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IMPROVING CORRECTED MODIFIED UNBIASED RIDGE REGRESSION ESTIMATOR IN THE LINEAR REGRESSION MODEL WITH CORRELATED ERRORS

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Abstract: This study introduces a novel bias estimator within the linear regression framework, specifically addressing models with correlated or heterogeneous errors. Recognizing the limitations of traditional estimation methods like Ordinary Least Squares (OLS) in the presence of multicollinearity and autocorrelated errors, the proposed estimator provides a robust alternative by leveraging bias corrections. The new estimator, termed Corrected Modified Unbiased Ridge Regression (CMURR), incorporates two parameters, *k* and *d*, to enhance accuracy. The properties of CMURR are derived, including its mean squared error (MSE), which serves as a key metric for performance evaluation. The CMURR estimator is rigorously compared against existing estimators, including the Modified Almost Unbiased Liu Estimator (MAULE), the Modified Almost Unbiased Two-Parameter Estimator (MAUTP), and the Mustafa Unbiased Ridge Regression (MURR). Simulation studies, conducted under varying degrees of multicollinearity and heteroscedasticity, demonstrate the superior performance of CMURR in reducing MSE across diverse conditions. Additionally, a numerical example utilizing real-world data from economic indicators underscores its practical applicability. Results indicate that the proposed estimator consistently outperforms others, particularly in scenarios involving high multicollinearity or correlated errors. The findings contribute to the ongoing advancement of statistical methods in regression analysis, providing a reliable tool for researchers handling complex error structures in data.

Keywords: Bias Estimation, Ridge Regression, Multicollinearity, Mean Squared Error, Heteroscedasticity, Correlated Errors.

1. Introduction

Consider a multiple linear regression model as follows,

$$Y=X\beta+\varepsilon, \qquad (1)$$

where Y is as n×1 vector dependent variable, X is as n×p known of the perdietor variables of p, β is as p×1 vector of the unknown regression coefficients, and ε is as n×1 vector of the errors distributed with E(ε)=0 and Var (ε)=σ^2 I. The Ordinary Least Squares (OLS) estimator is the most widely used method to estimate the unkown parameters in model (1) whivh defined as:

$$\hat{\beta}_{OLS} = (X'X)^{-1} X' Y \quad (2)$$

The OLS is the best when there is no correlation between independent variables in the regression matrix, but when there is a correlation between two or more variables in the regression matrix, the results are poor and unstable. The problem is called multicollinearity, and this problem gets worse as the correlation between the variables increases, because it will cause a large variance in addition to the unstable results. There are many ways to deal with this problem, such as the ordinary ridge regression estimator (ORR) introduced [1].

$$\hat{\beta}(k) = (X'X+kI)^{-1} X' Y \quad (3)$$

Based on the two-parameter estimator, [2] proposed in linear regression model, the modified almost unbiased Liu (MAULE) estimator.

$$\hat{\beta}_{MAULE} = [I-(1-d)^2 (X'X+kI)^{-2}] (X'X+kI)^{-1} X' Y = T_d W_k \hat{\beta}_{OLS}, \quad (4)$$

where $T_d = I-(1-d)^2 (X'X+kI)^{-2}$ and $W_k = (I-k(X'X)^{-1})^{-1}$.

Proposed [3] the Modified Almost Unbiased Two-Parameter (MAUTP) Estimator.

$$\hat{\beta}_{MAUTP} = [I-k^2 (1-d)^2 (X'X+kI)^{-2}] W_k \hat{\beta}_{OLS} = M_{(k,d)} W_k \hat{\beta}_{OLS}, \quad (5)$$

where $M_{(k,d)} = I-k^2 (1-d)^2 (X'X+kI)^{-2}$.

When the covariance of ε in (1) is $Cov(\varepsilon) = \sigma^2 I$, a term for this is called homoscedasticity. i.e $Var(\varepsilon_i) = \sigma^2$ which is uncorrelated i.e. $Cov(\varepsilon_i, \varepsilon_j) = 0$ for $i \neq j$ .Let us now extended our assumptions to include unequal error variances, which are known as heteroscedasticity as:

$$E(\varepsilon) = 0 \; ; \; Cov(\varepsilon) = \sigma^2 V,$$

where V is a well-known symmetric positive definite (p.d.) matrix of $n \times n$ dimensions, and $\sigma^2 > 0$ is an unknown. Since V is symmetric and (p.d) there is a symmetric nonsingular matrix P such that $P'P = V$. As a result, model (1) can be written as:

$$P^{-1} Y = P^{-1} X\beta + P^{-1} \varepsilon$$

Let $Y_* = P^{-1} Y$, $X_* = P^{-1} X$ and $\varepsilon_* = P^{-1} \varepsilon$, then $E(\varepsilon_*) = 0$ and $Cov(\varepsilon_*) = \sigma^2 I$. As a result, model (1) can be rewritten as:

$$Y_* = X_* \beta + \varepsilon_* \quad (6)$$

The assumption of homoscedasticity is met by model (6), i.e. $\varepsilon_* \sim N(0, \sigma^2 I)$. The OLS estimator in (6) is

$$\hat{\beta}_{GLS} = (X_*' X_*)^{-1} X_* Y_* = (X' V^{-1} X)^{-1} X' V^{-1} Y \quad (7)$$

It is called the generalized least squares estimator (GLS). The GLS is unbiased estimator. Since the GLS estimators deals with $(X'X)^{-1}$ The effects of multicollinearity on estimation will be signed. For this reason, researchers likes [5]) suggested ridge regression estimator. [6] introduced an estimator that can perform a Jackknife Ridge estimator (JRE). To reduce the bias of the ridge estimator, the jackknifed ridge estimator has been developed, the jackknifed ridge estimator has less bias than the ridge estimator. [7] proposed almost unbiased Liu (AUL) estimator. [8] studied the performance of the jackknifed modified ridge (JMRE) estimator. Our primary aim here is to introduce an estimator by using the jackknife procedure to the modified ridge estimator and provide an alternative method to overcome the multicollinearity problem. In Section 2, we introduce a new estimator and its properties. Performance against other estimators is given in Section 3. While a simulation study and a numerical example are given in Section 4.

