

On The Smarandache Semigroups

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Abstract:

We discusse in this paper *a Smarandache semigroups , a Smarandache normal subgroups and a Smarandache lagrange semigroup*. We prove some results about it and prove that the Smarandache semigroup Z_{p^n} with multiplication modulo p^n where p is a prime has the subgroup of order $p^n - p^{n-1}$ and we prove that if p is an odd prime then Z_{p^n} is a *Smarandache weakly Lagrange semigrup* and if p is an even prime then Z_{p^n} is a *Smarandache Lagrange semigrup*

الخلاصة: ناقشنا في هذا البحث أشباه زممر سماراندش، زممر سماراندش الناضمية وأشباه زممر لاكرانج سماراندش حيث برهنا مجموعة من النتائج المتعلقة بها. كما أثبتنا أن شبه الزمرة Z_{p^n} مع عملية الضرب معيار p^n حيث p عدد أولي تحوي مجموعة جزئية تكون زمرة ذات رتبة $p^n - p^{n-1}$ كما أثبتنا أنه إذا كانت p عدد أولي فردي فإن شبه الزمرة أعلاه تكون شبه زمرة لاكرانج سماراندش بضعف أما إذا كانت p عدد أولي زوجي فإن شبه الزمرة أعلاه تكون شبه زمرة لاكرانج سماراندش.

1.Introduction :

Padilla Raul introduced the notion of *Smarandache semigroups*[1], in the year 1998 in the paper entitled *Smarandache Algebraic Structures*. since groups are prefect structures under a single closed associative binary operation ,it has become infeasible to define Smarandache groups . Smarandache semigroups are the analog in the Smarandache ideologies of the groups where Smarandache semigroup is defined to be the semigroup A such that a proper subset of A is a group (with respect to the same binary operation).

In this paper we prove some results in *a Smarandache normal subgroups*[1],[2], *a Smarandache lagrange semigroups*[1],[2], *a Smarandache direct product semigroups*[2], *a Smarandache strong internal direct product semigeoups*[2] , *the S-semigroup homomorphism*[1],[2] and prove research problems in the references about the semigroup Z_{p^n} we solve then using Matlab programming to check our results in this open problems .

2.Definitions and Notations:

Definition 2.1: The *Smarandache semigroup* (S-semigroup) is defined to be a semigroup A such that a proper subset of A is a group (with respect to the same binary operation),[2].

Example 2.1: Let $z_{12} = \{0, 1, \dots, 11\}$ be the semigroup under multiplication module 12. Clearly the set $A = \{1, 11\} \subset z_{12}$ is a group under multiplication modulo 12, so z_{12} is the Smarandache semigroup,[2] .

Definition 2.2: Let S be a S-semigroup. Let A be a proper subset of S which is a group under the operation of S . We say S is a *Smarandache normal subgroup* of the S-semigroup if $xA \subseteq A$ and $Ax \subseteq A$ or $xA = \{0\}$ and $Ax = \{0\} \quad \forall x \in S$ and if 0 is an element in S then we have $xA = \{0\}$ and $Ax = \{0\}$,[2].

Definition 2.3: Let S be a S -semigroup we say the proper subset $M \subset S$ is the maximal subgroup of S that is N if a subgroup such that $M \subset N$ then $M=N$ is the only possibility,[2].

Definition 2.4: Let S be a S -semigroup.If S has only one maximal subgroup we call S a Smarandache maximal semigroup,[2] .

Example 2.2: Let $z_7=\{0,1, \dots ,6\}$ be the S -semigroup under multiplication module 7,the only maximal semigroup of z_7 is $G=\{1,2, \dots ,6\} \subset z_7$ so z_7 is a Smarandache maximal semigroup ,[2].

Definition 2.5: Let $S_1, \dots ,S_n$ be n S -semigroup , $S=S_1 \times S_2 \times \dots \times S_n=\{(s_1,s_2, \dots ,s_n):s_i \in S_i \text{ for } i=1, \dots ,n\}$ is called the Smarandache direct product of the S -semigroups $S_1, \dots ,S_n$ if S is a Smarandache maximal semigroup, and G is got from the $S_1, \dots ,S_n$ as $G=G_1 \times G_2 \times \dots \times G_n$ where each G_i is the maximal subgroup of the S -semigroup S_i for $i=1, \dots ,n$,[2].

Definition 2.6: Let S be a S -semigroup.If $S=B \cdot A_1 \cdot \dots \cdot A_n$ where B is a S -semigroup and $A_1, \dots ,A_n$ are maximal subgroup of S then we say S is a Smarandache strong internal direct product,[2].

