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The Moments for Some Hyperbolic Stochastic Differential Equations

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Abstract

This paper investigates moments for Ito's integral formula involving general form of hyperbolic stochastic functions, hyperbolic stochastic functions, which combine the deterministic structure of hyperbolic functions with stochastic elements such as noise and random fluctuations. The formulation of these functions through stochastic differential equations (SDEs) is discussed, along with methods for solving and analyzing such equations., with a specific focus on stochastic processes governed by Brownian motion. By utilizing Ito's integral calculus, we derive moments for integrals of hyperbolic sine and cosine functions. These results extend the classical Ito integral formula to hyperbolic trigonometric functions, providing a comprehensive understanding of their stochastic properties. These outcomes have applications in various fields such as mathematical finance, physics, and engineering, where modeling using hyperbolic functions under random effects is prevalent. Some examples are given to illustrate the theoretical results.

Keywords:
Moments, hyperbolic Stochastic differential equation, integral Ito- formula, Expectation, transformation Method.

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I. Introduction

The Ito's integral formula, As we know is the fundamental method to find the moments , after we find the stochastic differential equations for the terms of hyperbolic stochastic functions, and using some general relations and formulas for hyperbolic trigonometric functions .we use the ito-formula , then we applied the definition of the moment, which is the expected value of the m-order and can be solved using Ito's integral formula .

$$\begin{cases} dX(t) = F\big(t, \mathcal{X}(t)\big)dt + G(t, \mathcal{X}(t))d\mathcal{w}(t) \\ X(0) = X_0 \end{cases}$$

$\mathcal{w}(t)$ *is the* Wiener process known as Brownian motion, where $F(.)$is the drift coefficient and $G(.)$is the diffusion coefficient

As we know, the expectation value of the stochastic part is zero

$$E[\int_0^t X(s)dw(s)] = 0 \qquad (1)$$

The first moment is the expected value, i.e.

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \qquad (2)$$

To find the variance, we need the to find$\big(E(X)\big)^2$, and to find the kth moments we need to find $\big(E(x^k)\big)^2$

II. Prerequisites and Results

In this research, we will find the Variance of the differential equations with the limits of hyperbolic stochastic functions using the general and special formulas found by Ito's formula, and to find the Variance, we will solve the equations to reach the first expectation and the second expectation of the required function and apply the fundamental law to obtain the desired results

III. The Ito's Formula [11]

Suppose equation $dX(t) = F(t, X(t))dt + G(t, X(t))dw(t)$ be given, If there exist real-valued function $U(t, Y(t))$ and has continuous partial derivatives.
$\frac{\partial y}{\partial t}, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}$, $t \in [0, T]$ and $x \in R$, then:

$$dy = \left(\frac{dy}{dt} + F\ \frac{dy}{dx} + \frac{1}{2}\frac{d^2y}{dx^2}\ G^2\right)dt + \frac{dy}{dx}\ G\ dw_t \qquad (3)$$

is called (Ito's integrated Formula).

Definition 1. (Wiener Process) [2][3]
Where $w(t)$ is a continuous-time stochastic wiener process (Brownian motion) over the interval $[0,\mathcal{T}]$, satisfying: the parameters listed below:
1: $w(0) = 0$
2: $If\ t, s \geq 0, then\ w(t) - w(s)$ is normally distributed with zero mean and variance $|t - s|$.
3: For $0 \leq s < t < k < j \leq \mathcal{T}$, $w(t)$- $w(s)$ and $w(j) - w(k)$ are independent increments.

Definition 2. (Stochastic Integral) [4]
A stochastic integral is an integral which is defined as a sum more then integration and it is increased by the rise in time on the Wiener process trajectory. That is :

$$\int_b^c a(t)dW(t) = \sum_{i=0}^{k-1} \delta_i (W_{t_i+1} - W_{t_i}). \qquad (4)$$

Definition 3. (Random Variable) [1]

A random variable is a real-valued function $x(\omega), \omega \in \Omega$that is measurable from sample space to the real line R.

Definition 4. (Continuous Random Variable) [5]

is said $x(\omega)$ to be a continuous random variable if there is at least a function $f(x)$such that:

$$f(x) \geq 0, \int_{-\infty}^{\infty} f(x)dx = 1, F(x) = \int_{-\infty}^{x} f(u)du \qquad (5)$$

Definition 5. (Expected Value) [1][7]
If x is a random variable defined in the probability space $(\Omega, \mathcal{F}, P)$, then the expected value or average value of μ from X is :

$$\mu = E(X) = \int X(\omega)dP(\omega) \qquad (6)$$

Provided there is an integral, i.e., an average of x in the entire probability space, for continuous random variables above R, the available value is :

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \qquad (7)$$

The expected value of the random variables is of the form

$$E[x] = \sum_i x_i f(x_i) \qquad (8)$$

Definition 6. (Variance) [6]

Variance is a measure of the dispersion or difference of data around the mean μ , given by:

$$Var(Y) = E((Y - \mu)^2) = E(Y^2) - \mu^2 \qquad (9)$$

The variance is denoted for simplicity byσ^2 and the positive square root of σ is called the standard deviation (standard deviation) of the random variable Y, where $\sigma = \sqrt{\mathrm{Var(Y)}}$.

Definition 7. (The Kth-order moment) [5]
The order -K moments of a continuous random variable are defined by the following formula:

$$E(X^k) = \int_{-\infty}^{\infty} x^k f(x)dx \qquad (10)$$

where $f(x)$ is the probability density function and the moment of order -k for the discrete random variable of the formula:

$$E(X^k) = \sum_i {x_i}^k p(x_i) \qquad (11)$$

Definition 8. (Brownian motion)[1]
Brownian motion is a continuous-time stochastic process $\{w(t), t \geq 0\}$ or
Wiener process The Wiener process is generally characterized by the following properties:
1. $w\_0 = 0\ \ w.p.1\ \ E(w(t))\ \ = 0$,

2. $Var\big(w(t)-w(s)\big)=|t-s|,\ \ 0\le s\le t.$

Definition 9. (moments)[8]
The term "moment" refers to a quantitative measure related to the shape of a function or a set of points. Moments are used extensively in various fields such as statistics, physics, and engineering to describe the characteristics of distributions and shapes,so Types of moments