2.Proposed Estimator and its properties.

Proposed[4] two-parameter estimator and defined it as Mustafa unbiased ridge regression (MURR) estimator.

$$\hat{\beta}_{MURR}=F_{(K,d)}\,\hat{\beta}_{OLS}+k^2 d A J, \quad (10)$$

where $F_{(K,d)}=(X' V^{-1} X+kI)^{-1} [kd(X' V^{-1} X+kI)^{-1}+I] X' V^{-1} X)$, $A=(X' V^{-1} X+kI)^{-1} (X' V^{-1} X+kI)^{-1}$ and J is a random vector such that $J \sim N(\beta,\sigma^2/k)$ for $k > 0$, $0 < d < 1$.

We recreate the linear model (1) in its canonical form to facilitate thinking about it. Since $X' X$ is symmetric, there exists a $p\times p$ orthogonal matrix L such that $L' X' XL = \Lambda$, Λ is as $p\times p$ diagonal matrix containing diagonal elements $\lambda_1,\ldots,\lambda_P$ that are the eigenvalues of $X' X$ and $\lambda_1 > \lambda_2 >,\ldots,> \lambda_p$. Consequently, model (1) will appear as follows:

$$Y=Z\alpha+\varepsilon,$$

where $Z = X L$ and $\alpha=L' \beta$. Then we could rewrite the GLS, GORR, GURR and MURR. As follows:

$$\hat{\alpha}_{GLS}=\Lambda^{-1} Z' Y \quad (11)$$

$$\hat{\alpha}_{GORR}=(\Lambda+kI)^{-1} Z' Y \quad (12)$$

$$\hat{\alpha}_{MAULE}=[I-(1-d)^2 (\Lambda+kI)^{-2}] (\Lambda+kI)^{-1} Z' Y\, \hat{\alpha}_{GLS} \quad (13)$$

$$\hat{\alpha}_{MAUTP}=[I-k^2 (1-d)^2 (\Lambda+kI)^{-2}] R_k\, \hat{\alpha}_{GLS} \quad (14)$$

and

$$\hat{\alpha}_{MURR}=F_{(K,d)}\,\hat{\alpha}_{GLS}+k^2 d A J \quad , \quad (15)$$

where $F_{(K,d)}=(\Lambda+kI)^{-1} [kd(\Lambda+kI)^{-1}+I]\Lambda$ and $A=(\Lambda+kI)^{-1} (\Lambda+kI)^{-1}$.

Now, we add the biased quantity to the MURR estimator to replace it with the last estimator, and we can get the new estimator as follows:

$$\hat{\alpha}_{MURR}=F_{(K,d)}\,\hat{\alpha}_{GLS}+k^2 d A J$$

The bias of MURR is given as:

$$Bias(\hat{\alpha}_{MURR})=(F_{(K,d)}+k^2 d A-I)\alpha$$

The idea here is to add t;

$$\hat{\alpha}^{*}=\hat{\alpha}_{MURR}+(F_{(K,d)}+k^2 d A-I)\,\alpha$$

$$\hat{\alpha}^{*}=\hat{\alpha}_{MURR}+(F_{(K,d)}+k^2 d A-I)\,\hat{\alpha}_{MURR}$$

$\hat{\alpha}^{*}=[I+F_{(K,d)}+k^2 d A-I]\,\hat{\alpha}_{MURR}$

$$\hat{\alpha}_{CMURR}=T_{(K,d)}\,\hat{\alpha}_{GLS}+k^2 d M J \qquad (16)$$

where $T_{(K,d)}=(S_k^{-1}[kd\,S_k^{-1}+I]\Lambda)(F_{(K,d)}+k^2 d A)$, $S_k^{-1}=(\Lambda+kI)^{-1}$ and $M=[((\Lambda+kI)^{-1})]^2(F_{(K,d)}+k^2 d A)$. We called corrected Mustafa unbiased ridge regression (CMURR) estimator.

The properties CMURR are given as:

$E[\hat{\alpha}_{CMURR}]=(T_{(K,d)}+k^2 d M)\alpha$; $B[\hat{\alpha}_{CMURR}]=[T_{(K,d)}+k^2 d M-I]\alpha$;

$$Var[\hat{\alpha}_{CMURR}]=\sigma^2[T_{(K,d)}\Lambda^{-1}T_{(K,d)}'+k^3 d^2 MM'] .$$

Therefore,

$MSE[\hat{\alpha}_{CMURR}]=\sigma^2[T_{(K,d)}\Lambda^{-1}T_{(K,d)}'+k^3 d^2 MM']+[T_{(K,d)}+k^2 d M-I]\alpha\alpha'[T_{(K,d)}+k^2 d M-I]'$ (17)

The properties MAULE are given as:

$E[\hat{\alpha}_{MAULE}]=T_d W_K\alpha$; $Bias[\hat{\alpha}_{MAULE}]=(T_d W_K-I)\alpha$; $Var[\hat{\alpha}_{MAULE}]=\sigma^2 T_d W_K\Lambda^{-1}T_d' W_K'$,
and

$MSE[\hat{\alpha}_{MAULE}]=\sigma^2 T_d W_K\Lambda^{-1}T_d' W_K'+(T_d W_K-I)\alpha\alpha'(T_d W_K-I)'$ (18)

The properties MAUTP are given as:

$E[\hat{\alpha}_{MAUTP}]=M_{(k,d)}W_K\alpha$; $Bias[\hat{\alpha}_{MAUTP}]=(M_{(k,d)}W_K-I)\alpha$;

$Var[\hat{\alpha}_{MAUTP}]=\sigma^2 M_{(k,d)}W_K\Lambda^{-1}M_{(k,d)}' W_K'$ and

$MSE[\hat{\alpha}_{MAUTP}]=\sigma^2 M_{(k,d)}W_K\Lambda^{-1}M_{(k,d)}' W_K'+(M_{(k,d)}W_K-I)\alpha\alpha'(M_{(k,d)}W_K-I)'$ (19)

The properties MURR are given as:

$E[\hat{\alpha}_{MURR}]=(F_{(K,d)}+k^2 d A)\alpha$; $Bias[\hat{\alpha}_{MURR}]=[F_{(K,d)}+k^2 d A-I]\alpha$;

$Var[\hat{\alpha}_{MURR}]=\sigma^2[F_{(K,d)}\Lambda^{-1}F_{(K,d)}'+k^3 d^2 AA']$ and then,

$MSE[\hat{\alpha}_{MURR}]=\sigma^2[F_{(K,d)}\Lambda^{-1}F_{(K,d)}'+k^3 d^2 AA']+[F_{(K,d)}+k^2 d A-I]\alpha\alpha'[F_{(K,d)}+k^2 d A-I]'$ (20)

We can rewrite it in the canonical form to make it simpler to think about with the linear model.

$$\hat{\alpha}_{GLS}=\Gamma^{-1}W'Y \qquad (21)$$

So,

$\hat{\alpha}_{GLS}=(W'W)^{-1}W'Y=\Gamma^{-1}PX'V^{-1}Y$,

where $W=XP=T^{-1}XP$, P is an orthogonal matrix, which means that each column is an eigenvector of the matrix $X'V^{-1}X$, and $W'W=\Gamma=diag\{\lambda_1,\ldots,\lambda_P\}$ is the diagonal matrix, and those elements are the eigenvalues of the matrix $X'V^{-1}X$.

3. Performance of Proposed Estimator.

We need to provide some lemmas that will help us to do and prove our theoretical results. We also take into account the new estimator and its properties.

Lemma 1: [9] Let $n \times n$ matrices A > 0 and B > 0 (or B > 0) the A is positive definite (p.d), then as B is a (p.d) the A > B if and only if $\lambda_{max}(BA^{-1}) < 1$,where $\lambda_{max}(BA^{-1}) < 1$ is the max eigenvalue for the matrix BA^{-1}.

Lemma 2 [10] Let $\alpha_i = A_i y$ i=1,2 be the given two linear estimators of α. Suppose that $D_1 =$ Cov $(\hat{\alpha}_1)$ −Cov $(\hat{\alpha}_2)$,where Cov $(\hat{\alpha}_i)$ i=1,2 is the covariance matrix of $\hat{\alpha}_i$ and b_i =Bias $(\hat{\alpha}_i) = (A_iX - I)\alpha$, i=1,2 consequently,

$$\Delta = \text{MSE}\ (\hat{\alpha}_1) - \text{MSE}\ (\hat{\alpha}_2) = \sigma^2 \text{D} + b_1' b_1 - b_2' b_2$$

if b_2' (σ^2D + $b_1'b_1)b_2$< 1, where MSE $(\hat{\alpha}_i) = \text{Cov}(\hat{\alpha}_i) + b_i'b_i$

Lemma 3. [11] Let B be a (p.d) , matrix and A be a (n.n.d) matrix. A-B > 0 if only if $A^{-1} - B^{-1} > 0$ and Λ = diag $(\lambda_i{}^B(A))$ the diagonal matrix of the eigenvalues of A in the B . Then there exists a non-singular matrix W such that, B = $W'\,W$, A = $W'\Lambda\, W$

3.1 The Comparison Between CMURR and MUALE Estimators.

Since the CMURR and MUALE estimators, the difference between these two estimators.

$$\Delta_1 = \text{MSE}\ (\hat{\alpha}_{\text{MUALE}}) - \text{MSE}\ (\hat{\alpha}_{\text{CMURR}}) = \sigma^2 D_1 + b_1'b_1 - b_2'b_2$$

$D_1 =$ Cov $(\hat{\alpha}_{\text{MUALE}}) -$ Cov $(\hat{\alpha}_{\text{CMURR}}) = T_d\, W_K\ \Gamma^{-1} T_d{}' W_K{}' - T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM'$

$b_1 = (T_d\, W_K - I)\alpha$, $b_2 = [\,T_{K,d} + k^2 d\, M - I]\alpha$

Let B = $T_d\, W_K\ \Gamma^{-1} T_d{}' W_K{}'$, A = $T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM'$

By Lemma 3, we may rewrite D_1 as:

$D_1 = B - A$ if and only if $D_1 = A^{-1} - B^{-1} > 0$

where $B^{-1} = W'^{\,-1}W^{-1}$, $A^{-1} = W'^{\,-1}\Lambda^{-1}W^{-1}$ and $D_1 = W^{-1}(\Lambda^{-1} - I)W'^{\,-1} > 0$ and

Λ = diag {$\lambda\, T_d\, W_K\, \Gamma^{-1} T_d{}' W_K{}' - (T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM')$ } the diagonal matrix of the eiagenvalues of $(T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM')$ in the $[T_d\, W_K\ \Gamma^{-1} T_d{}' W_K{}']$ and denote in as $\lambda_i(D_1)$ so,

Therefore if $\lambda_i(D_1) < 1$, then $\Lambda^{-1} - I > 0$ is p.d. As a result of using Lemma 2, the following theorem may be shown.