Definition 2.7:A homomorphism φ from a semigroup $(S,.)$ to a semigroup $(S',*)$ is a mapping φ from the set S into the set S' such that $\varphi(x.y)=\varphi(x)*\varphi(y)$ for every $x,y \in S$.If φ is also a surjective mapping then φ is called a homomorphism from S onto S' .In case the mapping φ above is injective, it is called a one to one homomorphism .An isomorphism from S to S' is a homomorphism which is both surjective and injective,[3].

Definition 2.8: Let S and S' be any two S -semigroups .A map φ from S to S' is said to be a S -semigroup homomorphism if φ restricted to a subgroup $A \subset S \rightarrow A' \subset S'$ is a group homomorphism .The S -semigroup homomorphism is an isomorphism if $\varphi:A \rightarrow A'$ is one to one and onto . Similarly,one can define S -semigroup automorphism on S ,[2].

Definition 2.9: Let S be a finite S -semigroup .We say S is a Smarandache non-Lagrange semigroup if the order of none of the subgroups of S divides the order of the S -semigroup,[2].

Definition 2.10: Let S be a finite S -semigroup.If the order of every subgroups of S divides the order of the S -semigroup S then we say S is a Smarandache Lagrange semigroup ,[2].

Definition 2.11: Let S be a finite S -semigroup.If there exist at least one subgroup A that is a proper subset ($A \subset S$) having the same operation of S whose order divides the order of S then we say that S is a Smarandache weakly Lagrange semigroup ,[2].

Definition 2.12: Two integers a and b , not both of which are zero, are said to be relatively prime whenever $\text{g.c.d.}(a,b)=1$,[4].

Definition 2.13: For $n \geq 1$,let $\varphi(n)$ denotes the number of positive integer not exceeding n that is relatively prime to n ,[4].

Definition 2.14: Let $(G,*)$ be a finite group and p a prime .A subgroup $(P,*)$ of $(G,*)$ is said to be a Sylow p -subgroup if $(P,*)$ is a p -group and is not properly contained in any other p -subgroup of $(G,*)$ for the same prime number p ,[5].

Theorem 2.1: If p is a prime and $k > 0$,then

$$\varphi(p^k)=p^k-p^{k-1},[4].$$

Theorem (Euler) 2.2: If n is a positive integer and $\text{g.c.d.}(a,n)=1$ then

$$a^{\varphi(n)} \equiv 1 \pmod{n},[4].$$

Theorem 2.3: If n is a positive integer then $ax \equiv 1 \pmod{n}$ has a unique solution iff $\text{g.c.d.}(a,n)=1$,[6].

Theorem 2.4: Let $a,b,n \in \mathbb{Z}$, $n > 0$.If $\text{g.c.d.}(a,n)=1$ then the congruence $az \equiv b \pmod{n}$ has a unique solution z ; moreover , any integer z is a solution iff $z \equiv z' \pmod{n}$,[7] .

Theorem (Sylow)2.5: Let $(G,*)$ be a finite group of order p^kq ,where p is a prime not dividing q Then $(G,*)$ has a Sylow p -subgroup of order p^k ,[5].

3.A Smarandache normal subgroups:

Proposition 3.1: Let S be a S -semigroup and A is a proper subgroup of S which is a Smarandache normal subgroup ,if B is another subgroup of S have the same identity element as in A then B is not a Smarandache normal subgroup.

Proof: Let A, B be a Smarandache normal subgroups of S with the identity element e . Let $x \in A \subset S$ so $0 \neq x$ since A is a group and $x = x.e \in B$ since B is a Smarandache normal subgroup so $A \subset B$, by similar way $B \subset A$ so $A = B$ and this contradiction with our hypotheses, so B is not a Smarandache normal subgroup ■

Remark: In particular case if $B \subset A$ clear that those subgroups have the same identity element so all subgroups of S which are subsets from a Smarandache normal subgroup of S will be not a Smarandache normal subgroup.

Proposition 3.2: Let S be S -semigroup with zero element (0) and A is a Smarandache normal subgroup of S such that the identity element in S such as in A then $A = S/\{0\}$ and A is a maximal subgroup of S .

