



## Strongly 2T - Clean Rings

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### Article information

#### Article history:

Received :21/5/2024

Accepted:30/6/2024

Available online:15/12/2024

### Abstract

An element  $a$  in a ring  $\mathcal{R}$  is referred to be strongly 2T-clean (2 – STC element for short),  $a = \Omega - \Lambda + u$ , where  $\Omega, \Lambda$  are idempotent elements and  $u$  is a unit elements of order three. A ring  $\mathcal{R}$  is considered to be 2 - STC ring if every member of  $\mathcal{R}$  are 2 - STC ring. This paper presents the idea of an strongly 2T-clean ring and lists some of its fundamental characteristics. Further more we consider 2 - STC ring with 3 is nilpotent, we demonstrate that this ring is equivalent to strongly 2- nil clean ring and the Jacobson radical and the right singular ideal of such ring with 3 is nilpotent is a nil ideal. Finally we exhibit that if  $\mathcal{R}$  is an 2 - STC ring and if 2 is nilpotent, then  $a^4 - a$  is nilpotent for every  $a$  in  $\mathcal{R}$ . We demonstare that if  $\mathcal{R}$  is 2 - STC ring, then  $a \in \mathcal{R}$  such that  $a = \Omega - h + u$ ,  $\Omega$  is idempotent,  $h$  is a unit of order two and  $u$  is a unit of order three.

#### Keywords:

Idempotent, nilpotent, unit, strongly 2- nil clean, good ring.

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## I. Introduction

To symbolize the set of units, idempotents, nilpotent elements, and Jacobson of  $\mathcal{R}$ , respectively, we use the notations  $U(\mathcal{R})$ ,  $ID(\mathcal{R})$ ,  $Nil(\mathcal{R})$  and  $J(\mathcal{R})$ .

W. K. Nicholson [1] determined a clean ring in 1977, that is a ring where each member of  $\mathcal{R}$  is a sum of an idempotent and a unit. Later in 1999 Nicholson [2] defined the strongly clean ring if the idempotent and the unit commute. Many authors worked in this kind of ring, see for example[3], [4] and [5].

Disel [6] and [7] first proposed the idea of a nil clean ring. One such rings elements are the sum of an idempotent and nilpotent. T. Kozan presented the concept of (SNC for short) in 2016 [8]. This type of ring is one in which every member may be presented as the sum of an idempotent and a nilpotent that commute.

In 2017, Chen and Sheibani [9],[10] and[11] defined or (2-SNC for short), which is a ring in which each member of  $\mathcal{R}$  is the sum of two idempotents and a nilpotent that commute.

Numerous authors have worked with this type of ring, see

for example[12] and [13].

Danchev [14] introduced the idea of an invo – clean ring in 2017 as a ring  $\mathcal{R}$  with each  $a$  in  $\mathcal{R}$ , there exists a unit  $u$  of order two and an idempotent  $\Omega$  so that  $a = \Omega + u$ . If further  $\Omega u = u\Omega$ ,  $\mathcal{R}$  is called (SIC-for short).

## II. Preliminaries

We shall provide certain definitions, examples, and theorems in this part we might need it in the sequel.

### Definition 2.1 [1]

An element  $a$  is considered as clean if  $a = \Omega + v$ , where  $\Omega \in ID(\mathcal{R})$  and  $v \in U(\mathcal{R})$ . When all members of  $\mathcal{R}$  are clean,  $\mathcal{R}$  is considered as clean. If  $\Omega v = v\Omega$ ,  $\mathcal{R}$  is referred to be a strongly clean ring.

### Example (1)

In the ring  $\mathcal{Z}_{10}$ , clearly  $ID(\mathcal{Z}_{10}) = \{0,1,5,6\}$  and  $U(\mathcal{Z}_{10}) = \{1,3,7,9\}$ . Evidently, each element of  $\mathcal{Z}_{10}$ , is the sum of idempotent and a unit. So  $\mathcal{Z}_{10}$  is a clean ring.

**Definition 2.2** [4]

If  $a$  in  $\mathcal{R}$ , with  $a = \Omega + n$ , where  $n \in \text{Nil}(\mathcal{R})$ , and  $\Omega \in \text{ID}(\mathcal{R})$ , then  $\mathcal{R}$  is called a nil clean ring. If  $\Omega n = n\Omega$ ,  $\mathcal{R}$  is regarded as (SNC).