Theorem 1. Under heteroscedastic and /or correlated errors, $\hat{\alpha}_{AUM}$ is superior to the $\hat{\alpha}_{\text{MEURR}}$ in the MSE sense, if $\lambda_i(D_1) < 1$ then Δ_1 p.d. if and only if,

$$b_2'(\sigma^2 D_1 + b_1 b_1')^{-1} b_2 \le 1$$

3.2 The Comparison Between CMURR and MUATP Estimators.

Since the CMURR and MUATP estimators, the difference between these two estimators.

$$\Delta_2 = \text{MSE}\ (\hat{\alpha}_{\text{MUATP}}) - \text{MSE}\ (\hat{\alpha}_{\text{CMURR}})$$

$= \text{Cov}(\hat{\alpha}_{\text{MUALE}}) - \text{Cov}(\hat{\alpha}_{\text{CMURR}})$

$= \sigma^2 M_{k,d} W_K \Gamma^{-1} M_{k,d}{}' W_K{}' - \sigma^2 T_{K,d}\Gamma^{-1}T_{K,d}{}' - k^3 d^2 MM'$

$= \sigma^2(\text{G} - \text{H})$

where $D_2 = (\text{G} - \text{H}) = M_{k,d} W_K \Gamma^{-1} M_{k,d}{}' W_K{}' - T_{K,d}\Gamma^{-1}T_{K,d}{}' - k^3 d^2 MM'$. The following Lemma1, this can be shown under any condition D_2 is p.d.

$$\text{G} - \text{H} \geq 0 \leftrightarrow \lambda_{max}(\text{GH}^{-1}) < 1$$

Then G and H is p.d. It can be said that $\text{G} - \text{H} \geq 0$, if and only if $\lambda_{max}(\text{GH}^{-1}) < 1$.

Since the matrix $M_{k,d} W_K \Gamma^{-1} M_{k,d}{}' W_K{}' \geq 0$, and D_2 is p.d. In light of this to $\lambda_{max}\big((M_{k,d} W_K \Gamma^{-1} M_{k,d}{}' W_K{}')(T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM')^{-1}\big) < 1$. As a result, we can now state following theorem.

Theorem 2. The CMURR estimator is superior to the MUATP estimator in the MSE sense, if and only if $\lambda_{max}\big((M_{k,d} W_K \Gamma^{-1} M_{k,d}{}' W_K{}')(T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM')^{-1}\big) < 1$.

3.3 The Comparison Between CMURR and MURR Estimators.

Since the CMURR and MURR estimators, the difference between these two estimators

$$\Delta_3 = \text{MSE}(\hat{\alpha}_{\text{MUATP}}) - \text{MSE}(\hat{\alpha}_{\text{CMURR}}) = \sigma^2 D_3 + b_4' b_4 - b_2' b_2$$

$$D_3 = \text{Cov}(\hat{\alpha}_{\text{MUALE}}) - \text{Cov}(\hat{\alpha}_{\text{CMURR}})$$

$$= F_{K,d}\Gamma^{-1}F_{K,d}{}' + k^3 d^2 AA' - T_{K,d}\Gamma^{-1}T_{K,d}{}' - k^3 d^2 MM'$$

where $b_4 = [\, F_{K,d} + k^2 d\, A - I]\alpha$

The theorem that follows can be stated.

Theorem 3. If $\lambda_{max}\big((F_{K,d}\Gamma^{-1}F_{K,d}{}' + k^3 d^2 AA')(T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM')^{-1}\big) < 1$,the estimator $\hat{\alpha}_{\text{CMURR}}$ is better than $\hat{\alpha}_{\text{MUALE}}$, in the MSE if and only if

$$b_2'(\sigma^2 D_3 + b_4 b_4')^{-1} b_2 \leq 1$$

where $\lambda_{max}\big((F_{K,d}\Gamma^{-1}F_{K,d}{}' + k^3 d^2 AA')(T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM')^{-1}\big)$ This is the greatest value of the matrix. $\big((F_{K,d}\Gamma^{-1}F_{K,d}{}' + k^3 d^2 AA')(T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM')^{-1}\big)$

proof: As a result to Lemma 1, it is possible to state that the matrix D_3 is p.d , if and only if $\lambda_{max}\big((F_{K,d}\Gamma^{-1}F_{K,d}{}' + k^3 d^2 AA')(T_{K,d}\Gamma^{-1}T_{K,d}{}' + k^3 d^2 MM')^{-1}\big) < 1$. Now according to Lemma 2, it is clear that $\text{MSE}(\hat{\alpha}_{\text{MUATP}}) - \text{MSE}(\hat{\alpha}_{\text{CMURR}}) \geq 0$, if and only if

$$b_2'(\sigma^2 D_3 + b_4 b_4')^{-1} b_2 \leq \sigma^2,$$

and the proof is completed.

The comparison results are based on unknown parameters of and therefore, we cannot rule out the possibility that the results obtained are preserved. They are replaced by their unbiased estimators. Given the rarity of V, an estimate of V can be utilized. Some estimates of the V matrix are provided by [5] and[12] . It contains the V matrices presented by [5] .

$$V = (W_{i,j}) \quad , \; W_{i,j} = \rho^{|i-j|} \quad , \quad i , j = 1,2,\ldots\ldots,n \tag{22}$$

and

$$V=\frac{1}{1+\tau^2}\begin{pmatrix} 1+\tau^2 & \tau & 0 & \cdots & 0 & 0 \\ \tau & 1+\tau^2 & \tau & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\tau^2 & \tau \\ 0 & 0 & 0 & \cdots & \tau & 1+\tau^2 \end{pmatrix}$$

The terms in the error vector are from the MA(1) process. $\varepsilon_i = u_i + \tau\varepsilon_{i-1} \quad ,|\tau| < 1$

[13] provides the second V matrix.

$$V=\frac{1}{1-\rho^2}\begin{pmatrix} 1 & \rho & \cdots & \rho^{n-1} \\ \rho & 1 & \cdots & \rho^{n-2} \\ \vdots & \vdots & \vdots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \cdots & 1 \end{pmatrix}$$

The terms in the error vector are from the AR (1) process. $\varepsilon_i = \rho\varepsilon_{i-1} + u_i$, $|\rho| < 1$ i=1, 2,…,n and u_i~N(0 , σ^2I) .

