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FEKETE-SZEGÖ INEQUALITIES FOR QUASI-SUBORDINATION FUNCTION USING NEW SUBCLASS

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Abstract: This study focuses on deriving Fekete-Szegö inequalities for a specific subclass of univalent functions with complex order, related to quasi-subordination. It investigates the geometric properties of these functions and the conditions for subordination between analytic functions within the open unit disk. The research also discusses generalized starlikeness and convexity as examples of subordination and explores the role of analytic Schwarz functions in this context. New results are provided for superior functions of various orders, contributing to the field of geometric function theory. The findings offer insights into quasi-subordination and its practical applications in pure mathematics, advancing the understanding of function behavior under subordination constraints.

Keywords: Fekete-Szegö inequality, Quasi-subordination, Majorization, Holomorphic functions.

1. Introduction

Let A be the class of functions $f\,(\upsilon)$ which are analytic in $\mathfrak{A} = \{\upsilon \in C : |\upsilon| < 1\}$ (open unit disc), and with the expansion of Taylor defined as follows:

$$f(\upsilon) = \upsilon + \sum_{k=2}^{\infty} a_k \upsilon^k \qquad (\upsilon \in \mathfrak{A}). \qquad (1.1)$$

Additionally, let S be a subclass that is made up of functions that are univalent. For two functions, it can be expressed that function f is subordinated to function g within $\mathfrak{A}$ (i.e., f is majorized by g, respectively). It can be written in $\mathfrak{A}$, if there is a Schwarz function w be analytic in$\mathfrak{A}$ with $w(0)$ equals 0 and such that (really, if an analytic function is existed such that $\upsilon \in \mathfrak{A}$). To be more specific, the subordination requirement is identical in the event that g is univalent in district $\mathfrak{A}$ (see [18]). In the literature, there the concept of subordination has been utilized rather frequently throughout the body of research; for instance, [3, 9, 10, 11, 12,15, 17,18, 19,21] is something that comes to mind. Consider the ideas of generalized starlikeness and convexity as an illustration. Let us define these terms. Consider an analytic function (φ) that has a positive real portion on $\mathfrak{A}$. This function is designed to map $\mathfrak{A}$ onto an area symmetric with respect to the real axis and is starlike with regard to 1. In accordance with [17], in $\mathfrak{A}$, let the functions family symbolized as (φ). This class is the standard class of functions, which is symbolized in a broad sense. In order to maintain the nomenclatur Furthermore, consider S as a subclass consisting of univalent functions.

2. Methods

Suppose w is a Schwarz function and analytic in $\mathfrak{A}$ with $w(0)$ equal 0 such that $\mathcal{v} \in \mathfrak{A}$, there is an analytic function with. In that case, it can be expressed that function f is subordinated to function g within $\mathfrak{A}$ (i.e., f is majorized by g, correspondingly). If function g is univalent in district $\mathfrak{A}$, the subordination criterion remains the same (see to [18] for additional details).

In the literature, the idea of subordination was used quite a bit; for instance, the references [9, 10, 11, 12, 17, 18] have one example. Think about convexity and generalized starlikeness as two examples. Here, we will define the words. We will think about a function φ that is analytic and has a positive real part on $\mathfrak{A}$. This function aims to project $\mathfrak{A}$ onto a real-dimensional surface that is symmetric with respect to the real axis and has a star-like shape with respect to 1. The sign (φ) is used to represent the family of functions in $\mathfrak{A}$, as stated by Ma and Minda [17]. This is the generically represented standard class of functions. ∏(φ) represents the class of functions that are univalent in $\mathfrak{A}$. It should be considered so that we can keep the terminology consistent.

