mathcal{S}, +)$ is an additive group, we must prove that $\mathcal{S}$ is closed under multiplication,

$(\mathrm{g} + \mathrm{g}^2 + \cdots + \mathrm{g}^{m-1})(1 + \mathrm{g} + \mathrm{g}^2 + \cdots + \mathrm{g}^{m-1}) =$

$1 + \mathrm{g} + \cdots + \mathrm{g}^{m-1} + \mathrm{g} + \mathrm{g}^2 + \cdots + \mathrm{g}^{m-1} + 1 + \cdots + \mathrm{g} + \mathrm{g}^2 + \cdots + \mathrm{g}^{m-1} + 1$

$= (m-1) + (m-1)\mathrm{g} + (m-1)\mathrm{g}^2 + \cdots + (m-1)\mathrm{g}^{m-1} = 0$, since $m-1 \equiv 0 \pmod 2$. Hence $\mathcal{S}$ is a subring of $\mathbb{Z}_2$G, but $\{0, 1\} \subset \mathcal{S}$, thus $\mathcal{S}$ is a S-subring. Now

$$(\mathrm{g}+\mathrm{g}^2+\cdots+\mathrm{g}^{\mathrm{m}-1})^2 = \mathrm{g}^2+\mathrm{g}^4+\cdots+\left(\mathrm{g}^{\frac{\mathrm{m}-1}{2}}\right)^2+\left(\mathrm{g}^{\frac{\mathrm{m}-1}{2}+1}\right)^2+\cdots+(\mathrm{g}^{\mathrm{m}-1})^2$$
$$= \mathrm{g}^2+\mathrm{g}^4+\cdots+\mathrm{g}^{\mathrm{m}-1}+\mathrm{g}+\cdots+\mathrm{g}^{\mathrm{m}-2} \text{ and also}$$

$(1+\mathrm{g}+\mathrm{g}^2+\cdots+\mathrm{g}^{\mathrm{m}-1})^2 = 1+\mathrm{g}+\mathrm{g}^2+\cdots+\mathrm{g}^{\mathrm{m}-1}$.

Therefore $\mathbb{Z}_2\mathrm{G}$ is an S-weakly Boolean ring.

3. Smarandache elements in rings

In this section we study some types of Smarandache elements in rings such as Smarandache SS-element, Smarandache super idempotent and Smarandache semi idempotent. Some results about them are obtained in this section.

Definition 3.1 [2]

Let $\mathcal{R}$ be a ring. An element $x \in \mathcal{R}$ is said to be a Smarandache SS $-$ element of $\mathcal{R}$, if there exists $y \in \mathcal{R}\backslash\{x\}$ with $xy = x+y$. An element $\mathrm{a} \in \mathcal{R}\backslash\{0,2\}$ is an SS-element if $a^2 = a+a$.

[Definition 3.2 [2]

Let $\mathcal{R}$ be a ring. If $\mathcal{R}$ has at least one nontrivial Smarandache SS-element we call $\mathcal{R}$ a Smarandache SS $-$ ring.

Definition 3.3 [2]

Let $\mathcal{R}$ be a ring. An element $0 \neq x \in \mathcal{R}$ is a Smarandache idempotent (S-idempotent) of $\mathcal{R}$ if

1) $x^2 = x$.

2) There exists $y \in \mathcal{R}\backslash \{0, 1, x\}$

i) $y^2 = x$ and

ii) $xy = y$ $(yx = y)$ or $yx = x$ $(xy = x)$.

Definition 3.4 [2]

Let $\mathcal{R}$ be a ring. An element $0 \neq \alpha \in \mathcal{R}$ is called a Smarandache super idempotent (S- super idempotent) of $\mathcal{R}$, if $\alpha^2 - \alpha$ is an S-idempotent of $\mathcal{R}$.

Proposition 3.5

If $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\ldots{p_m}^{\alpha_m}$ is the prime factorization of n, not all $\alpha_\mathrm{i} = 1$, and n is not of the form 2^{α_i}, then $\mathbb{Z}_\mathrm{n}$ is Smarandache SS- ring.

Proof: First if $\alpha_\mathrm{i} \neq 1$ for each$(1 \leq i \leq m)$, take

$$x = {p_1}^{\alpha_1-1}{p_2}^{\alpha_2-1}\ldots{p_m}^{\alpha_m-1}, \ y = n-x.$$

Thus $x+y \equiv 0 \pmod n$ and

$$xy = x\,n - n\,{p_1}^{\alpha_1-2}{p_2}^{\alpha_2-2}\ldots{p_m}^{\alpha_m-2} \equiv 0 \pmod n.$$

Therefore $x+y = xy$. Hence x is Smarandache SS-element.