First Moment (Mean) The first moment about the origin is the mean μ of the distribution:

$$\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx \tag{12}$$

Second Moment (Variance) The second central moment is the variance σ^2

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2\, f(x) dx \tag{13}$$

IV. Method of Solution [12]

let

$$dx_t = f(t, x(t))dt + g(t, x(t))dw_t \tag{14}$$

Into a linear for $y(t)$ stochastic differential equation

$$dy(t) = (f_1\, y(t) + f_2)dt + (g_1\, y(t) + g_2)dw(t) \tag{15}$$

Where f_1, f_2, g_1, g_2 If there exist real-valued function and has continuous partial derivatives
the solution of this equation is given by

$$y_1 = \Phi_1[y_0 + (f_2 - g_1 g_2)\int_0^t \Phi^{-1}\, ds + g_2 \int_0^t \Phi^{-1}\, dw(s)] \tag{16}$$

$$\Phi_1 = \exp\left[f_1 t - \frac{1}{2} g_1^2 t + g_1 w\right] \tag{17}$$

V. (Transformation Method):[9][10]

And the analytic solution Transformation Method, By using into formula, that we have a function $F(.) = y$ and assuming it satisfies Ito's formula, that is, the partial derivatives exist and are continuous,

$$dY(t,x) = \left(Y_t(t,x) + \frac{dY(t,x)}{dx} f(t,w_t) + \frac{1}{2}\frac{d^2Y(t,x)}{dx^2} g^2(t,w_t)\right)dt + \left(\frac{dY(t,x)}{dx} g(t,w_t)\right)dw(t) \tag{18}$$

Assuming there is a function

$$Y(t,x) = p(t)x + q(t) \tag{19}$$

By finding the partial derivatives of $y(t,x)$

$$\frac{dY}{dt} = Y_t(t,x) = \frac{dp}{dt}x + \frac{dq}{dt}\ ;\ \frac{dY(t,x)}{dx} = p(t)$$

$$;\ \frac{d^2Y(t,x)}{dx^2} = 0$$

Substituting the first part of the equation (18)

$$Y_t(t,x) + \frac{dY(t,x)}{dx} f\,(t,w_t) + \frac{1}{2}\frac{d^2Y(t,x)}{dx^2} g^2(t,w_t) = \frac{dp}{dt}x + \frac{dq}{dt} + p(t)f(t,w) \tag{20}$$

We choose $p(t)$ so that we get the coefficients of dt, dw_t which is a function of t only

$$dY(t,x) = dw_t \tag{21}$$

Using the stochastic integration of the right-hand side, we find the solution using the transformation method.

VI. Results

In this paragraph we will find the moment for hyperbolic $y = \sinh(tx)$, $y = \sinh(t)$ *and* $y = \sinh(x_t)$ to find the moment(mean $= E(y), var = E(y^2) - (E(y))^2$ the k_th moment $= E(y^m)$)

And Suppose

$$dy_t = F(t,x_t)dt + G(t,x_t)dw_t, \text{ be given}$$

1) Let the function y_t be a function of x and t , that is ($\boldsymbol{sinh(tx)}$) and satisfy the following ito- stochastic differential equation

$$d(\sinh(tx)) = \left(\frac{dy}{dt} + F\,\frac{dy}{dx} + \frac{1}{2}\frac{d^2y}{dx^2}\,G^2\right)dt + \frac{dy}{dx}\,G\,dw$$

$$dy = \left(x\,cosh(tx) + F(t\,cosh(tx)) + \frac{1}{2}G^2(t^2\,sinh(tx))\right)dt + G(t\cosh(tx))dw_t \tag{22}$$

Integrating both sides, from 0 to t we get

$$\sinh(tx) - \sinh(0) =$$

$$\int_0^t (x\cosh(sx) + F\,(s\cosh(sx)) + \frac{1}{2}G^2(s^2\sinh(sx)))ds + \int_0^t G(s\cosh(sx))dw_s \quad (23)$$

$$\sinh(tx) = \sinh(0) + \int_0^t (x\cosh(sx) + F\,(s\cosh(sx)) + \frac{1}{2}G^2(s^2\sinh(sx)))ds + \int_0^t G(s\cosh(sx))dw_s \quad (24)$$

a) The (expected value) of equation (24)

$$E(\sinh(tx)) = \mathrm{E}\int_0^t (x\cosh(sx) + F\,(s\cosh(sx)) + G^2(s^2\sinh(sx)))ds + E\int_0^t G(s\cosh(sx))dw_s \quad (25)$$

$$E(\sinh(tx)) = (\mathrm{E}\int_0^t (x\cosh(sx)) + E\int_0^t F\,(s\cosh(sx)) + \frac{1}{2}E\int_0^t G^2(s^2\sinh(sx)))\,ds + E\int_0^t G(s\cosh(sx))dw_s \quad (26)$$

Sice $E\{dw\} = 0$ then

$$E(\sinh(tx)) = (\mathrm{E}\int_0^t (x\cosh(sx)) + E\int_0^t F(s\cosh(sx)) + \frac{1}{2}E\int_0^t G^2(s^2\sinh(sx)))\,ds \quad (27)$$

$$dy = (2x\,sinh(tx)\cosh(tx) + F(2t\sinh(tx)\ cosh(tx)) + \frac{1}{2}G^2(2t(t\ sinh(tx)\sinh(tx) + t\cosh(tx)\cosh(tx))))dt + G(2t\sinh(\mathrm{tx})\cosh(tx))dw_t \quad (28)$$

Integrating both sides, we get

$$\sinh^2(tx) - \sinh^2(0) = \int_0^t (2x\,sinh(sx)\cosh(sx) + F(2s\,sinh(sx)\ cosh(sx)) + \frac{1}{2}G^2(2s(s\sinh(sx)\sinh(sx) + s\,cosh(sx)\cosh(sx))))ds + G\int_0^t (2s\,sinh(sx)\,cosh(sx))\,dw_s \quad (29)$$

$$\sinh^2(tx) = sinh^2(0) + \int_0^t (2x\,sinh(sx)\,cosh(sx) + F(2s\,sinh(sx)\ cosh(sx)) + \frac{1}{2}G^2(2s(s\,sinh(sx)\ sinh(sx) + s\,cosh(sx)\cosh(sx))))ds + G\int_0^t (2s\,sinh(sx)\,cosh(sx))\,dw_s \quad (30)$$