Let the symbol ∗(φ) denotes the class of functions that are univalent in $\mathfrak{A}$.

$$\frac{\mathcal{v} f'(\mathcal{v})}{f(\mathcal{v})} \prec \varphi(\mathcal{v}) \quad (\mathcal{v} \in \mathfrak{A}),$$

C(φ) is a class of functions $f(z) \in$ A so as

$$1 + \frac{\mathcal{v} f''(\mathcal{v})}{f(\mathcal{v})} \prec \varphi(\mathcal{v}) \quad (\mathcal{v} \in \mathfrak{A}),$$

Two kinds of functions are encompassed in the particular cases of C(φ). These functions are called starlike functions of order α (represented as $\delta(\alpha)$ and C(α) respectively) or extremely starlike and convex functions of order (represented as and). The last examples match the criteria for selection.

$\varphi(\mathcal{v}) = \frac{1+(1-2\alpha)\mathcal{v}}{1-\mathcal{v}}$, and $\varphi(\mathcal{v}) = \left(\frac{1+\mathcal{v}}{1-\mathcal{v}}\right)^{\gamma}$ where $\alpha \in [0,1)$, $\gamma \in (0,1]$ (see

Robertson in [21] suggested quasi-subordination notion, a hybrid of majorization and subordination (see, for example [2,6,7,16]). We consider the function f to be quasi-subordinate to the function g if both f and g are analytic. If w and ϕ are analytic functions that satisfy the following conditions: $|\phi(\mathcal{v})|$ is less than or equal to 1, $w(0)$ is equal to zero, and $w(\mathcal{v})$ is less than one, then we may write $f(\mathcal{v})$ as ≺q g, where $\phi(\mathcal{v})$ is generated by $g(w(\mathcal{v}))$. It can be seen that in the set $\mathfrak{A}$, f and g are identical when ϕ(z) equals 1 since g(w($\mathcal{v}$)) equals $f(\mathcal{v})$. Often, g is said to be the most important element in $\mathfrak{A}$. Thus, $f(\mathcal{v})$ equals $\phi(\mathcal{v})g(\mathcal{v})$ when $w(\mathcal{v})$ *equals* $\mathcal{v}$.

The Fekete-Szegö functional $|a_3 - \mu a_2^2|$ which is renowned in the theory of geometric functions, is univalent with for normalized of the type (1.1). Its basis was Fekete and Szego's disproof of Paley's and Littlewood 1932 hypothesis that the estmaites of odd univalent functions have a unity-bounded distribution. Subsequently, the functional has garnered significant interest, especially in several subcategories within the univalent function family (see to, for example, [5, 8, 13, 20, 22, 1]). Please refer to the latest publication by Srivastava et al. [23] for a brief history of the Fekete-Szegö issue for the class of convex, starlike, and close-to-convex functions. We derive the Fekete-Szegö inequalities for functions in the subclass $\Sigma(\alpha, \beta, \varphi)$, in the current paper.

Definition 1.1: - A function $f(\mathcal{v})$ in $\mathcal{A}$ is considered to be in the family $\Sigma(\alpha, \beta, \varphi)$. If

$$(1-\beta)\frac{\mathcal{v} f'(\mathcal{v})}{f(\mathcal{v})} + \frac{\mathcal{v} f'(\mathcal{v}) + \alpha \mathcal{v}^2 f''(\mathcal{v})}{(1-\alpha)f(\mathcal{v}) + \alpha \mathcal{v} f'(\mathcal{v})} \prec_q \varphi(\mathcal{v}) \ (\mathcal{v} \in \mathfrak{A}) \quad (1.2)$$

where $0 \le \alpha \le 1$ *and* $0 \le \beta \le 1$

Lemma 1.2: - ([14])$If\ P(\mathcal{v}) = 1 + b_1\mathcal{v} + b_2\mathcal{v}^2 + \cdots$ has a positive real part, then $|b_1| \le 2$, and for any $\mu \in C$),

$$|b_2 - \mu b_1^2| \le 2\max\{1, |2\mu - 1|\}, \qquad (1.3)$$

For the functions, the result is sharp and written as follows:

$$P(\mathcal{v}) = \frac{1 + \mathcal{v}^2}{1 - \mathcal{v}^2} \quad P(\mathcal{v}) = \frac{1 + \mathcal{v}}{1 - \mathcal{v}}$$