**Example (2)**

The ring  $\mathcal{Z}_8$ , note that  $\text{ID}(\mathcal{Z}_8) = \{0,1\}$  and  $\text{Nil}(\mathcal{Z}_8) = \{0,2,4,6\}$ . Clearly,  $\mathcal{Z}_8$  is a nil clean.

**Definition 2.3** [9]

If every elements of  $\mathcal{R}$ ,  $a = \Omega + \Lambda + n$ , where  $\Omega, \Lambda \in \text{ID}(\mathcal{R})$  and  $n \in \text{Nil}(\mathcal{R})$ , which commute with one another then the element  $a$  is considered as (2-SNC). If each member in  $\mathcal{R}$  is (2-SNC), the ring  $\mathcal{R}$  is considered to be (2-SNC).

**Example (3)**

The ring  $\mathcal{Z}_{12}$ . Note that  $\text{ID}(\mathcal{Z}_{12}) = \{0,1,4,9\}$  and  $\text{Nil}(\mathcal{Z}_{12}) = \{0,6\}$ , so each element in  $\mathcal{Z}_{12}$  is (2-SNC).

**Definition 2.4** [14]

A ring  $\mathcal{R}$  is considered to be invo – clean if all element  $a \in \mathcal{R}$ , then  $a = \Omega + v$ , where  $\Omega$  is an idempotent element and  $v$  is a unit element of order two in  $\mathcal{R}$ ,  $\mathcal{R}$  is called (SIC) if  $\Omega v = v\Omega$ .

**Example (4)**

The rings  $\mathcal{Z}_2, \mathcal{Z}_3, \mathcal{Z}_4, \mathcal{Z}_6, \mathcal{Z}_8$ , are all invo – clean rings but  $\mathcal{Z}_5$  and  $\mathcal{Z}_7$  are not invo – clean.

**Definition 2.5** [10]

An element  $t$  is called a tripotent if  $t = t^3$ . A ring  $\mathcal{R}$  is called tripotent if each member of  $\mathcal{R}$  is a tripotent.

**Example (5)**

The ring  $\mathcal{Z}_6$  is a tripotent ring, since  $1^3 = 1, 2^3 = 2, 3^3 = 3, 4^3 = 4, 5^3 = 5$ .

**Lemma 2.6**

If  $\Omega$  and  $\Lambda \in \text{ID}(\mathcal{R})$ , and  $\Omega \Lambda = \Lambda \Omega$ , then  $\Omega - \Lambda$  is a tripotent.

**Proof:-** Let  $\Omega, \Lambda \in \text{ID}(\mathcal{R})$ , then  $\Omega^2 = \Omega$  and  $\Lambda^2 = \Lambda$  since  $\Omega \Lambda = \Lambda \Omega$  by an assumption then  $(\Omega - \Lambda)^3 = \Omega^3 - 3\Omega^2\Lambda + 3\Omega\Lambda^2 - \Lambda^3 = \Omega - 3\Omega\Lambda + 3\Omega\Lambda - \Lambda = \Omega - \Lambda$ . ■

**Lemma 2.7** [15]

If  $m$  is nilpotent,  $v$  is a unit and  $vn = nv$  then

1.  $1 + m$  is a unit.
2.  $v + m$  is a unit.

**Definition 2.8** [8]

A ring  $\mathcal{R}$  referred to be a  $r$  – good ring if each member  $d = u_1 + u_2 + u_3 + \dots + u_r$ , where  $u_i, i = 1,2,3, \dots, r$  are unit elements.

**Lemma 2.9**

If  $\Omega \in \text{ID}(\mathcal{R})$  and  $2 \in U(\mathcal{R})$ , then  $1 + \Omega$  is a unit.

**Proof:-** Suppose  $(1 + \Omega)(2 - \Omega) = 2 - \Omega + 2\Omega - \Omega$   
 $= 2 - 2\Omega + 2\Omega$   
 $= 2$

Since  $2 \in U(\mathcal{R})$ , then  $(1 + \Omega) \in U(\mathcal{R})$ . ■

**Definition 2.10**

An ideal  $A$  is said to be essential ideal if  $A \cap B \neq 0$  for each ideal  $B$  of  $\mathcal{R}$ .

**Example (6)**

In the ring  $\mathcal{Z}_8$ . Note that ideal  $A = \{0,2,4,6\}$ ,  $B = \{0,4\}$ , clearly  $A \cap B = \{4\}$  and  $B \cap A = \{4\}$ , thus  $A, B$  is an essential ideal.

