Al-Rafidain Journal of Computer Sciences and Mathematics (RJCM)

www.csmj.mosuljournals.com

On SITN Rings

Rafal R Dhanoon[1, *] and Nazar H. Shuker[2]

[1,2]Department of Mathematics, College of Computer Sciences and Mathematics, University of Mosul, Mosul, Iraq
Emails: rafal.23csp31@student.uomosul.edu.iq, nazarhamdoon7@gmail.com

Article information

Article history:
Received:29/5/2024
Accepted:16/7/2024
Available online :15/12/2024

Abstract

An element is considered as a strong SITN, if it is the sum of idempotent, tripotent and nilpotent, that commute with one another. A ring $\mathcal{R}$ is referred to be SITN ring if each member of $\mathcal{R}$ is a strongly SITN. In this paper additional properties of a strongly SITN – ring is given. We prove that if $\mathcal{R}$is a strongly SITN – ring, then a^5-5a^3+4a is a nilpotent for every a in $\mathcal{R}$, we also give a necessary and sufficient condition for a strongly SITN ring to be a strongly nil clean ring. Among other results, we show that the Jacobson radical of a strongly SITN is a nil ideal. Finally, we are considering a special SITN-ring.

Keywords:

Nilpotent, idempotent, tripotent, strongly nil clean.

Correspondence:
Author: Rafal R Dhanoon
Email: rafal.23csp31@student.uomosul.edu.iq

I. Introduction

Each ring is associated with an identity. The set of units, idempotent, tripotent, nilpotent elements and the Jacobson radical represented by the corresponding, symbols $U(\mathcal{R})$, $Id(\mathcal{R})$, $Tri\ (\mathcal{R})$, $NiL\ (\mathcal{R})$, and $J(\mathcal{R})$.

A clean ring, as defined by W. K. Nicholson [1]in 1997, is one in which each member of $\mathcal{R}$ is a sum of an idempotent and a unit. Later in 1999 Nicholson [2] defined the strongly clean ring when the idempotent and the unit commute. Many authors worked in this kind of ring see for example [3], [4],and [5]. In 2013, Disel [6] presented the idea of a nil clean, which is a ring whose members are the sum of an idempotent and a nilpotent.

A strongly nil clean ring was presented by T. kozan [7] in 2016. This type of ring is one in which each member may be expressed as the sum of an idempotent and nilpotent that commute. Many authors, including [8], [9] and [10], have examined these concepts. In 2017, Chen and Sheibani [11], presented a strongly 2 – nil clean ring, which is a ring in which each member of $\mathcal{R}$ is the sum of two idempotents and a nilpotent that commute, many authors worked in this kind of rings see for example [12] and [13].

II. Preliminaries

We shall provide certain definitions, examples and theorems in this part we might need in the sequel.

Definition 2.1 [14]

If ϑ in $\mathcal{R}$, and if $\vartheta^2=\vartheta$, then ϑ is an idempotent element of $\mathcal{R}$. Clearly if ϑ is an idempotent element then $(1-\vartheta)$ is also idempotent, since $(1-\vartheta)^2=1-2\vartheta+\vartheta^2=1-\vartheta$

Definition 2.2 [15]

An element τ in a ring $\mathcal{R}$ is called tripotent if $\tau^3=\tau$. If each member of a ring $\mathcal{R}$ is tripotent $\mathcal{R}$ is refered to be a tripotent.

Lemma 2.3 [16]

If $\tau=\tau^3$ then 1. τ^2 is a idempotent.
2. $\tau^2+\tau-1$ is a unit of order two.

Definition 2.4 [17]

A given element a is nilpotent if there exists a positive integer ℓ such that $a^\ell=0$.

Lemma 2.5

If ϑ is idempotent, then $2\vartheta-1$ is of order two unit.

Theorem 2.6 [6]
A ring $\mathcal{R}$ is clean and $c^2 - c$ is nilpotent if and only if $\mathcal{R}$ is a strongly nil clean.