Lemma 1.3: - ($P.198\ in\ [4],$) In open unit disk $\mathfrak{A}$, if the function analytic w satisfies $w(0)\ equals\ 0$ and$|\,w(\mathcal{v})| < 1$, and let $w(\mathcal{v}) = d_1\mathcal{v} + d_2\mathcal{v}^2 + \cdots$. Then $|d_1| \le 1$, and $n \ge 2$ for any natural number.

$$|d_n| \le 1 - |d_1|^2.$$

For the function, the result is sharp and written as follows.

$$w(\mathcal{v}) = \mathcal{v} \quad or \quad w(\mathcal{v}) = \mathcal{v}^n$$

Lemma 1.4: - ([14]) In open unit disk $\mathfrak{A}$, if the function analytic $\emptyset$ satisfies $(|\emptyset(\mathcal{v})|) < 1$, and $\emptyset(\mathcal{v}) = c_0 + c_1\mathcal{v} + c_2\,\mathcal{v}^2 + \cdots$. Then $|c_0| \le 1\ and\ |c_1| \le 1 - |c_0|^2$.

3- Results and Discussion

In this paper, unlike other studies, we assume that f is as the form in (1.1),

$w(\mathcal{v}) = d_1\mathcal{v} + d_2\mathcal{v}^2 + \cdots$, $\emptyset(\mathcal{v}) = c_0 + c_1\mathcal{v} + c_2\,\mathcal{v}^2 + \cdots$,

and $\varphi(\mathcal{v}) = 1 + b_1\mathcal{v} + b_2\mathcal{v}^2 + \cdots$

Theorem 2.1: - Let $0 \le \alpha \le 1\ and\ 0 \le \beta \le 1$. If f(z) $\in \Sigma\,(\alpha,\beta,\varphi)$, then

$$|a_2| \le \frac{b_1}{(2 - \beta + \alpha)} \qquad (2.1)$$

For $\mu \in c$ (any complex number).

$$|a_3 - \mu a_2^2| \le \frac{1}{2 + 4\alpha}\,\{|b_1|(|c_0| + 1) + 2|c_0|\max\left\{1, \left|\frac{2(2 - \beta + 2\alpha + \alpha^2 - \mu(2 + 4\alpha)}{(2 - \beta + \alpha)^2}\;c_0 - 1\right|\right\} \quad (2.2)$$

The result is a sharp.

Proof: Let $f \in \Sigma(\alpha,\beta,\varphi)$. Then, there exists $\emptyset$ and w (analytic function) in $\mathfrak{A}$, with $|\emptyset| \le 1, w(0) = 0\ and\ |w(\mathcal{v})| < 1\ in\ \mathfrak{A}$, and such that

$$(1 - \beta)\frac{\mathcal{v}f'(\mathcal{v})}{f(\mathcal{v})} + \frac{\mathcal{v}f'(\mathcal{v}) + \alpha\mathcal{v}^2 f''(\mathcal{v})}{(1 - \alpha)f(\mathcal{v}) + \alpha\mathcal{v}f'(\mathcal{v})} = \emptyset(\mathcal{v})[(\varphi(w(\mathcal{v})) - 1)] \qquad (2.3)$$

Since

$$(1 - \beta)\frac{\mathcal{v}f'(\mathcal{v})}{f(\mathcal{v})} + \frac{\mathcal{v}f'(\mathcal{v}) + \alpha\mathcal{v}^2 f''(\mathcal{v})}{(1 - \alpha)f(\mathcal{v}) + \alpha\mathcal{v}f'(\mathcal{v})} = 2 + (2 - \beta + \alpha)a_2\mathcal{v} + (\beta a_2^2 - 2a_2^2 - 2\alpha\,a_2^2 - \alpha^2 a_2^2 + 2a_3 + 4\alpha a_3)\mathcal{v}^2 + \cdots$$

and

$$\emptyset(\mathcal{v})[(\,\varphi(w(\mathcal{v}) - 1)] = b_1\,c_0\,d_1\mathcal{v} + (b_1c_1d_1 + c_0(b_1d_2 + b_2d_1^2)) \quad \mathcal{v}^2 + \cdots$$