4. Simulation Study

In this section, we study the performance of GMAULE, GMAUTP, MURR, and CMURR estimators using Monte Carlo simulation under different factors such as degrees of multicollinearity, different values of the shrinkage parameter d and number of explanatory variables. Different parameters are used with some specified value. The correlated explanatory variables are generated by considering the work of Kibria (2003).

$$X_{ij} = (1-\gamma^2)^{\frac{1}{2}} Z_{ij} + \gamma Z_{ip} \; , \quad i=1,\ldots\ldots,n \quad , j=1,\ldots\ldots,n \tag{23}$$

where Z_{ij} are considered an independent standard normal pseudo-random number. The explanatory variables counter represents p = 4, while the value of σ is chosen (1 and 5). The value of ρ is chosen (0.10 and 0.50), and d is (0.2 and 0.7), the sample size n is (50 and 100) and γ is represented the correlation between the explanatory variables where it is chosen (0.85 and 0.95). Standardization of explanatory variables is done when the matrix is $X'X$ in the form of a correlation. In the simulation study, the form of the matrix V used by the equation (22), the parameters β_0, β_1, β_2,……, β_p are chosen as eigenvectors that correspond to the largest eigenvalue of the $X'V^{-1}X$ matrix. The dependent variable is determined by the following equation:

$$Y_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \ldots\ldots + \beta_p X_{ip} + \epsilon_i \quad , i=1, 2,\ldots,n \tag{24}$$

where ϵ_i, which represents independent normal random variables, is in the form of a normal distribution (0,σ^2). So β_0 is taken as zero. The experiment is repeated 5000 times under the new error conditions. This is done according to the estimated mean squared error (EMSE), which is calculated as:

$$\text{EMSE}\,(\beta^*) = \frac{1}{5000}\sum_{i=1}^{5000}(\beta^* - \beta)'\,(\beta^* - \beta\;) \tag{25}$$

where β^* will be any estimators (GMAULE, GMAUTP, MURR, CMURR,).

Table 1. Estimated mse values of the MAULE, MAUTP, MURR and CMURR.

k =1	$d = 0.01$	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.4$	$d = 0.5$	$d = 0.6$	$d = 0.8$
MAULE	0.64418	0.67619	0.72116	0.7716	0.82349	0.87331	0.91812	0.98352
MAUTP	0.64272	0.65764	0.67848	0.70381	0.73362	0.76792	0.80671	0.89776
MURR	0.45975	0.45961	0.6018	0.89382	1.3357	1.9274	2.6689	4.6014
CMURR	0.59193	0.56983	0.54178	0.53821	0.60459	0.80317	1.2129	3.0659

Table 2. Estimated mse values of the MAULE, MAUTP, MURR and CMURR.

k =0.5	$d = 0.001$	$d = 0.01$	$d = 0.1$	$d = 0.2$	$d = 0.3$	$d = 0.5$	$d = 0.6$	$d = 0.8$
MAULE	0.72873	0.73716	0.83895	0.97996	1.1370	1.4519	1.5901	1.7916
MAUTP	0.92015	0.92484	0.97553	1.04000	1.1130	1.2846	1.3832	1.6060
MURR	1.7043	1.6786	1.5477	1.6699	2.0741	3.7281	4.9779	8.3233
CMURR	0.56423	0.56281	0.54533	0.53697	0.58847	1.2115	2.025	5.4766

Table 3. Estimated mse values of the MAULE, MAUTP, MURR and CMURR.

d =0.5	$k = 0.5$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 4.5$	$k = 5$	$k = 7$
MAULE	1.4519	0.87331	0.67167	0.63173	0.61909	0.61612	0.61424	0.61125
MAUTP	1.2846	0.76792	0.63453	0.61614	0.61212	0.61153	0.61132	0.61232
MURR	3.7281	1.9274	1.0099	0.68969	0.52468	0.46876	0.42367	0.26884
CMURR	1.2115	0.80317	0.58415	0.50777	0.46881	0.45569	0.44516	0.40929

Table 4 . Estimated mse values of the MAULE, MAUTP, MURR and CMURR .

d =0.3	$k = 0.1$	$k = 0.5$	$k = 0.9$	$k = 1$	$k = 1.3$	$k = 1.5$	$k = 2$	$k = 3$
MAULE	4.1358	1.137	0.80716	0.7716	0.70512	0.67961	0.64501	0.6211
MAUTP	6.2519	1.113	0.73524	0.70381	0.65283	0.63667	0.61904	0.61157
MURR	22.460	2.0741	1.0099	0.89382	0.66455	0.56764	0.41639	0.27223
CMURR	1.1192	0.58847	0.54276	0.53821	0.52978	0.5265	0.52182	0.51798