Taking the expectation to eq.30

$$\mathrm{E}(\sinh^2(tx)) = E\int_0^t (2x\sinh(sx)\cosh(sx) + F(2s\sinh(\mathrm{sx})\ cosh(sx)) + \frac{1}{2}G^2(2s(s\ sinh(sx)\ sinh(sx) + s\,cosh(sx)\ cosh(sx)))))\,ds + E\int_0^t G\,(2s\,sinh(sx)\,cosh(sx))\,dw_s \quad (31)$$

$$\mathrm{E}(\sinh^2(tx)) = E\int_0^t (2x\,sinh(sx)\cosh(sx)) + E\int_0^t F(2s\,sinh(sx)\ cosh(sx)) + \frac{1}{2}E\int_0^t G^2(2s(s\,sinh(sx)\ sinh(sx) + s\,cosh(sx)\cosh(sx))))\,ds + E\int_0^t G\,(2s\,sinh(sx)\,cosh(sx))\,dw_s \quad (32)$$

Wherein $E\{dw\} = 0$ then

$$\mathrm{E}(\sinh^2(tx)) = E\int_0^t (2x\,sinh(sx)\cosh(sx)) + E\int_0^t F(2s\,sinh(sx)\ cosh(sx)) + \frac{1}{2}E\int_0^t G^2(2s(s\ sinh(sx)\ sinh(sx) + cosh(sx)\,cosh(sx)))\,ds \quad (33)$$

From equation (27) and equation (33), we get the moments

$$V(\sinh(tx)) = E(\sinh^2(tx)) - \left(E(\sinh(tx))\right)^2 \quad (34)$$

$$= (\mathrm{E}\int_0^t (2x\,sinh(sx)\cosh(sx)) + \mathrm{E}\int_0^t (F\ 2s\,sinh(sx)\ cosh(sx) + \frac{1}{2}E\int_0^t G^2(2s(s\ sinh(sx)\,sinh(sx)) + s\,cosh(sx)\ cosh(sx)))ds) - (E\int_0^t (x\,cosh(sx)) + E\int_0^t F(s\,cosh(sx)) + \frac{1}{2}E\int_0^t G^2(s^2\,sinh(sx)))\,ds)^2 \quad (35)$$

c) The m-th moments $y = (\sinh(tx))^n$

$$dy = \left(\frac{dy}{dt} + F\frac{dy}{dx} + \frac{1}{2}\frac{d^2y}{dx^2}G^2\right)dt + \frac{dy}{dx}Gdw \quad (36)$$

So the general formal

$$d(sinh(tx))^n = (n\,sinh(tx)^{n-1}\,cosh(tx)\,F +$$
$$\frac{1}{2}G^2(n^2\,sinh(tx)^{n-2}\,cosh(tx)^2 - n\,sinh(tx)^n - n$$
$$*\,sinh(tx)^{n-2}\,cosh(tx)^2))dt$$
$$+G(n\,sinh(tx)^{n-1}\,cosh(tx))dw_t \quad (37)$$

Integrating both sides, we get

$$(sinh(tx))^n - (\sinh(0))^n = \int_0^t (\,n\,sinh(sx)^{n-1}\,cosh(sx)\,F$$
$$+\frac{1}{2}G^2(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2 - n\,sinh(sx)^n$$
$$-n\,sinh(sx)^{n-2}\,cosh(sx)^2))ds$$
$$+\int_0^t G\,(n\,sinh(sx)^{n-1}\,cosh(sx))dw_s \quad (38)$$

$$(sinh(tx))^n = (sinh(0))^n \int_0^t n\,sinh(sx)^{n-1}\ cosh(sx)\,F$$
$$+\int_0^t \frac{1}{2}G^2(n^2\,sinh(sx)^{n-2}\ cosh(sx)^2 - n\,sinh(sx)^n$$
$$-n\,sinh(sx)^{n-2} * cosh(sx)^2)ds + \int_0^t G(n\,sinh(sx)^{n-1}$$
$$cosh(sx))dw_s \quad (39)$$

$$(sinh(tx))^n = \int_0^t n\,sinh(sx)^{n-1}\ cosh(sx)\,F + \frac{1}{2}G^2$$
$$(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2 - n\,sinh(sx)^n - n\,sinh(sx)^{n-2}$$
$$*\,cosh(sx)^2)ds + \int_0^t G\,(n\,sinh(sx)^{n-1}\,cosh(sx))dw_s \quad (40)$$

The expected value of equation (40)

$$E(sinh(tx))^n = E\int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)\,F + E$$
$$\int_0^t \frac{1}{2}G^2(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2 - n\,sinh(sx)^n$$
$$-n\,sinh(sx)^{n-2}\,cosh(sx)^2)ds +$$
$$E\int_0^t G(n\,sinh(sx)^{n-1}\,cosh(sx))\,dw_s \quad (41)$$

since $E\{dw\} = 0$ then

$$E(sinh(tx))^n = E\int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)\,F +$$
$$E\int_0^t \frac{1}{2}G^2(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2 - n\,sinh(sx)^n$$
$$-n\,sinh(sx)^{n-2}\,cosh(sx)^2)ds \quad (42)$$

2) Let the function y_t be a function of x only, i.e. [$y = (\sinh(x_t))^n$], by using ito formula , we get

$$d(\sinh(x_t))^n = [n\,cosh(x_t)\ (sinh(x_t)\ )^{n-1}\,F + \frac{1}{2}G^2$$
$$[n(n-1)(cosh(x_t))^2\ (sinh(x_t)\ )^{n-2} + n\ (sinh(x_t)\ )^n]]dt$$
$$+G[n\,cosh(x_t)\ (sinh(x_t)\ )^{n-1}]dw(t) \quad (43)$$