Then, comparing both sides of (2.3), we see that $a_2 = \frac{b_1\,c_0\,d_1}{2-\beta-1\alpha}$

from which, by the inequalities $|c_0| \le 1\,, |d_1| \le 1$, we immediately obtain (2.1)

$-2a_2^2 + \beta a_2^2 - 2\alpha a_2^2 - \alpha^2\,a_2^2 + 2a_3 + 4\alpha a_3 = b_1c_1d_1 + c_0(b_1d_2 + b_2d_1^2)$

Or equivalently

$$a_3 = \frac{1}{(2 + 4\alpha)}\left[b_1c_1d_1 + c_0(b_1d_2 + b_2d_1^2) + (2 - \beta + 2\alpha + \alpha^2)\frac{{b_1}^2{c_0}^2\,{d_1}^2}{(2 - \beta + \alpha)^2}\right]$$

Further,

$$a_3-\mu a_2^2=\frac{1}{2+4\alpha}\left[b_1c_1d_1+c_0(b_1d_2+b_2d_1^2)+\left(2-\beta+2\alpha+\alpha^2-\mu(2+4\alpha)\right)\frac{{b_1}^2{c_0}^2\,{d_1}^2}{(2-\beta+\alpha)^2}\right] \qquad (2.4)$$

Then applying $|d_n|<1$ *and* $|c_1|\le 1,$ *we have*

$$|a_3-\mu a_2^2|\le\frac{1}{(2+4\alpha)}\left[|b_1|(|c_0|+1)+|c_0||d_1|^2\left|b_2+\frac{(2-\beta+2\alpha+\alpha^2-\mu(2+4\alpha)}{(2-\beta+\alpha)^2}c_0\,{b_1}^2\right|\right]$$

Utilizing lemma (1.2) and the estimate $|d_1|\le 1$, the above can be written as follows:

$$|a_3-\mu a_2^2|\le\frac{1}{(2+4\alpha)}\left[|b_1|(|c_0|+1)+2|c_0|\max\left\{1,\left|2\frac{(2-\beta+2\alpha+\alpha^2-\mu(2+4\alpha)}{(2-\beta+\alpha)^2}c_0-1\right|\right\}\right]$$

The result is sharp for functions.

$$(1-\beta)\frac{\upsilon f'(\upsilon)}{f(\upsilon)}+\frac{\upsilon f'(\upsilon)+\alpha\upsilon^2f''(\upsilon)}{(1-\alpha)f(\upsilon)+\alpha\upsilon f'(\upsilon)}=\emptyset(\upsilon)(\varphi(\upsilon^2)-1), \qquad (2.5)$$

and

$$(1-\beta)\frac{\upsilon f'(\upsilon)}{f(\upsilon)}+\frac{\upsilon f'(\upsilon)+\alpha\upsilon^2f''(\upsilon)}{(1-\alpha)f(\upsilon)+\alpha\upsilon f'(\upsilon)}=\emptyset(\upsilon)(\varphi(\upsilon)-1), \qquad (2.6)$$

The proof of theorem2.1 is completes.

Corollary: - let $0\le\alpha\le 1$ *and* $0\le\beta\le 1$. If $f(\upsilon)\in\Sigma(\alpha,\varphi)$, then

$$|a_2|\le\frac{b_1}{2+\alpha}$$

For $\mu\in c$ (any complex number).

$$|a_3-\mu a_2^2|\le\frac{1}{(2+4\alpha)}\left[|b_1|(|c_0|+1)+2|c_0|\max\left\{1,\left|2\frac{(2+2\alpha+\alpha^2-\mu(2+4\alpha)}{(2+\alpha)^2}c_0-1\right|\right\}\right]$$

Corollary: - let $0\le\alpha\le 1$ *and* $0\le\beta\le 1$. If $f(\upsilon)\in\Sigma(\beta,\varphi)$, then

$$|a_2|\le\frac{b_1}{2-\beta}$$

For $\mu\in C$ (any complex number).