Example (1)
In the ring $\mathcal{Z}_8$, note that $Id\ (\mathcal{Z}_8) = \{0,1\}$ and $NiL\ (\mathcal{Z}_8) = \{0,2,4,6\}$. Clearly, $\mathcal{Z}_8$ is a nil clean ring.

Definition 2.7
A ring with any a in $\mathcal{R}$ is called a strongly 2 – nil clean or briefly strongly 2 - NC if $a = \vartheta_1 + \vartheta_2 + n$ where $\vartheta_1,\ \vartheta_2 \in Id(\mathcal{R})$ and $n \in\ Nil\ (\mathcal{R})$ that commute with each other.

Example (2)
In the ring $\mathcal{Z}_{12}$, note that $Id\ (\mathcal{Z}_{12}) = \{0,1,4,9\}$ and $NiL\ (\mathcal{Z}_{12}) = \{0,6\}$, so $\mathcal{Z}_{12}$ is strongly 2 - NC.

Theorem 2.8 [11]
Suppose $\mathcal{R}$ is a ring, and let $c \in \mathcal{R}$ then $c^3 - c$ is nilpotent if and only if $\mathcal{R}$ is a strongly 2 – NC.

Lemma 2.9 [18]
If m is nilpotent, v is a unit and $v\,m = m\,v$

1. $1 \pm m$ is a unit.
2. $v + m$ is a unit.

Definition 2.10 [19]
A ring $\mathcal{R}$ is referred as a m – good ring if all elements $c = \mathcal{U}_1 + \mathcal{U}_2 + \cdots + \mathcal{U}_m$ where $\mathcal{U}_i,\ i = 1,2,3, \dots m$ are unit elements.

Lemma 2.11
If $\vartheta \in Id(\mathcal{R})$ and $2 \in\ U(\mathcal{R})$, then $1 + \vartheta$ is a unit.
Proof: Consider
$$\begin{aligned}(1 + \vartheta)(2 - \vartheta) &= 2 - \vartheta + 2\vartheta - \vartheta\\ &= 2 - 2\vartheta + 2\vartheta\\ &= 2.\end{aligned}$$
Since $2 \in\ U(\mathcal{R})$, then $1 + \vartheta \in U(\mathcal{R})$.

Lemma 2.12
If ϑ_1 and $\vartheta_2 \in Id(\mathcal{R})$ and if $\vartheta_1\vartheta_2 = \vartheta_2\vartheta_1$, then $\vartheta_1 - \vartheta_2$ is a tripotent.

III. Basic Properties of A Strongly Sitn – Rings

In this part, we give an overview of the strongly SITN-rings, highlighting some of its basic properties and offering some examples.

Definition 3.1
Let $\mathcal{R}$ be a ring, then $\mathcal{R}$ is considered to be a strongly SITN-ring if for every a in $\mathcal{R}$, $a = \vartheta + \tau + n$, where ϑ is idempotent, τ is a tripotent and n is a nilpotent that commutes with one another.

Example (1)
The ring $\mathcal{Z}_{16}$ is an SITN- ring. Clearly $Id\ (\mathcal{Z}_{16}) = \{0,1\}$ and $Tri\ (\mathcal{Z}_{16}) = \{0,1,7,9,15\}$ and $NiL\ (\mathcal{Z}_{16}) = \{0,2,4,6,8,10,12,14\}$. Evidently, every member of $\mathcal{Z}_{16}$ is the a sum of an element in $Id\ (\mathcal{R})$ and an element in $Tri(\mathcal{R})$ and an element in $NiL\ (\mathcal{R})$. So $\mathcal{Z}_{16}$ is a strongly SITN- ring.