Second if $\alpha_\mathrm{i} = 1$ for some $(1 \leq i \leq m)$, let $\mathcal{A} = \{\alpha_\mathrm{j} : \alpha_\mathrm{j} = 1\}$.

Take $x = \prod_{\alpha_t \notin \mathcal{A}} {p_t}^{\alpha_t-1} \prod_{\alpha_k \in \mathcal{A}} p_{\alpha_k}$, $y = n-x$.

Hence $x+y \equiv 0 \pmod n$ and

$$xy = x\,n - k\,n, \text{ for some integer } k,$$
$$\equiv 0 \pmod n.$$

Hence x is a Smarandache SS-element.

Proposition 3.6

Let $\mathbb{Z}_2$ be the ring of integers modulo 2 and $G = \langle \mathrm{g} \mid \mathrm{g}^\mathrm{n} = 1 \rangle$ be the cyclic group of order n. Then the group ring $\mathbb{Z}_2G$ is a Smarandache SS-ring.

Proof: Let $a = 1+\mathrm{g}$ and $b = 1+\mathrm{g}^{\mathrm{n}-1}$. Then $a+b = \mathrm{g}+\mathrm{g}^{\mathrm{n}-1}$ and $ab = (1+\mathrm{g})(1+\mathrm{g}^{\mathrm{n}-1}) = \mathrm{g}+\mathrm{g}^{\mathrm{n}-1}$, this implies that $x+y = xy$. Hence $\mathbb{Z}_2$G has nontrivial Smarandache SS-element, therefore $\mathbb{Z}_2$G is a Smarandache SS-ring.

When we search for Smarandache SS-elements of $\mathbb{Z}_p$, we form an opinion that $\mathbb{Z}_p$ has the maximum number of pairs of Smarandache SS-elements, by using MATLAB program for testing many p. This program is very important because implementing Smarandache SS-element of $\mathbb{Z}_p$ for large p is difficult, but due to this program we obtain the result in a short period of time. For a prime number of the form $n^2 + n - 1 (n \geq 4)$, we obtain six pairs of Smarandache SS-elements as it is shown in the following theorem.

Theorem 3.7

Let p be a prime of the form $n^2 + n - 1$ $(n \geq 4)$. Then each of the following pairs $\left(\frac{p+1}{2}, p-1\right)$, $\left(3, \frac{p+1}{2} + 1\right)$,$(n+1, n+2)$, $(n+3, n^2+1)$, $(n, n^2-2)$and $(n^2 - 1, n^2)$ is Smarandache SS-elements of $\mathbb{Z}_p$.

Proof: Let $x = \frac{p+1}{2}$ and $y = p - 1$. Thus $xy - x - y = \frac{p+1}{2}(p-1) - \frac{p+1}{2} - (p-1)$ $= \frac{p-3}{2}\, p \equiv 0 \pmod{p}$, since $\frac{p-3}{2} \in \mathbb{Z}^+$.Hence $xy \equiv x + y \pmod{p}$.
Let $x = 3$ and $y = \frac{p+1}{2} + 1$. Then $xy - x - y = 3\frac{p+1}{2} + 3 - 3 - \frac{p+1}{2} - 1 \equiv 0 \pmod{p}$.
This implies that $xy \equiv x + y \pmod{p}$.Let $x = n + 1$ and $y = n + 2$. Then $xy - x - y = n^2 + 3n + 2 - 2n - 3 \equiv 0 \pmod{p}$. Therefore, $xy \equiv x + y \pmod{p}$.
Let $x = n + 3$ and $y = n^2 + 1$.Thus $xy - x - y = n^3 + 3n^2 + n + 3 - n - 3 - n^2 - 1$ $= (n+1)p \equiv 0 \pmod{p}$. Hence $xy \equiv x + y \pmod{p}$.
Let $x = n$ and $y = n^2 - 2$. Thus $xy - x - y = n^3 - 2n - n - n^2 + 2 =$ $(n-2)p \equiv 0 \pmod{p}$. Therefore $xy \equiv x + y \pmod{p}$.Let $x = n^2 - 1$ and $y = n^2$.
Then $xy - x - y = n^4 - n^2 - n^2 + 1 - n^2$

$$= n^2 + n - 1 + n^4 - 4n^2 - n + 2 \equiv 0 \pmod{p}.$$

This implies that $xy \equiv x + y \pmod{p}$.

The following example shows the existence of more than six pairs of Smarandache SS-elements.