**Definition 2.11**

Let  $c \in \mathcal{R}$ , then  $r(c) = \{g \in \mathcal{R} : cg = 0\}$ , is called the right annihilator of  $c$ .

**Definition 2.12**

Let  $d \in \mathcal{R}$ , then  $Y(\mathcal{R}) = \{d \in \mathcal{R} : r(d) \text{ is essential right ideal}\}$  represent the right singular ideal.

**Definition 2.13** [16]

Two idempotent elements  $\Omega$  and  $\Lambda$  are said to be orthogonal if  $\Omega \Lambda = \Lambda \Omega = 0$ .

### III. 2 – STC RINGS

This part presents an overview of the 2 – STC ring, highlighting some of its basic properties and offering some examples.

**Definition 3.1**

An element  $a \in \mathcal{R}$  is referred as a 2 – STC ring if each element  $a$  in  $\mathcal{R}$ , existing two idempotent elements  $\Omega, \Lambda$  in  $\mathcal{R}$  and a unit element of order three  $u$  in  $\mathcal{R}$  that commute, such that  $a = \Omega - \Lambda + u$ .

**Example (1)**

In the ring  $\mathcal{Z}_{18}$ . Note that  $\text{ID}(\mathcal{Z}_{18}) = \{0,1,9,10\}$  and  $U(\mathcal{Z}_{18}) = \{1,5,7,11,13,17\}$ , and the unit element of  $\mathcal{Z}_{18}$ ,  $U_3(\mathcal{Z}_{18}) = \{1,7,13\}$  is easy to check that each member of  $\mathcal{Z}_{18}$  can be expressed as a sum element two idempotents and unit of order three, so  $\mathcal{Z}_{18}$  is 2 – STC ring.

**Note that:**

The ring  $\mathcal{Z}_7, \mathcal{Z}_{14}, \mathcal{Z}_{27}, \mathcal{Z}_{32}$ , are not 2 – STC ring.

**Proposition 3.2**

Assume  $\mathcal{R}$ , is 2 – STC ring and  $3 \in \text{NiL}(\mathcal{R})$ , then  $\mathcal{R}$  is a strongly 2 – nil clean ring.

**Proof:-** Take  $a \in \mathcal{R}$ , then  $a + 1 \in \mathcal{R}$ , thus  $a + 1 = \Omega - \Lambda + u$ , where  $\Omega, \Lambda \in ID(\mathcal{R})$  and  $u$  is a unit of order three that commute with one another.  $a = \Omega - \Lambda + u - 1$ . Now consider  $(u - 1)^3 = u^3 - 3u^2 + 3u - 1 = 1 - 3u^2 + 3u - 1 = 3u(1 - u)$ ,

Since  $3 \in \text{NiL}(\mathcal{R})$ , then  $1 - u \in \text{NiL}(\mathcal{R})$ , thus  $(u - 1)^3 \in \text{NiL}(\mathcal{R})$ , implies  $u - 1 \in \text{NiL}(\mathcal{R})$ , so  $a = \Omega - \Lambda + n$ , where  $u - 1 = n$ .  $\mathcal{R}$  is therefore a strongly 2 – nil clean. ■

**Proposition 3.3**

If  $\mathcal{R}$  is strongly 2 – nil clean with  $n^3 = 3n^2 - 3n$  for every nilpotent element  $n$ , then  $\mathcal{R}$  is 2 – STC ring.

**Proof:-** Take  $a \in \mathcal{R}$ , then  $1 + a = \Omega - \Lambda + n$  where  $\Omega, \Lambda \in ID(\mathcal{R})$  and  $n \in \text{NiL}(\mathcal{R})$ , so  $a = \Omega - \Lambda + n - 1$ , by Lemma 2.7, then  $n - 1$  is a unit, say  $u$  observe that  $u^3 = n^3 - 3n^2 + 3n + 1 = 1$ , so  $u$  is unit of order three, thus  $a = \Omega - \Lambda + u$ . Therefore  $\mathcal{R}$  is 2 – STC ring. ■

**Example (2)**

In the ring  $Z_9$ . Note that  $ID(Z_9) = \{0,1\}$ ,  $U(Z_9) = \{1,2,4,5,7,8\}$ , the unit elements of  $Z_9$ ,  $U_3(Z_9) = \{1,4,5,7,8\}$ , and  $\text{NiL}(Z_9) = \{0,3,6\}$ . It is easy to check that each of  $Z_9$  member is a sum of two idempotents and unit of order three.