Proof: For every $0 \neq x \in S$, $x = x.e \in A$ since A is a Smarandache normal subgroup of S and $x.e \neq 0$ where $x.e = 0$ iff $x = 0$ since e is the identity element of S and A , so $S/\{0\} \subset A$ and since A is a group of S so $A \subset S/\{0\}$ therefore $A = S/\{0\}$ and by the definition of a maximal subgroup of S then A is a maximal subgroup of S ■

Proposition 3.3: Let S be a S -semigroup without zero element then every proper subgroup of S have the same identity element as in S is not a Smarandache normal subgroup of S .

Proof: Suppose that A is a Smarandache normal subgroup of S and e is the identity element of S and A .

Let $x \in S$, $x = x.e \in A$ since A is a Smarandache normal subgroup of S , where $x.e \neq 0$ since $0 \notin S$ and e is the identity element of S , so $S \subset A$ therefore $A = S$ and this is contradiction , so A is not a Smarandache normal subgroup of S ■

proposition 3.4: Let $S = S_1 \times S_2 \times \dots \times S_n$ is a Smarandache direct product of the S -semigroup $S_1, \dots, S_n$ where no one of S_i has zero element and let $G = G_1 \times G_2 \times \dots \times G_n$ is the maximal subgroup of S where each G_i is a maximal subgroup of S_i and a Smarandache normal subgroup of S_i with identity element different from the identity element of S_i then G is a Smarandache normal subgroup of S , $i = 1, \dots, n$.

proof: Let $S = S_1 \times S_2 \times \dots \times S_n$ is Smarandache direct product of the S - semigroup $S_1, \dots, S_n$, $G = G_1 \times G_2 \times \dots \times G_n$ is a maximal subgroup of S where each G_i is a maximal subgroup of S_i and a Smarandache normal subgroup of S_i with identity elements different from the identity element of S_i , $i = 1, \dots, n$.

Let $x \in S, y \in G$

$xy = (x_1, \dots, x_n)(g_1, \dots, g_n)$ where $g_i \in G_i$ and $x_i \in S_i, i = 1, \dots, n$.

So $xy = (x_1g_1, \dots, x_ng_n) \in G_1 \times G_2 \times \dots \times G_n = G$

since G_i is a Smarandache normal subgroup of S_i and no one of $x_i g_i = 0$ because that S_i has no zero element, $i=1, \dots, n$, so $xG \subseteq G$,

by similar way $Gx \subseteq G$ so G is a Smarandache normal subgroup of S ■

Remark: It is important to know that the condition in above proposition where S_i doesn't have the same identity elements of G_i where if S_i has the same identity element of G_i we have $G_i = S_i$ by proposition 3.3 and this contradiction, also it is important to note that each S_i must not have zero element since, for example if S_2 has zero element so $x=(x_1, 0, \dots, x_n) \in S$

let $y=(g_1, \dots, g_n) \in G$ where $g_i \in G_i$ and $x_i \in S_i, i=1, \dots, n$.

but $xy=(x_1 g_1, 0, \dots, x_n g_n) \notin G_1 \times G_2 \times \dots \times G_n = G$, since $0 \notin G_2$ where G_2 is a group.

proposition 3.5: Let S and S' be the S -semigroups and let $f:S \rightarrow S'$ be a homomorphism from S onto S' , if f is a Smarandache onto homomorphism from A to A' where $A \subset S$ is a group, $A' \subset S'$ is a group and A is a Smarandache normal subgroup of S then A' is a Smarandache normal subgroup of S' .

proof: Let $f:S \rightarrow S'$ is a Smarandache onto homomorphism from A to A' so $f(A)=A'$.

Let $f:S \rightarrow S'$ is the onto homomorphism on the semigroup S' ,

let $y \in S'$ and $a' \in f(A)$,

so $\exists x \in S$ such that $f(x)=y$,

since f is a Smarandache onto homomorphism from A to A'

$\exists a \in A$ such that $f(a)=a'$,

$y \cdot a' = f(x) \cdot f(a) = f(x \cdot a)$, now since A is a Smarandache normal subgroup of S then

either $x \cdot a \in A$ so $f(x \cdot a) \in f(A)$ so $y \cdot a' \in f(A)$ and by similar way $a' \cdot y \in f(A)$

or $x \cdot a = 0$ if S has zero element (0) so $y \cdot a' = f(x \cdot a) = f(0) = 0'$ where $f(0)=0'$ is the zero element of S' if

S has 0 as a zero element. Since for all $y \in S' \exists x \in S$ such that $f(x)=y$ so if S has (0) then