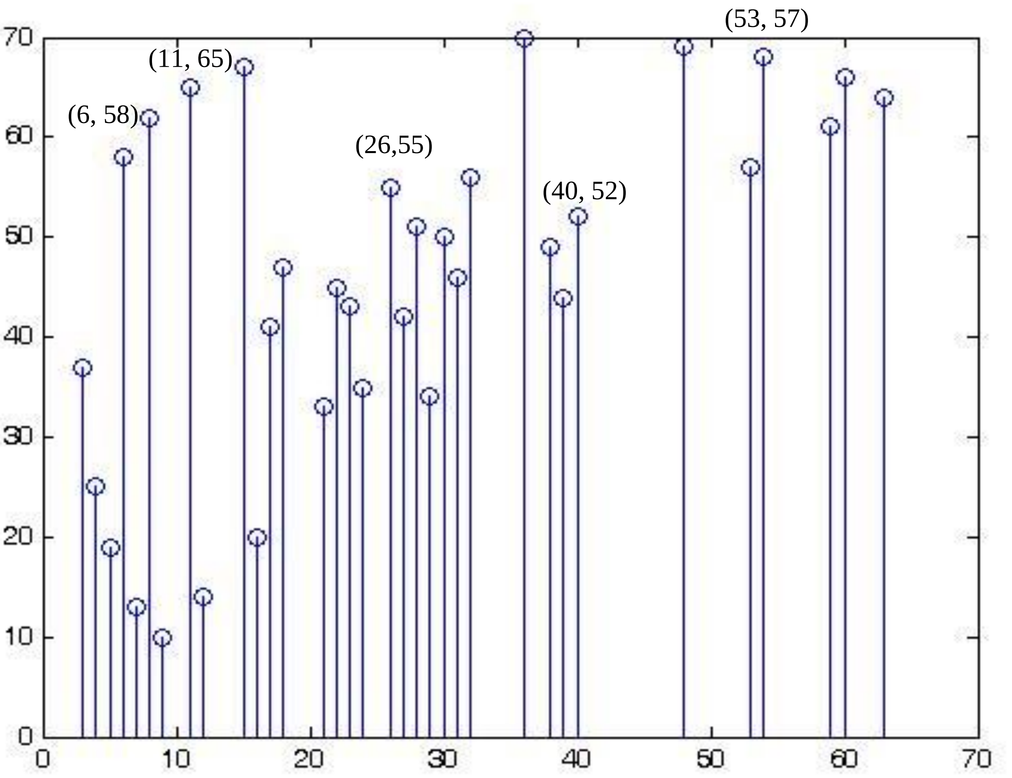
Example 3.1

Consider $\mathbb{Z}_{71}$. We use the following MATLAB program for evaluating the pairs of Smarandache SS-elements, and we plot them.

MATLAB program

```
function g = ss-element(p)
k=0;
for i = 3 :p-2
  for j = i+1:p-1
    if mod(i +j, p)==mod(i∗j, p)
      k = k+1;
      a(k, :)=[i j];
    end
  end
end
a
g1=a(:,1);
g2=a(:,2);
plot(g1,g2)
end
```

In the following figure we plot all Smarandache SS-elements



Hence we obtain 34 pairs of SS-elements, they are
(3, 37), (4, 25),(5, 19), (6, 58), (7, 13), (8, 62), (9, 10), (11, 65), (12, 14), (15, 67), (16, 20), (17, 41), (18, 47), (21, 33), (22, 45), (23, 43), (24, 35), (26, 55), (27, 42), (28, 51), (29, 34), (30, 50), (31, 46), (32, 56), (36, 70), (38, 49), (39, 44), (40, 52), (48, 69), (53, 57), (54, 68), (59, 61), (60, 66) and (63, 64).

Definition 3.8 [2]

Let $\mathcal{R}$ be a ring. An element $\alpha \in \mathcal{R}\setminus\{0\}$ is said to be a Smarandache semi idempotent (S- semi idempotent), if the ideal generated by $(\alpha^2-\alpha)$ that is $\mathcal{R}(\alpha^2-\alpha)\mathcal{R}$ is an S-ideal and $\alpha \notin \mathcal{R}(\alpha^2-\alpha)\mathcal{R}$ or $\mathcal{R} = \mathcal{R}(\alpha^2-\alpha)\mathcal{R}$.