Table 6. The EMSE values of the MAULE, MAUTP, MURR and CMURR when d=0.2

$n = 50$, $\gamma = 0.85$, $\rho = 0.10$										
σ	k	MAULE	MAUTP	MURR	CMURR	σ	MAULE	MAUTP	MURR	CMURR
	0.001	1.18273	0.36005	0.35972	0.24830		12.10690	1.01116	1.00628	2.08118
	0.005	1.15267	0.36081	0.35915	0.24086		11.71959	1.02220	0.99778	2.01273
	0.01	1.11736	0.36179	0.35846	0.23207		11.26511	1.03636	0.98750	1.93230
	0.05	0.90584	0.37153	0.35415	0.17788		8.55898	1.17006	0.91701	1.44704
	0.09	0.78099	0.38665	0.35147	0.14343		6.98225	1.36371	0.86310	1.14630
	0.1	0.75949	0.39175	0.35100	0.13689		6.71045	1.42732	0.85164	1.08947
	0.5	0.30497	0.96152	0.36134	0.08831		0.85201	3.64385	0.67267	0.48209
	0.9	0.19650	0.19743	0.38667	0.12643		0.49924	0.51488	0.64880	0.45307

σ	k	MAULE	MAUTP	MURR	CMURR	σ	MAULE	MAUTP	MURR	CMURR
1	1	0.17634	0.17634	0.39321	0.13787	5	0.47030	0.47030	0.64828	0.45678
	5	6.28431	4.87970	0.56544	0.44941		2.80504	2.34951	0.73934	0.66533
	10	1.64542	1.41383	0.65357	0.58178		1.32035	1.20483	0.79541	0.75126
	15	1.33671	1.19476	0.69724	0.63989		1.17470	1.10031	0.82263	0.78770
	20	1.22740	1.12233	0.72326	0.67263		1.12003	1.06400	0.83863	0.80795
	30	1.13776	1.06744	0.75280	0.70845		1.07379	1.03580	0.85659	0.82987
	40	1.09878	1.04562	0.76912	0.72774		1.05327	1.02439	0.86642	0.84158

Table 7. The EMSE values of the MAULE, MAUTP, MURR and CMURR when d=0.7

$n = 50$, $\gamma = 0.95$, $\rho = 0.50$										
σ	k	MAULE	MAUTP	MURR	CMURR	σ	MAULE	MAUTP	MURR	CMURR
1	0.001	0.46439	0.40692	0.40676	0.14077	5	19.14068	0.87904	0.87365	1.24317
	0.005	0.45770	0.40741	0.40662	0.13912		17.34411	0.89698	0.86845	1.20478
	0.01	0.45038	0.40811	0.40642	0.13709		15.59749	0.92299	0.86161	1.15924
	0.05	0.41730	0.41835	0.40468	0.12311		14.23186	1.62029	0.80969	0.89527
	0.09	0.40715	0.45627	0.40317	0.11296		76.51874	19.08013	0.77300	0.74970
	0.1	0.40723	0.48373	0.40286	0.11092		9.58096	4.19058	0.76589	0.72378
	0.5	0.37376	0.37952	0.40391	0.09063		0.65735	0.66533	0.68452	0.46308
	0.9	0.31917	0.31910	0.41521	0.10204		0.62412	0.62422	0.67673	0.44150
	1	0.30956	0.30956	0.41864	0.10618		0.61793	0.61793	0.67638	0.44067
	5	0.34404	0.31462	0.59327	0.35098		0.20127	0.20804	0.69544	0.49458
	10	2.96953	2.88912	0.78192	0.64222		2.53868	2.48451	0.71072	0.53160
	15	1.67586	1.64296	0.91836	0.86366		1.42313	1.40499	0.71931	0.54973
	20	1.40408	1.38187	1.01815	1.03091		1.24369	1.23160	0.72479	0.56019
	30	1.22330	1.20915	1.15212	1.26258		1.13167	1.12391	0.73137	0.57154
	40	1.15416	1.14359	1.23710	1.41398		1.09018	1.08433	0.73518	0.57749

Table 8 . The EMSE values of the MAULE, MAUTP, MURR and CMURR when d=0.2

$n = 100$, $\gamma = 0.85$, $\rho = 0.10$										
σ	k	MAULE	MAUTP	MURR	CMURR	σ	MAULE	MAUTP	MURR	CMURR
	0.001	10.25112	0.34642	0.34578	0.19579		24.42169	0.89659	0.89334	1.30665
	0.005	9.82924	0.34789	0.34469	0.18485		23.50290	0.90399	0.88773	1.26179
	0.01	9.35483	0.34984	0.34342	0.17248		22.45336	0.91359	0.88101	1.21027
	0.05	7.09150	0.37243	0.33606	0.10869		16.99842	1.01213	0.83744	0.92676
	0.09	6.72445	0.42383	0.33223	0.07851		15.08009	1.18874	0.80706	0.77541
	0.1	6.92733	0.44714	0.33164	0.07366		15.05367	1.25731	0.80094	0.74903
	0.5	0.25436	0.25354	0.34709	0.06256		0.64249	0.84290	0.72686	0.53653
	0.9	0.19854	0.19684	0.37714	0.10649		0.59773	0.59968	0.73002	0.56140
	1	0.18419	0.18419	0.38481	0.11875		0.58406	0.58406	0.73288	0.57051
	5	22.65070	16.8928	0.62241	0.50921		2.62175	2.24054	0.85156	0.80536
	10	1.86273	1.57253	0.78190	0.73104		1.24491	1.16079	0.92313	0.90527

1	15	1.42071	1.25258	0.87257	0.84566	5	1.13189	1.07761	0.96040	0.95210
	20	1.27744	1.15480	0.93059	0.91653		1.09022	1.04920	0.98315	0.97976
	30	1.16484	1.08338	1.00031	1.00037		1.05527	1.02732	1.00947	1.01134
	40	1.11721	1.05571	1.04063	1.04865		1.03984	1.01854	1.02423	1.02898