Integrating both sides, we get

$$(sinh(x_t))^n - (\sinh(0))^n \int_0^t (n\,cosh(x_s)\ (sinh(x_s)\ )^{n-1}\,F$$
$$+\frac{1}{2}G^2[n(n-1)\ (cosh(x_s))^2(sinh(x_s)\ )^{n-2} +$$
$$n(sinh(x_s)\ )^n])ds +$$
$$\int_0^t G\ \ [n\,cosh(x_s)\ (sinh(x_s)\ )^{n-1}]dw(s) \quad (44)$$

$$(sinh(x_t))^n = (sinh(0))^n + \int_0^t n\,cosh(x_s)\ (sinh(x_s)\ )^{n-1}$$
$$F\ ds + \int_0^t \frac{1}{2}G^2\ [n(n-1)\ (cosh(x_s))^2\ (sinh(x_s)\ )^{n-2} + n$$
$$(sinh(x_s)\ )^n]ds + \int_0^t G\ \ [n\,cosh(x_s)\ (sinh(x_s)\ )^{n-1}]dw(s)$$
$$,(sinh(0))^m = 0 \quad (45)$$

The expected value of equation (45)

$$E(sinh(x_t))^n = E\int_0^t n\,cosh(x_s)\,(sinh(x_s))^{n-1}\,F\,ds +$$

$$E\int_0^t \frac{1}{2}G^2\,[n(n-1)\,(cosh(x_s))^2\,(sinh(x_s))^{n-2} + n(sinh(x_s))^n]$$

$$ds + E\int_0^t G\,[n\,cosh(x_s)\,(sinh(x_s))^{n-1}]\,dw(s) \qquad (46)$$

Wherein $E\{dw\} = 0$ then

$$E(sinh(x_t))^n = E\int_0^t n\,cosh(x_s)\,(sinh(x_s))^{n-1}\,F\,ds$$

$$E\int_0^t \frac{1}{2}G^2\,[n(n-1)\,(cosh(x_s))^2\,(sinh(x_s))^{n-2}$$

$$+n\,(sinh(x_s))^n]\,ds \qquad (47)$$

3) Let the function y_t be a function of x only, i.e. $[y = (\sinh(t))^n]$, by using ito formula , we get

$$d(\sinh(t))^n = [n(sinh(t))^{n-1}cosh\,(t)]dt \qquad (48)$$

Integrating both sides, we get

$$(sinh(t))^n - (\sinh(0))^n = \int_0^t [n(sinh(s))^{n-1}\cosh(s)]ds \quad (49)$$

$$(sinh(t))^n = (sinh(0))^n + \int_0^t [\,n(sinh(s))^{n-1}\,cosh\,(s)]ds \quad (50)$$

The (expected value) of equation (50)

$$E(sinh(t))^n = E\int_0^t [n(sinh(s))^{n-1}cosh\,(s)]ds \qquad (51)$$

VII. Examples we explain the above method by simple examples with different coefficients of stochastic equation (F and G)

Example (1):
Let we have the following stochastic different equation $dx = dt$ there $(F = 1, G = 0)$, and suppose $y = \sinh(tx)$
Solution:
From eq (27) after substitute with the values of F and G, we get

$$E(\sinh(tx)) = (\mathrm{E}\int_0^t (x\cosh(sx)) + E\int_0^t F(\,s\cosh(sx))$$

$$+\frac{1}{2}E\int_0^t G^2(s^2\sinh(sx)))\,ds \qquad (52)$$

Since $F = 1, G = 0$

$$E(\sinh(tx)) = E(\int_0^t (x\cosh(sx)) + E\int_0^t (\,s\cosh(sx)))\,ds \quad (53)$$

$$E(\sinh(tx)) = \sinh(tx) + \frac{t\sinh\,(tx)}{x} + \frac{-\cosh(tx)+1}{x^2} \qquad (54)$$

The variance In order we need to find $E[\sinh^2(tx)]$ from equation(33)

$$\mathrm{E}(\sinh^2(tx)) = E\int_0^t (2x\,sinh(sx)\cosh(sx)) +$$

$$E\int_0^t F(2s\,sinh(sx)\;\;cosh(sx)) + \frac{1}{2}E\int_0^t G^2\,(2s(s\,sinh(sx)$$

$$sinh(sx) + s\,cosh(sx)\,cosh\,(sx))ds \qquad (55)$$

Substitute $F = 1, G = 0$

$$\mathrm{E}(\sinh^2(tx)) = (E\int_0^t (2x\,sinh(sx)\,cosh(sx)) +$$

$$E\int_0^t (2s\,sinh(sx)\;\;cosh(sx))\;\;ds \qquad (56)$$

$$E(sinh^2(tx)) = \frac{\cosh\,(2tx)}{2} + \frac{t}{2x}\cosh(2tx) - \frac{1}{4x^2}$$

$$\sinh(2tx) \qquad (57)$$

Then from equation (54) and equation (57), we get

$$V(\sinh(tx)) = E(\sinh^2(tx)) - \left(E(\sinh(tx))\right)^2 \qquad (58)$$

$$= \left(\frac{\cosh(2tx)}{2} + \frac{t}{2x}\cosh(2tx) - \frac{1}{4x^2}\sinh(2tx)\right) -$$

$$\left(sinh(tx) + \frac{t\sinh(tx)}{x} + \frac{-cosh(tx)+1}{x^2}\right)^2 \qquad (59)$$

$V(\sinh(tx)) =$

$$\frac{(-2t^2x^2 - 2tx^3 - 2)\cosh(tx)^2 + (4 + (4tx + 3x^2)\sinh(tx))\cosh(tx) - 4x(t+x)\sinh(tx) + 2t^2x^2 + 3tx^3 + x^4 - 2}{2x^4} \quad (60)$$

Let the m-th moments $y = (\sinh(tx))^n$ there
So the general formal

$$d(sinh(tx))^n = (n\,sinh(tx)^{n-1}\,cosh(tx)\,F + +\frac{1}{2}G^2$$

$$(n^2\,sinh(tx)^{n-2}\,cosh(tx)^2 - n\,sinh(tx)^n - n\,sinh(tx)^{n-2}$$

$$cosh(tx)^2))\,dt + G(n\,sinh(tx)^{n-1}\,cosh(tx))dw_t \quad (61)$$

Integrating both sides, we get

$$(sinh(tx))^n - (\sinh(0))^n = \int_0^t (n\,sinh(sx)^{n-1}\,cosh(sx)\,F$$

$$+\frac{1}{2}G^2(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2 - n\,sinh(sx)^n - n$$

$$sinh(sx)^{n-2}\,cosh(sx)^2))ds + \int_0^t G\,(n\,sinh(sx)^{n-1}$$

$$cosh(sx))dw_s \quad (62)$$

Since $F = 1, G = 0$

$$(sinh(tx))^n = (sinh(0))^n + \int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)$$

$$,\sinh(0) = 0 \quad (63)$$

$$(sinh(tx))^n = \int_0^t n\,sinh(sx)^{n-1}\,cosh(sx) \quad (64)$$

The expected value of equation (64)

$$E(sinh(tx))^n = E\int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)\;\;ds \quad (65)$$