Example 3.2

Let $\mathbb{Z}_{24}$ be the ring of integers modulo 24. Take $\alpha = 5 \in \mathbb{Z}_{24}$, and consider the ideal generated by $\alpha^2-\alpha$. Thus $\langle \alpha^2-\alpha \rangle = \langle 20 \rangle = \{0, 20, 16, 12, 4, 8\} = \mathcal{J}$ is an S-ideal, since $\mathcal{F} = \{0, 8, 16\} \subset \mathcal{J}$ is a field. Hence $5 \in \mathbb{Z}_{24}$ is an S-semi idempotent elements of $\mathbb{Z}_{24}$.

Theorem 3.9

If $\mathcal{R}$ has S-semi idempotent elements, then $\mathcal{R}$ has an S-ideal.

Proof: The proof is an immediate consequence of the definition of S-semi idempotent element.

The converse of this theorem is not true in general, for example, if we take $\mathcal{R} = \mathbb{Z}_{18}$, then $\mathcal{R}$ has an S-ideal but has no S-semi idempotent element.

Theorem 3.10

Let $\mathbb{Z}_n$ be the ring of integers modulo n and $n = p_1^{\alpha_1} \dots p_m^{\alpha_m} q$ be the prime factorization of n not of the form pq. If there exists i $(1 \le i \le m)$ such that $(p_i - 1, n) = 1$, then $\mathbb{Z}_n$ has an S-semi idempotent element.

Proof: Choose p_i such that $(p_i - 1, n) = 1$ and let $\alpha = n - (p_i - 1)$.

Thus $\langle \alpha^2 - \alpha \rangle = \langle (n - (p_i - 1))^2 - n + (p_i - 1) \rangle$

$= \langle {p_i}^2 - p_i \rangle = \langle p_i(p_i - 1) \rangle$.

We will show that $\langle p_i \rangle = \langle p_i(p_i - 1) \rangle$. By [7] the ideal $\langle p_i \rangle$ generated by p_i contains ${p_1}^{\alpha_1}\, {p_2}^{\alpha_2} \ldots {p_{i-1}}^{\alpha_i - 1} \ldots {p_m}^{\alpha_m}\, q$ elements. Also the ideal $\langle\, p_i\,(p_i - 1) \rangle$ generated by $p_i\,(\,p_i - 1\,)$ likewise contains ${p_1}^{\alpha_1} \ldots {p_{i-1}}^{\alpha_i - 1} \ldots {p_m}^{\alpha_m}\, q$ elements, hence $o(\langle p_i \rangle) = o(\langle p_i(p_i - 1) \rangle)$. Then by [5], $\langle p_i \rangle = \langle p_i(p_i - 1) \rangle$. Clearly $\alpha \notin \langle p_i \rangle$ and $\langle {p_1}^{\alpha_1}\, {p_2}^{\alpha_2} \ldots {p_m}^{\alpha_m} \rangle \subset \langle p_i \rangle$, hence $\langle p_i \rangle$ is an S-ideal, therefore $\mathbb{Z}_n$ has S-semi idempotent.

Remark 3.11

A Boolean ring has no S-semi idempotent elements, since $\langle \alpha^2 - \alpha \rangle = \langle 0 \rangle$ is not an S-ideal.

Theorem 3.12

If $\mathbb{Z}_n$ has an S-idempotent element which is the product of two consecutive numbers, then $\mathbb{Z}_n$ has super idempotent.

Proof: Let $k = l(l+1)$ be an S-idempotent. Thus $k^2 = l(l+1)$. Since

$(l+1)^2 - (l+1) = l^2 + l = k$, then $l + 1$ is super idempotent which is a S-idempotent, also $n - l$ is super idempotent since

$(n - l\,)^2 - (n - l\,) = n^2 - 2nl + l^2 - n + l = l(l+1) = k.$

Theorem 3.13

If α is an S- super idempotent of $\mathbb{Z}_n$, then $1 - \alpha$ is super idempotent.

Proof: Suppose α is an S-super idempotent, then $\alpha^2 - \alpha$ is an S-idempotent hence $(\alpha^2 - \alpha\,)^2 = \alpha^2 - \alpha$.

Now, $((1-\alpha)^2 - (1-\alpha)\,)^2 = (1 - 2\alpha + \alpha^2 - 1 + \alpha\,)^2 = (\alpha^2 - \alpha\,)^2 = \alpha^2 - \alpha$.

Therefore $1 - \alpha$ is super idempotent.

Finally, we have the following result.

Theorem 3.14

The group ring $\mathbb{Z}_2 G$, where $G = \langle g \mid g^m = 1 \rangle$ is a cyclic group of an odd order $m > 1$ has no S-super idempotent element.

Proof: The proof is an immediate consequence of [8].

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