Proposition 3.2
If $\mathcal{R}$ ring a strongly SITN- ring. Then for every $a \in \mathcal{R}, a = \vartheta - \tau + n$, for every a in $\mathcal{R}$.
Proof: Suppose $1 - a \in \mathcal{R}$, then $1 - a = \vartheta + \tau + n$, where ϑ is idempotent, τ is tripotent and n is nilpotent that commute with one another, then $1 - \vartheta - \tau - n = a$, since $(1 - \vartheta)$ is idempotent then $a = (1 - \vartheta) - \tau + (-n)$, and $-n$ is a nilpotent.■

Proposition 3.3
If $\mathcal{R}$ is a strongly SITN – ring, then $a^5 - 5a^3 + 4a \in NiL(\mathcal{R})$.
Proof: Take $a \in \mathcal{R}$, then $a = \vartheta - \tau + n$, where ϑ is idempotent, τ is tripotent and n is nilpotent that commute with one another. Thus
$a^3 = (\vartheta - \tau)^3 + 3(\vartheta - \tau)^2 n + 3(\vartheta - \tau)n^2 + n^3$
$a^3 = (\vartheta - \tau)^3 + n'$, where
$n' = 3(\vartheta - \tau)^2 n + 3(\vartheta - \tau)n^2 + n^3$. Observe that
$a^3 = \vartheta^3 - 3\vartheta\tau + 3\vartheta\tau^2 - \tau^3 + n'$
$a^3 = \vartheta - 3\vartheta\tau + 3\vartheta\tau^2 - \tau + n'$,
$a^5 = (\vartheta - \tau)^5 + 5(\vartheta - \tau)^4 n + 10(\vartheta - \tau)^3 n^2 + 10(\vartheta - \tau)^2 n^3 + 5(\vartheta - \tau)n^4 + n^5$
$a^5 = (\vartheta - \tau)^5 + n_1$, where
$n_1 = 5(\vartheta - \tau)^4 n + 10(\vartheta - \tau)^3 n^2 + 10(\vartheta - \tau)^2 n^3 + 5(\vartheta - \tau)n^4 + n^5$,
$a^5 = \vartheta^5 - 5\vartheta^4\tau + 10\vartheta^3\tau^2 - 10\vartheta^2\tau^3 + 5\vartheta\tau^4 - \tau^5 + n_1$,
$a^5 = \vartheta - 5\vartheta\tau + 10\vartheta\tau^2 - 10\vartheta\tau + 5\vartheta\tau^2 - \tau + n_1$,
$a^5 = \vartheta - 15\vartheta\tau + 15\vartheta\tau^2 - \tau + n_1$,
$a^5 - 5a^3 + 4a = \vartheta - 15\vartheta\tau + 15\vartheta\tau^2 - \tau + n_1 - 5\vartheta + 15\vartheta\tau - 15\vartheta\tau^2 + 5\tau - 5n' + 4\vartheta - 4\tau + 4n$
$= n_1 - 5n' + 4n \in NiL(\mathcal{R})$.

Corollary 3.4
If $\mathcal{R}$ is a strongly SITN- ring, then $120 \in NiL(\mathcal{R})$.
Proof: Take a in $\mathcal{R}$, then by **Proposition 3.3** $a^5 - 5a^3 + 4a \in NiL(\mathcal{R})$. Observe that, $3^5 - 5.3^3 + 4.3 = 120$. Then $120 \in NiL(\mathcal{R})$.

Proposition 3.5
If $\mathcal{R}$ is a strongly SITN- ring if $2 \in NiL(\mathcal{R})$, every element of $\mathcal{R}$ is a sum of two idempotents and two units.
Proof: Take $a \in \mathcal{R}$, then $a - 1 = \vartheta + \tau + n$, where ϑ is idempotent, τ is tripotent and n is nilpotent, that commute with one another then $a - 1 = \vartheta + (1 - \tau^2) + (\tau^2 + \tau - 1) + n$, so $a = \vartheta + (1 - \tau^2) + (\tau^2 + \tau - 1) + 1 + n$. Clearly $1 + n$ is unit, $1 - \tau^2$ is idempotent and $\tau^2 + \tau - 1$ is unit, gives $a = \vartheta_1 + \vartheta_2 + \mathcal{U}_1 + \mathcal{U}_2$ where $\vartheta_2 = 1 - \tau^2, \mathcal{U}_1 = \tau^2 + \tau - 1,\ \mathcal{U}_2 = 1 + n$.■