Example (2):
Let we have the following stochastic different equation $dx = dt + dw(t)$ that is $(F = 1, G = 1)$, and suppose $y = \sinh(tx)$

Solution:
From equation (27) after substitute with the values of F and G, we get

$$E(\sinh(tx)) = (E\int_0^t (x\cosh(sx)) + E\int_0^t F(\,s\cosh(sx)) +$$

$$\frac{1}{2}\,E\int_0^t G^2(s^2\sinh(sx)))\,ds \quad (66)$$

$F = 1, G = 1$

$$E(\sinh(tx)) = (\mathrm{E}\int_0^t (x\cosh(sx)) + E\int_0^t (\,s\cosh(sx)) +$$

$$\frac{1}{2}\,E\int_0^t (s^2\sinh(sx)))\,ds \quad (67)$$

$$E(sinh(tx)) = \sinh(tx) + \frac{t\sinh(tx)}{x} - \frac{\cosh(tx)+1}{x^2} +$$

$$\frac{x^2\,t^2\cosh(tx) - 2(tx\sinh(tx) - \cosh(tx)) - 2}{2x^3} \quad (68)$$

The variance In order we need to find $\mathrm{E}[\sinh^2(tx)]$ from equation(33)

$$\mathrm{E}(\sinh^2(tx)) = E\int_0^t (2x\,sinh(sx)\cosh(sx)) + E$$

$$\int_0^t F(2s\,sinh(sx)\;\;cosh(sx)) \;+\frac{1}{2}E\int_0^t G^2$$

$$\big(2s(s\;\;sinh(sx)\,sinh(sx) + s\,cosh(sx)\,cos\,h(sx))\big)ds \quad (69)$$

$F = 1, G = 1$

$$\mathrm{E}(\sinh^2(tx)) = E\int_0^t (2x\,sinh(sx)\cosh(sx)) + E\int_0^t (2s(s$$

$$sinh(sx)\,sinh(sx) + s\,cosh(sx)\,cosh\,(sx)))\,ds \quad (70)$$

$$\mathrm{E}(\sinh^2(tx)) = \frac{1}{2}(\cosh(2tx) - 1) + \frac{2tx\cosh(2tx) - \sinh(2tx)}{4x^2}$$

$$+\int_0^t s(s\sinh(sx)^2 + s\cosh(sx)^2)ds \quad (71)$$

From equation (68) and equation (71), we get the moment

$$V(\sinh(tx)) = E(\sinh^2(tx)) - (E(\sinh(tx)))^2$$

$$V(\sinh(tx)) = (\frac{1}{2}(cosh(2tx) - 1) + \frac{2tx\,cosh(2tx) - \sinh(2tx)}{4x^2}$$

$$+\frac{1}{2}\int_0^t s(s\,sinh(sx)^2 + s\,cosh(sx)^2)\,ds) - (sinh(tx)$$

$$+\frac{t\sinh(tx)}{x}-\frac{cosh(tx)+1}{x^2}+$$

$$\frac{x^2\,t^2\,cosh(tx)-2(tx\,sinh(tx)-cosh(tx))-2}{2x^3})^2 \quad (72)$$

$$V(\sinh(tx))=\frac{1}{4x^6}(4(\int_0^t s(2s\cosh(sx)^2-s)ds)x^6+$$

$$(2x^6+2txx^4)\cosh(2tx)-\sinh(2tx)\,x^4+(-4x^6-8tx^5$$

$$+(-t^4-4t^2)x^4+(-4t^2+8tx)x^3+(-4t^2+8ttx-4)x^2$$

$$-8x-4tx^2-4)\cosh(tx)^2-4(x^2t^2+2x+2)$$

$$\left((x^2t+x^3-tx)\sinh(tx)+x-1\right)\cosh(tx)$$

$$-8(x-1)(x^2t+x^3-tx)\sinh(tx)+2x^6+8tx^5+4t^4x^2$$

$$-8ttx^3+(-8txx-4)x^2+8x+4tx^2-4) \quad (73)$$

Let m-th moments $y=(\sinh(tx))^n$

So the general formal

$$d(sinh(tx))^n=(n\,sinh(tx)^{n-1}\,cosh(tx)\,F+\frac{1}{2}G^2(n^2$$

$$sinh(tx)^{n-2}\,cosh(tx)^2-n\,sinh(tx)^n-n\,sinh(tx)^{n-2}$$

$$cosh(tx)^2)dt+G(n\,sinh(tx)^{n-1}\,cosh(tx))dw_t \quad (74)$$

Integrating both sides, we get

$$(sinh(tx))^n-(\sinh(0))^n+\int_0^t(\,n\,sinh(sx)^{n-1}\,cosh(sx)\,F$$

$$+\frac{1}{2}G^2(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2-n\,sinh(sx)^n$$

$$-n\,sinh(sx)^{n-2}\,cosh(sx)^2))\,ds+\int_0^t G\,(n\,sinh(sx)^{n-1}$$

$$cosh(sx))dw_s \quad (75)$$

Since $(F=1,G=1)$

$$(sinh(tx))^n=(sinh(0))^n+\int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)$$

$$+\int_0^t\frac{1}{2}(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2-n\,sinh(sx)^n$$

$$-n\,sinh(sx)^{n-2}*cosh(sx)^2)ds+\int_0^t(n\,sinh(sx)^{n-1}$$

$$cosh(sx)\,dw_s\ ,\sinh(0)=0 \quad (76)$$

$$(sinh(tx))^n=\int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)+\int_0^t\frac{1}{2}$$