**Proposition 3.4**

If  $\mathcal{R}$  is 2 – STC ring, then  $a = \Omega + \Lambda + u$ .

**Proof:-** Suppose  $a \in \mathcal{R}$ , then  $a - 1 \in \mathcal{R}$ , thus  $a - 1 = \Omega - \Lambda + u$ , where  $\Omega, \Lambda \in ID(\mathcal{R})$  and  $u$  is a unit of order three, that commute with one another, thus  $a = \Omega - \Lambda + 1 + u$ ,  $a = \Omega + (1 - \Lambda) + u$ , where  $\Omega \in ID(\mathcal{R})$ ,  $(1 - \Lambda) \in ID(\mathcal{R})$  and  $u$  is a unit of order three, implies  $a = \Omega + \Lambda' + u$ , where  $\Lambda' = 1 - \Lambda$ . ■

**Proposition 3.5**

Suppose  $\mathcal{R}$  is 2 – STC ring, then  $a \in \mathcal{R}$  such that  $a = \Omega + h + u$ , where  $h$  is a unit of order two and  $u$  is a unit of order three.

**Proof:-** Let  $a \in \mathcal{R}$ , then  $a = \Omega - \Lambda + u$ , where  $\Omega, \Lambda \in ID(\mathcal{R})$  and  $u$  is a unit of order three, that commute with one another, by Lemma 2.6 then  $(\Omega - \Lambda) = (\Omega - \Lambda)^3$ , the element  $a$  can be written as  $a = 1 - (\Omega - \Lambda)^2 + (\Omega - \Lambda)^2 + (\Omega - \Lambda) - 1 + u$ , since  $(\Omega - \Lambda)$  is tripotent by Lemma [ If  $t$  is tripotent then  $1 - t^2$  is idempotent and  $t^2 + t - 1$  is a unit of order two ]. So  $a = \Omega' + h + u$ , where  $\Omega' = 1 - (\Omega - \Lambda)^2$ ,  $h = (\Omega - \Lambda)^2 + (\Omega - \Lambda) - 1$ ,  $h^2 = 1$ . ■

**3.6 Proposition**

If  $\mathcal{R}$  is 2 – STC ring, and  $3 \in \text{NiL}(\mathcal{R})$ , then  $J(\mathcal{R})$  is a nil ideal.

**Proof:-** Take  $a \in \mathcal{R}$ , then  $a + 1 \in \mathcal{R}$ ,  $a + 1 = \Omega - \Lambda + u$  where  $\Omega, \Lambda \in ID(\mathcal{R})$  and  $u$  is a unit of order three that commute with one another, so  $a = \Omega - \Lambda + u - 1$ , where

$(u - 1)^3 = u^3 - 3u^2 + 3u - 1 = 1 - 3u^2 + 3u - 1 = 3u(1 - u)$ , since  $3 \in \text{NiL}(\mathcal{R})$ ,  $(1 - u) \in \text{NiL}(\mathcal{R})$  Then  $(u - 1)^3 \in \text{NiL}(\mathcal{R})$ , so  $(u - 1) \in \text{NiL}(\mathcal{R})$ , say  $u - 1 = n$ , such that  $a = \Omega - \Lambda + n$ . Hence  $a \in J(\mathcal{R})$ , then  $a^2 \in J(\mathcal{R})$ , thus  $a^2 = (\Omega - \Lambda)^2 + 2(\Omega - \Lambda)n + n^2 = (\Omega - \Lambda)^2 + (2(\Omega - \Lambda) + n)n$ , since,  $(2(\Omega - \Lambda) + n)n \in \text{NiL}(\mathcal{R})$  say  $(2(\Omega - \Lambda) + n)n = n_1$ , thus  $a = (\Omega - \Lambda)^2 + n_1$ , as  $a^2 \in J(\mathcal{R})$  then  $1 - a^2 \in U(\mathcal{R})$ , hence  $1 - a^2 = 1 - (\Omega - \Lambda)^2 - n$ , thus  $u + n = 1 - (\Omega - \Lambda)^2$  by Lemma 2.7 then  $u + n$  is a unit, say  $u + n = u_1$ , so  $u_1 = 1 - (\Omega - \Lambda)^2$ , since  $1 - (\Omega - \Lambda)^2 \in ID(\mathcal{R})$  implies  $1 = 1 - (\Omega - \Lambda)^2$ , gives  $(\Omega - \Lambda)^2 = 0$ , thus  $\Omega - \Lambda = 0$ , so  $a = n$ . Therefore  $J(\mathcal{R})$  is nil ideal. ■

**Theorem 3.7**

Suppose  $\mathcal{R}$  is 2 – STC ring, with  $3 \in \text{NiL}(\mathcal{R})$ , then every element of  $\mathcal{R}$  is a sum of two difference orthogonal idempotents and a nilpotent.