$$|a_3-\mu a_2^2|\le\frac{1}{2}\left[|b_1|(|c_0|+1)+2|c_0|\max\left\{1,\left|2\frac{(2-\beta-2\mu)}{(2-\beta)^2}c_0-1\right|\right\}\right]$$

Corollary: - let $0\le\alpha\le 1$ *and* $0\le\beta\le 1$. If $f(\upsilon)\in\Sigma(1,\beta,\varphi)$ then

$$|a_2|\le\frac{b_1}{3-\beta}$$

and for any complex number $\mu\in c$

$$|a_3-\mu a_2^2|\le\frac{1}{6}\left[|b_1|(|c_0|+1)+2|c_0|\max\left\{1,\left|2\frac{(5-\beta-6\mu)}{(3-\beta)^2}c_0-1\right|\right\}\right]$$

Corollary: - let $0\le\alpha\le 1$ *and* $0\le\beta\le 1$. If $f(\upsilon)\in\Sigma(\alpha,1,\varphi)$, then

$$|a_2|\le\frac{b_1}{1+\alpha}$$

and for any complex number $\mu\in c$

$$|a_3-\mu a_2^2|\le\frac{1}{2+4\alpha}\left[|b_1|(|c_0|+1)+2|c_0|\max\left\{1,\left|2\frac{(1+2\alpha+\alpha^2-\mu(2+4\alpha)}{(1+\alpha)^2}c_0-1\right|\right\}\right]$$

Theorem: - let $0\le\alpha\le 1$ *and* $0\le\beta\le 1$. $f(\upsilon)\in\mathcal{A}$ sotisfies

$$\left\{(1-\beta)\frac{\upsilon f'(\upsilon)}{f(\upsilon)}+\frac{\upsilon f'(\upsilon)+\alpha\upsilon^2f''(\upsilon)}{(1-\alpha)f(\upsilon)+\alpha\upsilon f'(\upsilon)}\right\}-1\ll(\varphi(\upsilon)-1)\ \upsilon\in\mathfrak{A} \quad (2.7)$$

then

$$|a_2| = \frac{b_1}{(2-\beta+\alpha)} \tag{2.8}$$

For μ (any complex number).

$$|a_3 - \mu a_2^2| \le \frac{1}{2+4\alpha}\left[|b_1|(|c_0|+1) + 2|c_0| \max\left\{1, \left|2\frac{(2-\beta+2\alpha+\alpha^2-\mu(2+4\alpha)}{(2-\beta+\alpha)^2}c_0 - 1\right|\right\}\right] \tag{2.9}$$

Proof: - (2.7) is assumed to be hold. From the principle of majorization, $\eta(\mathscr{v})$ (a holomorphic function) is existed, such as:

$$\left\{(1-\beta)\frac{\mathscr{v}f'(\mathscr{v})}{f(\mathscr{v})} + \frac{\mathscr{v}f'(\mathscr{v}) + \alpha\mathscr{v}^2 f''(\mathscr{v})}{(1-\alpha)f(\mathscr{v}) + \alpha\mathscr{v}f'(\mathscr{v})}\right\} - 1 = \eta(\mathscr{v})(\varphi(\mathscr{v}) - 1), \mathscr{v} \in \mathfrak{A} \tag{2.10}$$

Setting $w(\mathscr{v}) \equiv \mathscr{v}$ $(so\ that\ w_1 = 1,\ \ w_n = 0, n \ge 2\ )$, the desired results of (2.8) and (2.9) can be obtained, following the theorem 2.1 proof. By defining $f(\mathscr{v})$, the sharpness can be exhibited as follows:

$$(1-\beta)\frac{\mathscr{v}f'(\mathscr{v})}{f(\mathscr{v})} + \frac{\mathscr{v}f'(\mathscr{v}) + \alpha\mathscr{v}^2 f''(\mathscr{v})}{(1-\alpha)f(\mathscr{v}) + \alpha\mathscr{v}f'(\mathscr{v})} = \varphi(\mathscr{v}), \mathscr{v} \in \mathfrak{A}$$

This completes the proof.

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