Proposition 3.5
If $2 \in NiL(\mathcal{R})$, then $\mathcal{R}$ is SITN- ring, if and only if $\mathcal{R}$ is a strongly nil clean.
Proof: Suppose $a \in \mathcal{R}, a = \vartheta + \tau + n$, where ϑ is idempotent, τ is tripotent and n is nilpotent, that commute with one another, we may write $a = \vartheta + \tau - \tau^2 + \tau^2 + n$, so $a = \vartheta + \tau^2 + \tau - \tau^2 + n$, note that $(\tau - \tau^2)^2 = \tau^2 - 2\tau + \tau^2 = 2(\tau^2 - \tau)$, as $2 \in NiL(\mathcal{R})$ then $\tau - \tau^2 \in NiL(\mathcal{R})$ then $a = \vartheta + \tau^2 + n' + n$, where $\tau - \tau^2 = n'$, we may write as $a = (\vartheta - \tau^2)^2 + 2\vartheta\tau^2 + n$. Observe that $(\vartheta - \tau^2)^2$ is idempotent, and $2\vartheta\tau^2 + n \in NiL(\mathcal{R})$, say n_1. Therefore $a = (\vartheta - \tau^2)^2 + n_1$.

Conversely, let $\mathcal{R}$ is a strongly nil clean. Take $a \in \mathcal{R}$, then $a = \vartheta + n$, where ϑ is idempotent and n is nilpotent. Since $0 \in Tri(\mathcal{R})$, so $a = \vartheta + 0 + n$. Therefore $\mathcal{R}$ is a strongly SITN- ring.■

Proposition 3.6
If $\mathcal{R}$ ring a strongly SITN- ring with $2 \in U(\mathcal{R})$. Then $\mathcal{R}$ is 5 – good ring.
Proof: Let $a \in \mathcal{R}, a = \vartheta + \tau + n$, where ϑ is idempotent, τ is tripotent and n is nilpotent, that commute with one another, we may write $a = (1 - \vartheta) + (1 - \tau^2) + 2\vartheta - 1 + \tau^2 + \tau - 1 + n$, clearly $1 - \vartheta$ is idempotent, $2\vartheta - 1$ is unit, $1 - \tau^2$ is idempotent and $\tau^2 + \tau - 1$ is unit, gives $a = \vartheta_1 + \vartheta_2 + \mathcal{U}_1 + \mathcal{U}_2 + n$, we may write $a = (\vartheta_1 + 1) + (\vartheta_2 + 1) + \mathcal{U}_1 + \mathcal{U}_2 + (n - 2)$, Now $a = \mathcal{U}_3 + \mathcal{U}_4 + \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_5$, since $1 + \vartheta_1, 1 + \vartheta_2 \in U(\mathcal{R})$ and $n - 2 \in U(\mathcal{R})$, since $2 \in U(\mathcal{R})$. Therefore $\mathcal{R}$ is 5 – good ring.■

Proposition 3.7
If $\mathcal{R}$ is a strongly SITN – ring and $2 \in U(\mathcal{R})$, then every element of $\mathcal{R}$ can be written as a sum of a tripotent and a unit.
Proof: Give $a \in \mathcal{R}$, then $a - 1 = \vartheta + \tau + n$, where ϑ is idempotent, τ is tripotent and n is nilpotent that commute with one another, so $a = \vartheta + 1 + \tau + n$, since $\vartheta + 1$ by (**Lemma 2.11**), is unit then $a = \mathcal{U} + \tau + n$, since $\mathcal{U} + n$ is unit by **Lemma (2.9)**. So $a = t + (\mathcal{U} + n)$ Therefore $\mathcal{R}$ is a sum of a tripotent and unit.■