**Proof:-** Let  $a \in \mathcal{R}$ , then  $a + 1 \in \mathcal{R}$ , thus  $a + 1 = \Omega - \Lambda + u$  where  $\Omega, \Lambda \in ID(\mathcal{R})$  and  $u$  is a unit of order 3 that commute with one another, so  $a = \Omega - \Lambda + u - 1$ , since  $(u - 1)^3 = u^3 - 3u^2 + 3u - 1 = 1 - 3u^2 + 3u - 1 = 3u(1 - u)$ , since  $3 \in \text{NiL}(\mathcal{R})$ , then  $(1 - u) \in \text{NiL}(\mathcal{R})$ , say  $n = u - 1$ , so  $a = \Omega - \Lambda + n$  by Lemma 2.6,  $(\Omega - \Lambda)^3 = (\Omega - \Lambda)$ , and  $3 \in \text{NiL}(\mathcal{R})$  by Lemma 2.7, implies  $2 \in U(\mathcal{R})$ , can be written as  $\Omega - \Lambda$ ,

$$\Omega - \Lambda = \frac{(\Omega - \Lambda)^2 + (\Omega - \Lambda)}{2} - \frac{(\Omega - \Lambda)^2 - (\Omega - \Lambda)}{2}$$
 Note that  $\frac{(\Omega - \Lambda)^2 + (\Omega - \Lambda)}{2}$  and  $\frac{(\Omega - \Lambda)^2 - (\Omega - \Lambda)}{2}$  are idempotent since  $[\frac{(\Omega - \Lambda)^2 + (\Omega - \Lambda)}{2}]^2 = \frac{(\Omega - \Lambda)^4 + 2(\Omega - \Lambda)^3 + (\Omega - \Lambda)^2}{4} = \frac{(\Omega - \Lambda)^2 + 2(\Omega - \Lambda) + (\Omega - \Lambda)^2}{4} = \frac{(\Omega - \Lambda)^2 + (\Omega - \Lambda)}{2}$  is idempotent and  $[\frac{(\Omega - \Lambda)^2 - (\Omega - \Lambda)}{2}]^2 = \frac{(\Omega - \Lambda)^4 - 2(\Omega - \Lambda)^3 + (\Omega - \Lambda)^2}{4} = \frac{(\Omega - \Lambda)^2 - 2(\Omega - \Lambda) + (\Omega - \Lambda)^2}{4} = \frac{(\Omega - \Lambda)^2 - (\Omega - \Lambda)}{2}$  is idempotent. Further more  $\frac{(\Omega - \Lambda)^2 + (\Omega - \Lambda)}{2} \cdot \frac{(\Omega - \Lambda)^2 - (\Omega - \Lambda)}{2} = \frac{(\Omega - \Lambda)^4 - (\Omega - \Lambda)^2}{4} = \frac{0}{4} = 0$ . Therefore  $a = \Omega' - \Lambda' + n$ , where  $\Omega', \Lambda'$  are two orthogonal idempotent and  $n$  is nilpotent. ■

**Proposition 3.8**

If  $\mathcal{R}$  is 2 – STC ring, and  $3 \in \text{NiL}(\mathcal{R})$ , then  $Y(\mathcal{R})$  is a nil ideal.