$$(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2-n\,sinh(sx)^n-n\,sinh(sx)^{n-2}$$

$$*\,cosh(sx)^2)ds+\int_0^t(n\,sinh(sx)^{n-1}\,cosh(sx))\,dw_s \quad (77)$$

The expected value of equation (77)

$$E(sinh(tx))^n=E\int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)+E\int_0^t\frac{1}{2}$$

$$(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2-n\,sinh(sx)^n-n\,sinh(sx)^{n-2}$$

$$cosh(sx)^2)ds+E\int_0^t(n\,sinh(sx)^{n-1}\,cosh(sx))\,dw_s \quad (78)$$

since $E\{dw\}=0$ then

$$E(sinh(tx))^n=E\int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)+E\int_0^t\frac{1}{2}$$

$$(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2-n\,sinh(sx)^n-n\,sinh(sx)^{n-2}$$

$$cosh(sx)^2)ds \quad (79)$$

Example (3): let we have the following stochastic different equation $dx(t)=dw(t), x=w$ that is $(F=0,G=1)$, and suppose $y=\sinh(tx)$
Solution: from equation (22) after substitute with the values of F and G, we get

$$dy=\left(x\,cosh(tx)+F\,(t\,cosh(tx))+\frac{1}{2}G^2(t^2\,sinh(tx))\right)$$

$$+G(t\cosh(tx))dw_t \quad (80)$$

And with compensation $F=0,G=1$

$$dy=\left(x\,cosh(tx)\ +\frac{1}{2}(t^2\,sinh(tx))\right)+(t\cosh(tx))dw_t \quad (81)$$

Applying the general formula for Ito's

$$dF=\left(F_t+fF_x+\frac{1}{2}g^2F_{xx}\right)dt+(gF_x)dw_t \quad (82)$$

then

$$f=x\,cosh(tx)\ +\frac{1}{2}(t^2\,sinh(tx)) \quad (83)$$

$$g = t\cosh(tx) \quad (84)$$

taking the values of f, g

$$dF = (F_t + (x\,cosh(tx) + \frac{1}{2}t^2 \sinh(tx))\,F_x$$

$$+\frac{1}{2}(t^2\,cosh^2(tx)\,F_{xx})dt + (t\,cosh(tx)\,F_x)dw_t \quad (85)$$

Let $p(t) = t\,cosh(tx)\,F_x$

$$F_x = \frac{p(t)}{tcosh\,(tx)} \quad (86)$$

As $p(t)$ is an arbitrary function, we assume it is equal to $p(t) = t^2$

$$F_x = \frac{t^2}{t\cosh\,(tx)} = t\,\mathrm{sech}(tx) \quad (87)$$

Integrating with respect to x, we get

$$F = \ln|\mathrm{sech}(tx) + \tanh(tx)| \quad (88)$$

$$F_{xx} = t^2\,\mathrm{sech}(tx)\tanh(tx) \quad (89)$$

$$F_t = x\,\mathrm{sech}(tx) \quad (90)$$

Substituting the value of F_x, F_{xx}, F_t into equation 7

$$dF = (xsec\,h(tx) + \left(x\,cosh(tx) + \frac{1}{2}t^2\sinh(tx)\right)$$

$$*\frac{t}{\cosh(tx)} + \frac{1}{2}(t^2\,cosh^2(tx)) * t^2\,sech(tx)\,tanh\,(tx))$$

$$dt + \left(t\,cosh(tx) * \frac{t}{cosh(tx)}\right)dw_t \quad (91)$$

$$d(\sinh(tx)) = (xsec\,h(tx) + xt + \frac{1}{2}t^3\,tanh(tx)$$

$$+\frac{1}{2}(t^4\,sinh(tx)))dt + (t^2)dw_t \quad (92)$$

Taking the integral

$$\sinh(tx) = \sinh(0) + \int_0^t (\,xsec\,h(sx) + xs + \frac{1}{2}s^3$$

$$tanh(sx) + \frac{1}{2}(s^4\,sinh(sx)))ds + \int_0^t (s^2)\,dw_s \quad (93)$$

Since every $x = w$

$$\sinh(tx) = \int_0^t (\,w\,sec\,h(sw) + ws + \frac{1}{2}s^3\,tanh(sw)$$

$$+\frac{1}{2}(s^4\,sinh(sw)))\,ds + \int_0^t (s^2)\,dw_s \quad (94)$$

The expectation for (94)

$$\mathrm{E}(\sinh(tx)) = E\int_0^T w\,sec\,h(sw)\,ds + E\int_0^t ws\,ds +$$

$$E\int_0^t \frac{1}{2}s^3\,tanh(sw)\,ds + E\int_0^t \frac{1}{2}(s^4\,sinh(sw))\,ds$$

$$+E\int_0^t (s^2)\,dw_s \quad (95)$$

Wherein $E\{dw\} = 0$ then

$$E(\sinh(tx)) = E\int_0^T w\,sec\,h(sw)\,ds + E\int_0^t ws\,ds +$$

$$E\int_0^t \frac{1}{2}s^3\,tanh(sw)\,ds + E\int_0^t \frac{1}{2}(s^4\,sinh(sw))\,ds \quad (96)$$

$$E(\sinh(tx)) = \arctan(\sinh(wt)) + \frac{wt^2}{2} + \frac{\tanh(sw)\,t^4}{8}$$

$$+\frac{\sinh(sw)\,t^5}{5} \quad (97)$$

Let, taking the function at $[\sinh^2(tx), m = 2]$ by taking the equation(28) and substituting in the values of F, G

$$dy = (2x\,sinh(tx)\cosh(tx) + F(2t\sinh(tx)\;cosh(tx))$$

$$+\frac{1}{2}G^2(2t(t\;sinh(tx)\sinh(tx) + t\cosh(tx)\cosh\,(tx))))dt$$

$$+G(2t\sinh(\mathrm{tx})\cosh(tx))dw_t \quad (98)$$

And with compensation $F = 0, G = 1$

$$dy = (2x\,sinh(tx)\cosh(tx) + \frac{1}{2}(2t(t\sinh(tx)\sinh(tx)) +$$

$$t\cosh(tx)\cosh\,(tx))))dt + (2t\sinh(\mathrm{tx})\cosh(tx))dw_t \quad (99)$$

Applying the general formula for Ito's

$$dF = \left(F_t + fF_x + \frac{1}{2}g^2F_{xx}\right)dt + (gF_x)dw_t \quad (100)$$