Table 9 . The EMSE values of the MAULE, MAUTP, MURR and CMURR when d=0.7

$n = 100,\ \gamma = 0.95,\ \rho = 0.50$										
σ	k	MAULE	MAUTP	MURR	CMURR	σ	MAULE	MAUTP	MURR	CMURR
1	0.001	1.95152	0.37930	0.37827	0.22884	5	2.07806	0.84733	0.84484	1.03768
	0.005	1.81000	0.38271	0.37727	0.22084		1.96179	0.85535	0.84247	1.01959
	0.01	1.66273	0.38754	0.37595	0.21122		1.83535	0.86634	0.83937	0.99764
	0.05	1.17808	0.47702	0.36545	0.15182		1.27806	1.02193	0.81363	0.85313
	0.09	2.59101	1.66671	0.35752	0.11632		1.16043	1.65599	0.79170	0.75665
	0.1	9.55333	7.75605	0.35597	0.10984		1.23967	2.21563	0.78702	0.73798
	0.5	0.29247	0.29379	0.34480	0.04843		0.70051	0.70920	0.72168	0.52058
	0.9	0.25610	0.25600	0.35562	0.05335		0.66868	0.66881	0.71371	0.49868
	1	0.24711	0.24711	0.35909	0.05627		0.66316	0.66316	0.71325	0.49760
	5	0.49266	0.45556	0.54167	0.28219		0.31321	0.31952	0.73018	0.54649
	10	3.47899	3.37642	0.74619	0.58690		2.75219	2.69234	0.74517	0.58281
	15	1.80901	1.76943	0.89753	0.82935		1.41263	1.39559	0.75394	0.60133
	20	1.47734	1.45104	1.00969	1.01642		1.23212	1.22096	0.75966	0.61231
	30	1.26125	1.24469	1.16194	1.27978		1.12364	1.11652	0.76667	0.62457
	40	1.17966	1.16734	1.25941	1.45408		1.08423	1.07887	0.77080	0.63119

5. Discussion of Simulated Results

1-Table (6) When the suggested bias estimator is best when they have the least variance, the MEURR estimator is best in all cases when (σ = 1). when it is (σ = 5), the suggested estimator is better at k = 0.5,... 40.

2-Table (7) when (σ = 1) and the suggested estimator is better at k = 0.001,... 15th

when it is (σ = 5), the suggested estimator is better at k = 0.09,..., 1 , but in the case of k = 5, the estimator MAULE is smaller than others.

3-Table (8) when (σ = 1) and the suggested estimator MEURR is better at k = 0.001,.. 20.

when it is (σ = 5), the suggested estimator is better at k = 0.09,..., 1 , but in the case of k = 5, the estimator MAULE is smaller than others.

4-Table (9) when (σ = 1) and the suggested estimator is better at k = 0.001,... 15th

when it is (σ = 5), the suggested estimator MEURR is better at k = 0.09,...1 , but in the case of k = 5, the estimator MAULE is smaller than others.

6. Numerical Example.

Based on the gross domestic product (GDP) of the United States as a y variable, personal disposable income (PDI) is X_1, personal consumption expenditure (PCE) is X_2, united states corporation tax after profits statistics

(Profits) is X_3, and net dividends to corporations (Dividends) is X_4 were derived from [14] for the years 1970 - 1991.data and information were also used by[15] . The set of data has been made uniform. To use the Durbin-Watson test to determine the presence of autocorrelation (DW), the DW value obtained from the new data is 0.52. which demonstrates the existence of positive autocorrelation at the significance level of 0.05 for n = 88, which confirms that the error structure is being used to select the new error structure trace data process ar (1). The process estimate is 0. 47045. As a result, V can be estimated using the relationships in (22). The number of the condition of $X'V^{-1}X$ is (246.395) despite the presence of a (GLS) file, which indicates the presence of a multicollinearity in the variables. It turns out that it is unstable in the presence of a multicollinearity in the data. So, we can say that many studies show that estimating the variance depends at least on the square estimator, and it depends more on other estimators. We've done the math by (mse). This value of σ^2 is calculated using the (GLS) of in (2). The eigenvalues of the matrix of $X'V^{-1}X$ are $\lambda_1 = 0.3095$, $\lambda_2 = 0.0433$, $\lambda_3 = 0.0030$ and $\lambda_4 = 0.0012$. The mse value is calculated for the proposed estimator murr, and it is compared to other estimators (MAULE, MAUTP, MURR) using MSE.

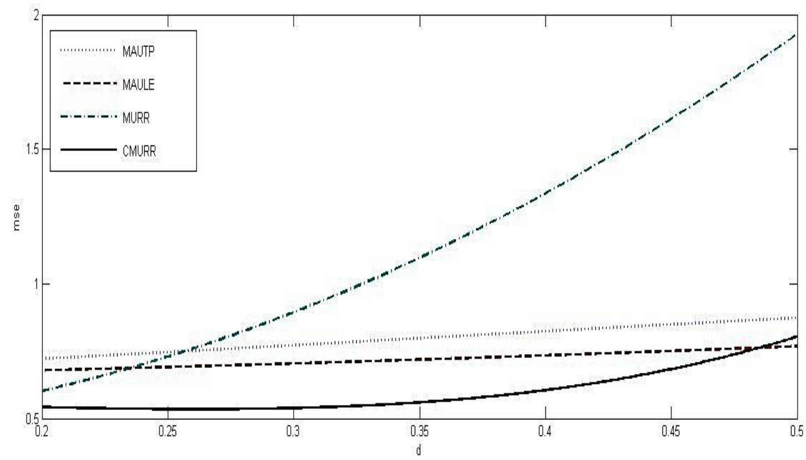

Figure (1)MAUTPThe MAULE, , MURR and CMURR estimated mse values for different d when k =1