**Proof:-** Let  $a \in \mathcal{R}$ , then  $a + 1 \in \mathcal{R}$ , thus  $a + 1 = \Omega - \Lambda + u$  where  $\Omega, \Lambda \in ID(\mathcal{R})$  and  $u$  is a unit of order three that commute with one another, so  $a = \Omega - \Lambda + u - 1$ , where  $(u - 1)^3 = u^3 - 3u^2 + 3u - 1 = 1 - 3u^2 + 3u - 1 = 3u(1 - u)$ , since  $3 \in \text{NiL}(\mathcal{R})$ , then  $(1 - u) \in \text{NiL}(\mathcal{R})$ , so  $a = \Omega - \Lambda + n$ ,  $n = u - 1$ , suppose  $b \in Y(\mathcal{R})$ , where then  $r(b)$  is essential ideal. Let  $x \in r(b) \cap (\Omega - \Lambda)\mathcal{R}$ ,  $rx = 0$  and  $x = (\Omega - \Lambda)r$ ,  $a(\Omega - \Lambda)r = 0$ ,  $[(\Omega - \Lambda) + n](\Omega - \Lambda)r = 0$ ,  $[(\Omega - \Lambda)^2 + (\Omega - \Lambda)n]r = 0$  by Lemma 2.6 then  $(\Omega - \Lambda)^3 = (\Omega - \Lambda)$ ,  $[(\Omega - \Lambda)^2 + (\Omega - \Lambda)^3n]r = 0$ ,  $(\Omega - \Lambda)^2[1 + (\Omega - \Lambda)n] = 0$  by Lemma 2.7 then  $1 + n$  is a unit, thus  $(\Omega - \Lambda)^2ur = 0$ , so  $u = 1$   $(\Omega - \Lambda)^2r = 0$  implies  $(\Omega - \Lambda)r = 0$  thus  $x = 0$ ,  $r(b) \cap (\Omega - \Lambda)\mathcal{R} = 0$

since  $r(b) \neq 0$  then  $(\Omega - \Lambda)\mathcal{R} = 0$ , so  $\Omega - \Lambda = 0$ , thus  $b = n$ . Therefore  $Y(\mathcal{R})$  is nil ideal. ■

**Proposition 3.9**

Suppose  $\mathcal{R}$  is 2 – STC ring, and  $3 \in \text{NiL}(\mathcal{R})$ , then  $\mathcal{R}$  is 3 – good.

**Proof:-** Suppose  $a \in \mathcal{R}$ , then  $a - 2 = \Omega - \Lambda + u$  where  $\Omega, \Lambda \in \text{ID}(\mathcal{R})$ ,  $u$  is a unit of order three, that commute with one another, since  $3 \in \text{NiL}(\mathcal{R})$  by Lemma 2.7, then  $2 \in U(\mathcal{R})$ , so  $a = \Omega + 1 + \Lambda + 1 + u$  by Lemma 2.9, then  $\Omega + 1$  is a unit, so  $a = u_1 + u_2 + u_3$ , where  $u_1 = \Omega + 1, u_2 = \Lambda + 1$  and  $u_3 = u$ . Therefore  $\mathcal{R}$  is 3 – good ring. ■

**Theorem 3.10**

If  $\mathcal{R}$  is 2 – STC ring and if  $2 \in \text{NiL}(\mathcal{R})$ , then  $a^4 - a \in \text{NiL}(\mathcal{R})$ .

**Proof:-** Take  $a \in \mathcal{R}$ , then  $a = \Omega - \Lambda + u$  where  $\Omega, \Lambda \in \text{ID}(\mathcal{R})$ ,  $u$  is a unit of order three, which commute with one another,

$a^4 = (\Omega - \Lambda)^4 + 2(\Omega - \Lambda)^3u + (\Omega - \Lambda)^2u^2 + 2(\Omega - \Lambda)^3u + 4(\Omega - \Lambda)^2u^2 + 2(\Omega - \Lambda)u^3 + (\Omega - \Lambda)^2u^2 + 2(\Omega - \Lambda)u^3 + u^4$ , by Lemma 2.6, then  $(\Omega - \Lambda)^3 = (\Omega - \Lambda)$ , thus  $a^4 = (\Omega - \Lambda) + 2(\Omega - \Lambda)u + 2(\Omega - \Lambda)u + 2(\Omega - \Lambda) + 2(\Omega - \Lambda) + (\Omega - \Lambda)^2u^2 + 4(\Omega - \Lambda)^2u^2 + (\Omega - \Lambda)^2u^2 + u = a + 2[2(\Omega - \Lambda)u + 2(\Omega - \Lambda) + 3(\Omega - \Lambda)^2u^2]$ , since  $2 \in \text{NiL}(\mathcal{R})$  then  $(2(\Omega - \Lambda)u + 2(\Omega - \Lambda) + 3(\Omega - \Lambda)^2u^2) \in \text{NiL}(\mathcal{R})$ . Therefore  $a^4 - a \in \text{NiL}(\mathcal{R})$ . ■

**IV. Conclusion**

In this work, we introduce the notion of 2 – STC ring, we dive some of its fundemaded proper, we also conclude that the Jacobson radical of this ring in a nil ideal with  $3 \in \text{NiL}(\mathcal{R})$ , and then right singular ideal.

**Acknowledgement**

The authors would like to express their gratitude to the College of Computer Science and Mathematics at the University of Mosul for its support of this report.

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