Then

$$f = 2x\,sinh(tx)\,cosh\,(tx) + \frac{1}{2}(2t(t\sinh(tx)\sinh(tx) + t\,cosh(tx)\,cosh\,(tx))) \quad (101)$$

$$g = 2t\,sinh(tx)\,cosh(tx) \quad (102)$$

Offsetting the value of f , g

$$dF = (F_t + (2x\,sinh(tx)\cosh(tx) + \frac{1}{2}(2t(t\sinh(tx)\sinh(tx)) + t\,cosh(tx)\cosh\,(tx))))F_x + (2t^2 sinh^2(tx)\,cosh^2(tx))F_{xx})\,dt + (2t\,sinh(tx)\,cosh(tx)\,F_x)dw_t \quad (103)$$

Let $p(t) = 2t\sinh(tx)\cosh\,(tx)F_x$

$$F_x = \frac{p(t)}{2t\sinh(tx)\cosh(tx)} \quad (104)$$

Assuming that $p(t)$ is an arbitrary function equal to $p(t) = 2t$

$$F_x = \frac{2t}{2t\sinh(tx)\cosh(tx)} \quad (105)$$

$$F_x = \frac{1}{sinh(tx)\,cosh\,(tx)} \quad (106)$$

$$F = \frac{x}{\sinh(tx)\cosh\,(tx)} \quad (107)$$

$F_{xx} = 0, F_t = 0$

Substituting into the F_x, F_{xx}, F_t equation(*9)

$$dF = (0 + (2x\,sinh(tx)\cosh(tx) + \frac{1}{2}(2t(t\,sinh(tx)\,\sinh\,(tx) + t\,cosh(tx)\cosh\,(tx)))) * \frac{1}{sinh(tx)\cosh(tx)} + 0)dt + \left(2t\,sinh(tx)\,cosh(tx) * \frac{1}{sinh(tx)\cosh(tx)}\right) dw_t$$

$$d(\sinh^2(tx)) = (2x\,sinh(tx)\,cosh(tx) + \frac{1}{2}(2t(t\;\,sinh(tx)\,sinh(tx) + t\,cosh(tx)\cosh\,(tx))) * \frac{1}{sinh(tx)\cosh(tx)})\,dt\; + (2t)dw_t \quad (108)$$

Taking the integral

$$\sinh^2(tx) = \sinh^2(0) + \int_0^t (\,2x\,sinh(sx)\,cosh(sx)\,\frac{1}{2}(2s(s\;\,sinh(sx)\,sinh(sx) + s\,cosh(sx)\,cosh\,(sx))) * \frac{1}{sinh(sx)\,cosh(sx)})ds + \int_0^t (2s)\,dw_s \quad (109)$$

Since every $x = w$

$$\sinh^2(tx) = \int_0^t (2w\,sinh(sw)\,cosh(sw) + \frac{1}{2}(2s(s\;\,sinh(sw)\,sinh(sw) + s\,cosh(sw)\,cosh\,(sw))) * \frac{1}{sinh(sw)\,cosh(sw)})ds + \int_0^t (2s)\,dw_s \quad (110)$$

Distributing the integral and taking the expectation for each term

$$\mathrm{E}(\sinh^2(tx)) = E\int_0^t 2w\,sinh(sw)\,cosh(sw)ds + E\int_0^t \frac{1}{2}(2s(s\;\,sinh(sw)\,sinh(sw) + s\,cosh(sw)\cosh\,(sw))) * \frac{1}{sinh(sw)\,cosh(sw)})ds + E\int_0^t (2s)\,dw_s \quad (111)$$

$$\mathrm{E}(\sinh^2(tx)) = \frac{1}{2}(\cosh(2wt) - 1) + \frac{(\cosh(sw)^2 + \sinh(sw)^2)t^3}{6\sinh(sw)\cosh\,(sw)} \quad (112)$$

From equation (97) and equation (112), we get the variance

$$V(\sinh(tx)) = E(\sinh^2(tx)) - \big(E(\sinh(tx))\big)^2 \quad (113)$$

$$V(\sinh(tx)) = (\frac{1}{2}(cosh(2wt) - 1) + \frac{(cosh(sw)^2 + sinh(sw)^2)t^3}{6\,sinh(sw)\,cosh\,(sw)} - (\arctan(\sinh(wt)) + \frac{wt^2}{2} + \frac{\tanh(sw)\,t^4}{8} + \frac{\sinh(sw)\,t^5}{5})^2 \quad (114)$$

$$V(\sinh(tx)) = cosh(wt)^2 - 1 - arctan(\sinh(wt))^2 - \frac{(2cosh(sw)^2 * \arctan\,(\sinh\,(wt))t^5)}{(5\sinh(sw))} - \frac{\arctan(\sinh(wt))\,t^4\cosh(sw)}{(4\sinh(sw))} + \frac{(2\arctan\,(\sinh\,(wt))t^5}{(5\sinh(sw))} - arctan\big(sinh(wt)\big)wt^2$$

$$+\frac{t^4\arctan(\sinh(wt))}{(4\sinh(sw)\cosh(sw))}-\frac{cosh(sw)^2t^{10}}{25}$$

$$-\frac{cosh(sw)^2t^5wt^2}{(5*\sinh(sw))}-\frac{\cosh(sw)\,t^9}{20}$$

$$-\frac{t^4wt^2\cosh(sw)}{(8\sinh(sw))}+\frac{t^3\cosh(sw)}{(3\sinh(sw))}+\frac{t^{10}}{25}-\frac{t^8}{64}$$