Proposition 3.8
If $\mathcal{R}$ is a strongly SITN- ring and $2 \in NiL(\mathcal{R})$, then $J(\mathcal{R})$ is a nil ideal.
Proof: Let $\mathcal{R}$ be a strongly SITN- ring and let $2 \in NiL(\mathcal{R})$. Then by **Proposition 3.5** $\mathcal{R}$ is a strongly nil clean ring. Take $a \in J(\mathcal{R})$, then $a = \vartheta + n$, where $\vartheta \in Id(\mathcal{R})$, $n \in NiL(\mathcal{R})$ and $\vartheta n = n\vartheta$. Thus $1 - a$ is a unit. So $1 - a = 1 - \vartheta - n$, the gives $(1 - a) + n = 1 - \vartheta$. Since $1 - a$ is a unit, then $(1 - a) + n \in U(\mathcal{R})$. So $-\vartheta \in U(\mathcal{R})$, since $1 - \vartheta$ is idempotent then $1 - \vartheta = 1$, gives $\vartheta = 0$. Hence $a = 0 + n \in NiL(\mathcal{R})$. Therefore $J(\mathcal{R})$ is a nil ideal of $\mathcal{R}$.■

IV. Special Strongly Sitn Rings

In this section, we consider a special type of strongly SITN ring, we give some of its properties.

Definition 4.1
A ring $\mathcal{R}$ is called a special strongly SITN- ring if for all $b \in \mathcal{R}, b = \vartheta - \tau + n$ and $(\vartheta\tau) = (\vartheta\tau)^2$, for every b in R.

Example (1)
Consider the ring $\mathcal{Z}_4$. So $Id\ (\mathcal{Z}_4) = \{0,1\}$, $Tri(\mathcal{Z}_4) = \{0,1,3\}$and $NiL\ (\mathcal{Z}_4) = \{0,2\}$. Then $\mathcal{Z}_4$ is a special strongly SITN-ring, since $0 = 0 + 0 + 0, (0 \cdot 0) = (0 \cdot 0)^2$, $1 = 1 + 0 + 0, (1 \cdot 0) = (1 \cdot 0)^2$, $2 = 1 + 1 + 0, (1 \cdot 1) = (1 \cdot 1)^2$, $3 = 1 + 0 + 2, (1 \cdot 0) = (1 \cdot 0)^2$.

Proposition 4.2
If $\mathcal{R}$ is a special strongly SITN – ring, then $b^3 - b$ is nilpotent, for every $b \in \mathcal{R}$.
Proof: Let $b \in \mathcal{R}$, so $b = \vartheta - \tau + n$, where ϑ is idempotent, τ is tripotent and n is nilpotent that commute with one another. So $b^3 = (\vartheta - \tau)^3 + 3(\vartheta - \tau)^2 n + 3(\vartheta - \tau)n^2 + n^3$, $b^3 = (\vartheta - \tau)^3 + n'$ where $n' = 3(\vartheta - \tau)^2 n + 3(\vartheta - \tau)n^2 + n^3$, $b^3 = \vartheta - 3\vartheta\tau + 3\vartheta\tau^2 - \tau + n'$, $b^3 = \vartheta - \tau + 3(\vartheta\tau^2 - \vartheta\tau) + n'$, since $\vartheta\tau = (\vartheta\tau)^2$then $\vartheta\tau = \vartheta\tau^2$,then $b^3 = \vartheta - \tau + n'$, given $b^3 - b = \vartheta - \tau + n' - \vartheta + \tau - n$, $b^3 - b = n' - n \in NiL(\mathcal{R})$.Therefore $b^3 - b \in NiL(\mathcal{R})$. ■