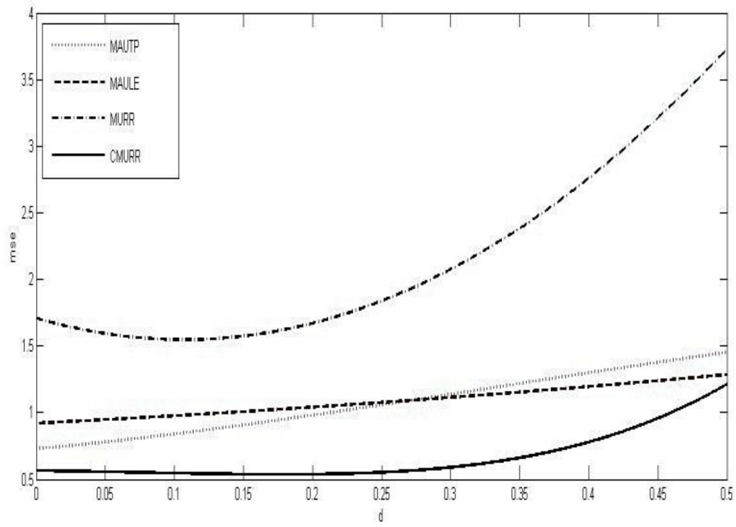

Figure (2) The MAULE, MAUTP, MURR and CMURR estimated mse values for different d when k =0.5

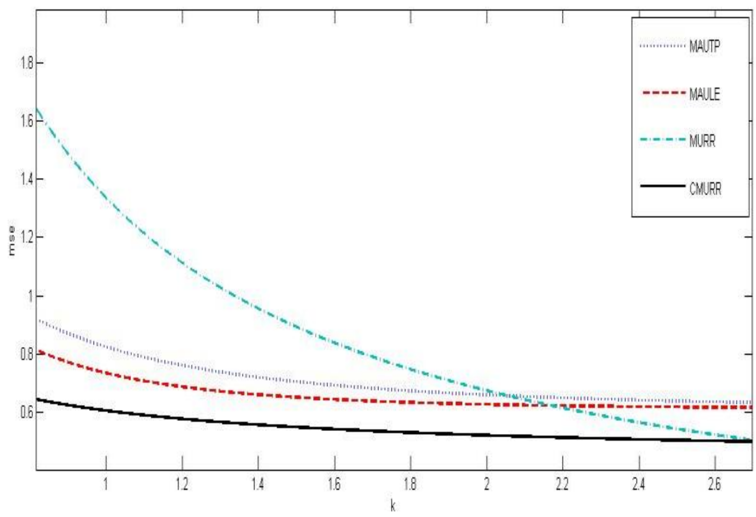

Figure (3) The MAULE, MAUTP, MURR and CMURR estimated mse values for different k when d =0.4

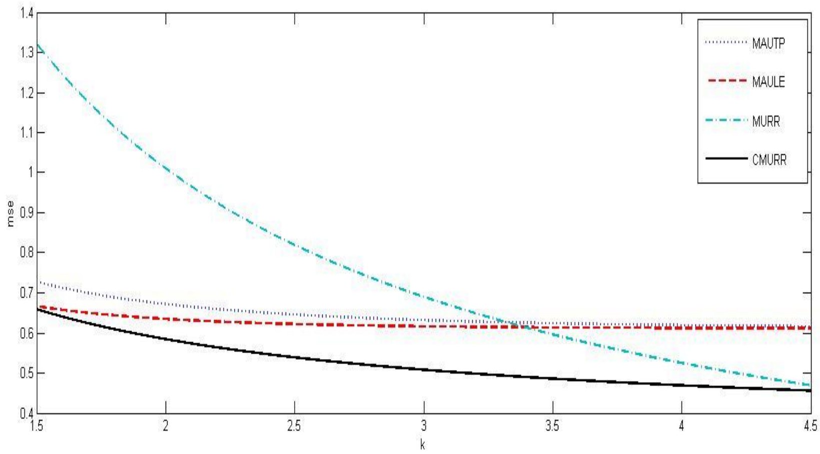

Figure 4. The MAULE, MAUTP, MURR and CMURR estimated mse values for different k when d =0.5

6. Interpretation the Results

1. The value of k in Table (1) is 1, which is constant, and the value of d is not fixed. Therefore, the MURR is better at d = 0.01, 0.1, and 0.2. The proposed estimator is better than the other estimators at d = 0.2, 0.3, and 0.4. whereas the MAUTP estimator is superior to the suggested estimator at d = 0.5, 0.6, and 0.8. As shown in Figure (1), at k = 1,

2. Table (3) is k = 0.5 and it is fixed, and the value of d is variable. The proposed estimator is better than other estimators when d = 0.001,... 0.5. whereas the MAUTP estimator is superior to the suggested estimator at d = 0.6 and 0.8. As shown in Figure (2) at k = 0.5,

3. Table (4) is d = 0.5 and it is fixed, and the value of k is variable. MAUTP is better at k = 0.5, 1 and the proposed estimator is better than other estimators at k = 2,..., 4.5. whereas the MURR estimator is superior to the proposed estimator at k = 5, 7. As shown in Figure (3), at d = 0.4.

4. Table (5) is d = 0.3 and it is fixed, and the value of k is variable, the proposed estimator is better than other estimators, when k = 0. 1,…,1.5. Whereas, the MURR estimator is superior to the proposed estimator at k=2,3. As shown in Figure (4), it has d = 0.5. Finally, the CMURR estimator is better than the compared estimators, which are MAUTP, MAULE, MURR, as shown in the example, tables and illustrations.

In this paper, a new biased estimator with two parameters (k and d) is proposed for linear regression model in the presence of multicollinearity problem when there are heterogeneous and/or correlated errors. Using the mean square error (MSE), the new (CMURR) estimator was compared to the other estimators (MAULE), (MAUTP), and (MURR), and the new estimator outperformed the others, as shown.

7. Acknowledgments

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