$$-\frac{wt^4}{4}+\frac{t^5wt^2}{(5*\sinh(sw))}+\frac{t^9}{(20*\cosh(sw))}$$

$$+\frac{t^4wt^2}{(8\sinh(sw)\cosh(sw))}-\frac{t^3}{(6*\sinh(sw)*\cosh(sw))}$$

$$+\frac{t^8}{(64*cosh(sw)^2)} \qquad (115)$$

The m-th moments $y=(\sinh(tx))^n$
So the general formal

$$d(sinh(tx))^n=(n\,sinh(tx)^{n-1}\,cosh(tx)\,F+\frac{1}{2}G^2(n^2$$

$$sinh(tx)^{n-2}\,cosh(tx)^2-n\,sinh(tx)^n-n*sinh(tx)^{n-2}$$

$$cosh(tx)^2)dt+G(n\,sinh(tx)^{n-1}\,cosh(tx))dw_t \qquad (116)$$

Integrating both sides, we get

$$(sinh(tx))^n-(\sinh(0))^n=\int_0^t(n\,sinh(sx)^{n-1}\,cosh(sx)\,F$$

$$+\frac{1}{2}G^2(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2-n\,sinh(sx)^n$$

$$-n\,sinh(sx)^{n-2}\,cosh(sx)^2))ds+\int_0^t G\,(n\,sinh(sx)^{n-1}$$

$$cosh(sx))dw_s \qquad (117)$$

Since $F=0, G=1$

$$(sinh(tx))^n=(sinh(0))^n+\int_0^t\frac{1}{2}(n^2\,sinh(sx)^{n-2}$$

$$cosh(sx)^2-n\,sinh(sx)^n-n\,sinh(sx)^{n-2}\,cosh(sx)^2)ds$$

$$+\int_0^t(n\,sinh(sx)^{n-1}\,cosh(sx))\,dw_s \qquad (118)$$

$$(sinh(tx))^n=\int_0^t\frac{1}{2}(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2$$

$$-n\,sinh(sx)^n\;-n\,sinh(sx)^{n-2}*cosh(sx)^2)ds$$

$$+\int_0^t(n\,sinh(sx)^{n-1}\,cosh(sx))\,dw_s \qquad (119)$$

The expected value of equation (119)

$$E(sinh(tx))^n=E\int_0^t\frac{1}{2}G^2\,(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2$$

$$-n\,sinh(sx)^n-n\,sinh(sx)^{n-2}\,cosh(sx)^2)\,ds$$

$$+E\int_0^t G(n\,sinh(sx)^{n-1}\,cosh(sx))\,dw_s \qquad (120)$$

since $E\{dw\}=0$ then

$$E(sinh(tx))^n=E\int_0^t n\,sinh(sx)^{n-1}\,cosh(sx)\,F+E$$

$$\int_0^t\frac{1}{2}G^2\,(n^2\,sinh(sx)^{n-2}\,cosh(sx)^2-n\,sinh(sx)^n$$

$$-n\,sinh(sx)^{n-2}\,cosh(sx)^2)ds \qquad (121)$$

Example (4):
Let we have the following stochastic different equation for $y=(\sinh(tx))^3$by Ito's integral
Solution:

$$\text{Let } dy=\left(\frac{dy}{dt}+F\frac{dy}{dx}+\frac{1}{2}\frac{d^2y}{dx^2}G^2\right)dt+\frac{dy}{dx}Gdw \qquad (122)$$

$$d(\sinh(tx))^3=(3x\sinh^2(tx)\cosh(tx)+F(3t\sinh^2(tx)$$

$$\cosh(tx))+\frac{1}{2}G^2(6t^2\cosh^2(tx)\sinh(\mathrm{tx})+3\mathrm{t}^2\sinh^2(\mathrm{tx}))dt$$

$$+\mathrm{G}(3\mathrm{t}\sinh^2(tx)\cosh(tx))dw(t) \qquad (123)$$

Integrating both sides, we get

$$(sinh(tx))^3-(\sinh(0))^3=\int_0^t(3x\,sinh^2(sx)\,cosh(sx)$$

$$+F(3s\,sinh^2(sx)\,cosh(sx))+\frac{1}{2}G^2(6s^2\,cosh^2(sx)\,sinh(sx)$$

$$+3s^2\,sinh^2(sx))ds+\int_0^t G\,(3s\,sinh^2(sx)\;\;cosh(sx))dw(s) \qquad (124)$$

$$(sinh(tx))^3=\int_0^t(3x\,sinh^2(sx)\,cosh(sx))\,ds+\int_0^t F$$

$$(3s\,sinh^2(sx)\,cosh(sx))ds+\int_0^t\frac{1}{2}G^2\,(6s^2\,cosh^2(sx)$$

$$sinh(sx) + 3s^2\, sinh^2(sx))ds + \int_0^t G\,(3s\, sinh^2(sx)$$

$$cosh(sx))\; dw(s) \qquad (125)$$

a) The (expected value) of equation (125)

$$E(sinh(tx))^3 == E\int_0^t (3x\, sinh^2(sx)\, cosh(sx))\, ds$$

$$+E\int_0^t F(3s\, sinh^2(sx)\, cosh(sx))ds + E\int_0^t \frac{1}{2}G^2$$

$$(6s^2\, cosh^2(sx)\, sinh(sx) + 3s^2\, sinh^2(sx))ds$$

$$+E\int_0^t G(3s\, sinh^2(sx)\, cosh(sx))\; dw(s) \qquad (126)$$

Wherein $E\{dw\} = 0$ then

$$E(sinh(tx))^3 = E\int_0^t (3x\, sinh^2(sx)\, cosh(sx))\, ds$$

$$+E\int_0^t F(3s\, sinh^2(sx)\, cosh(sx))ds + E\int_0^t \frac{1}{2}G^2$$

$$(6s^2\, cosh^2(sx)\, sinh(sx) + 3s^2\, sinh^2(sx))\; ds \qquad (127)$$

$$E(sinh(tx))^3 = \sin\text{h}^3(tx) + \frac{3\sinh(sx)\cosh(sx)\, t^2}{2}$$

$$+\frac{\left(3\cosh(sx)^2\sinh(sx) + \frac{3\sinh(sx)^2}{2}\right)}{3} \qquad (128)$$

VIII. Conclusion

Stochastic differential equations consist of two parts, the first part represents the specific part, while the second part represents the stochastic part, or what is sometimes called random, and by using the general law of Ito's formula to obtain the general formulas for the hyperbolic trigonometric functions, which verify the existence of the solution to the stochastic equation at period $[0, T]$, and then find the moments represented by With the expected value, variance and moment of order m applied to the equations in hyperbolic trigonometric function form.

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