Proposition 4.3
If $\mathcal{R}$ is a special SITN – ring, then every nonzero divisor is a unit.
Proof: Take $b \in \mathcal{R}$, then $b = \vartheta - \tau + n$, where ϑ is idempotent, τ is tripotent and n is nilpotent that commute with one another by **Proposition (4.2)**. So $(b^3 - b)^s = 0$, $b^s(b^2 - 1)^s = 0$, since b is a non - zero divisor, then b^s is a non - zero divisor. $(b^2 - 1)^s = 0$, gives $b^2 - 1 \in NiL(\mathcal{R})$, say, n , thus $b^2 = n + 1$ by **Lemma (2.9)**, then $b^2 = \mathcal{U} \in U(\mathcal{R})$, gives $b \in U(\mathcal{R})$. Therefore b is unit.■

Proposition 4.4
Suppose $\mathcal{R}$ is a special strongly SITN – ring and if 2 is nilpotent, then $\mathcal{R}$ is a strongly nil clean ring.
Proof: Let $\mathcal{R}$ be a special strongly SITN – ring, then by **Proposition (4.2)** $\mathcal{R}$ is a strongly 2-nil clean ring. Hence for any $b \in \mathcal{R}$, then $b = \vartheta_1 - \vartheta_2 + n$, where ϑ_1, ϑ_2 are idempotent, and n is a nilpotent that commute with one another. We may write b as $b = (\vartheta_1 - \vartheta_2)^2 + 2\vartheta_1\vartheta_2 + n$. Clearly $(\vartheta_1 - \vartheta_2)^2$ is idempotent, since

$$\begin{aligned}(\vartheta_1 - \vartheta_2)^4 &= \vartheta_1^4 - 4\vartheta_1^3\vartheta_2 + 6\vartheta_1^2\vartheta_2^2 - 4\vartheta_1\vartheta_2^3 + \vartheta_2^4 \\ &= \vartheta_1 - 4\vartheta_1\vartheta_2 + 6\vartheta_1\vartheta_2 - 4\vartheta_1\vartheta_2 + \vartheta_2 \\ &= \vartheta_1 - 2\vartheta_1\vartheta_2 + \vartheta_2 \\ &= (\vartheta_1 - \vartheta_2)^2 .\end{aligned}$$

And since $2\vartheta_1\vartheta_2 \in NiL(\mathcal{R})$, then $2\vartheta_1\vartheta_2 + n \in NiL(\mathcal{R})$, say n_1. Thus $b = (\vartheta_1 - \vartheta_2) + n_1$. Therefore $\mathcal{R}$ is a strongly nil clean.■

Proposition 4.5
If $\mathcal{R}$ is a special strongly SITN – ring. Then $J(\mathcal{R})$ is a nil ideal.
Proof: Take $b \in J(\mathcal{R})$. Then **Proposition (4.2)**, $b^3 - b \in$

$NiL(\mathcal{R})$. Thus there exist a positive integer ℓ, such that $(b^3-b)^\ell=0$. So $b^\ell(b^2-1)^\ell=0$. Since $b\ \in J(\mathcal{R})$, then $b^2\ \in J(\mathcal{R})$, thus b^2-1 is a unit . Hence $b^\ell=0$, this means that b is nilpotent. Therefore $J(\mathcal{R})$ is a nil ideal.■

V. Conclusion

In this work, further properties of a strongly SITN- rings are given, we conclude that the Jacobson radical of a strongly SITN- ring is a nil ideal with every $d \in \mathrm{J}(\mathcal{R}), d^3=0$. In addition, consider a strongly SITN- rings with 2 units, and 2 is nilpotent.

Acknowledgement

The authors are grateful for the support provided by the College of Computer Science and Mathematics at the University of Mosul.

References

[1] W. K. Nicholson, “Lifting idempotents and exchange rings,” Trans. Am. Math. Soc., vol. 229, pp. 269--277, 1977.

[2] W. K. Nicholson, “Strongly clean rings and fitting’s lemma,” Commun. Algebr., vol. 27, no. 8, pp. 3583–3592, 1999, doi: 10.1080/00927879908826649.

[3] D. D. Anderson and V. P. Camillo, “Commutative rings whose elements are a sum of a unit and idempotent,” Commun. Algebr., vol. 30, no. 7, pp. 3327–3336, 2002, doi: 10.1081/AGB-120004490.

[4] V. P. Camillo and H. P. Yu, “Exchange Rings, Units and Idempotents,” Commun. Algebr., vol. 22, no. 12, pp. 4737–4749, 1994, doi: 10.1080/00927879408825098.

[5] J. Han and W. K. Nicholson, “EXTENSIONS OF CLEAN RINGS,” Commun. Algebr., vol. 29, no. 6, pp. 2589–2595, Apr. 2001, doi: 10.1081/AGB-100002409.

[6] A. J. Diesl, “Nil clean rings,” J. Algebr., vol. 383, pp. 197–211, 2013, doi: 10.1016/j.jalgebra.2013.02.020.

[7] T. Koşan, Z. Wang, and Y. Zhou, “Nil-clean and strongly nil-clean rings,” J. Pure Appl. Algebr., vol. 220, no. 2, pp. 633–646, 2016, doi: 10.1016/j.jpaa.2015.07.009.

[8] M. S. Abdolyousefi, N. Ashrafi, and H. Chen, “On Unit Nil-Clean Rings,” Mediterr. J. Math., vol. 16, no. 4, pp. 1–9, 2019, doi: 10.1007/s00009-019-1369-z.

[9] H. Chen and M. Sheibani, “On strongly nil clean rings,” Commun. Algebr., vol. 45, no. 4, pp. 1719–1726, 2017, doi: 10.1080/00927872.2016.1222411.

[10] M. T. Koşan and Y. Zhou, “On weakly nil-clean rings,” Front. Math. China, vol. 11, no. 4, pp. 949–955, 2016, doi: 10.1007/s11464-016-0555-6.

[11] H. Chen and M. Sheibani, “Strongly 2-nil-clean rings,” J. Algebr. Its Appl., vol. 16, no. 09, p. 1750178, Sep. 2017, doi: 10.1142/S021949881750178X.

[12] H. Chen and M. S. Abdolyousefi, “Strongly 2-nil-clean rings with involutions,” Czechoslov. Math. J., vol. 69, no. 2, pp. 317–330, 2019, doi: 10.21136/CMJ.2018.0291-17.

[13] A. N. Abyzov, “Strongly q-Nil-Clean Rings,” Sib. Math. J., vol. 60, no. 2, pp. 197–208, 2019, doi: 10.1134/S0037446619020022.

[14] D. M. Burton, Introduction to Modern Abstract Algebra. Addison-Wesley, 1967.

[15] M. S. Abdolyousefi and H. Chen, “Sums of tripotent and nilpotent matrices,” Bull. Korean Math. Soc., vol. 55, no. 3, pp. 913–920, 2018, doi: 10.4134/BKMS.b170384.

[16] G. Călugăreanu, “Tripotents: A class of strongly clean elements in rings,” Analele Stiint. ale Univ. Ovidius Constanta, Ser. Mat., vol. 26, no. 1, pp. 69–80, 2018, doi: 10.2478/auom-2018-0003.

[17] J. J. Watkins, Topics in commutative ring theory, vol. 45, no. 08. Princeton University Press, 2009. doi: 10.5860/CHOICE.45-4443.

[18] T. H. Lam, A first course in non- commutative rings, vol. 131. Springer, 1991. doi: 10.1017/s0013091500019003.

[19] D. Zelinsky, “Every Linear Transformation is a Sum of Nonsingular Ones,” Proc. Am. Math. Soc., vol. 5, no. 4, p. 627, Aug. 1954, doi: 10.2307/2032048.