$f(x) \cdot f(0) = f(x \cdot 0) = f(0) = f(0) \cdot f(x) \forall y \in S'$, so $f(A)=A'$ is a Smarandache normal subgroup of S' ■

proposition 3.6: Let S and S' be the S -semigroups. Let $f:S \rightarrow S'$ be an isomorphism from S to S' , if f is a Smarandache onto homomorphism from A to A' where $A \subset S$ is a group, $A' \subset S'$ is a group and A is a Smarandache normal subgroup of S then $f^{-1}(A')$ is a Smarandache normal subgroup of S .

proof: Let $f:S \rightarrow S'$ is an isomorphism so $f:A \rightarrow A'$ is a one to one.

let f is a Smarandache onto homomorphism from A to A' ,

so $f(A)=A'$ and $f^{-1}(f(A))=A$.

Let $a \in A$ and $x \in S$,

since A is a Smarandache normal subgroup of S then either $f(x) \cdot f(a) \in A'$

so $f(x \cdot a) \in A'$ therefore $x \cdot a \in A$, by similar way $a \cdot x \in A$,

or $f(x) \cdot f(a) = f(0) = 0'$ if S has zero element (0),

so $f(x \cdot a) = f(0)$ then $x \cdot a = 0$ where $f(0)=0'$ is the zero element of S' if S has 0 as a zero element, by

similar way $a \cdot x = 0$ also $x \cdot 0 = 0 \cdot x = 0$ for each $x \in S$ if S has zero element (0) therefore $f^{-1}(A')$ is a

Smarandache normal subgroup of S ■

proposition 3.7: Let S be a Smarandache strong internal direct product semigroup where $S=B \cdot A_1 \cdot \dots \cdot A_n$ such that $A_1, \dots, A_n$ are the maximal subgroups of S and B is the S -semigroup, if

- 1- $A_i, i=1, \dots, n$; is a Smarandache normal subgroup of S commutative with each other.
- 2- If S has identity element it must be different from the identity elements of A_i which is also different from one to another, $i=1, \dots, n$.

3- S has zero

Then $G = A_1 \cdot \dots \cdot A_n = S/\{0\}$ and G is a Smarandache normal subgroup of S.

Proof: Since $A_1, \dots, A_n$ are the maximal subgroup of S which are commutative with each other so $G = A_1 \cdot \dots \cdot A_n$ is a group.

Let $0 \neq x \in S$, $x = b \cdot x_1 \dots \cdot x_n$ where $x_i \in A_i$ and $b \in B$ such that B is the S-semigroup, $i=1, \dots, n$, so $x \in G$ since A_i is a Smarandache normal subgroup of S, $i=1, \dots, n$, where if $b \cdot x_i = 0$ for some i , then $x=0$ contradiction.

So $\forall 0 \neq x \in S$ we have $x \in G$ therefore $S/\{0\} \subset G$ so $S/\{0\} = G$.

To prove G is a Smarandache normal subgroup of S.

Let $x \in S$ and $y \in G$ such that $x = b \cdot x_1 \dots \cdot x_n$ and $y = x'_1 \cdot \dots \cdot x'_n$, where $x_i, x'_i \in A_i$ and $b \in B$. since $A_1, \dots, A_n$ are commutative with each other

so $x \cdot y = b \cdot x_1 \dots \cdot x_n \cdot x'_1 \dots \cdot x'_n = b \cdot x''_1 \cdot \dots \cdot x''_n$, where $x''_i = x_i \cdot x'_i$, $i=1, \dots, n$.

Now since A_i a Smarandache normal subgroup of S so either

$x \cdot y \in G$ if $b \cdot x''_i \in A_i$ for some i or $x \cdot y = 0$ if $b \cdot x''_i = 0$ for some i ,

also $x \cdot 0 = 0 \cdot x = 0 \quad \forall x \in S$.

So G is a Smarandache normal subgroup of S ■

Remark: It is important to know that the condition in above proposition where A_i does not have the same identity because in such case we will have contradiction by proposition 3.1, also it is important the condition that S must have not the same identity as in A_i and S has zero element since if S has the same identity as in A_i and does not have zero element we have contradiction with proposition 3.3.

4-The S-semigroup (Z_{p^n}, p^n)

Proposition 4.1: Let Z_{p^n} be the semigroup (p is a prime, $n > 1$) under multiplication modulo p^n , then Z_{p^n} has subset of order $p^n - p^{n-1}$ which is a subgroup under multiplication modulo p^n .

Proof: Let (Z_{p^n}, p^n) be the semigroup under multiplication modulo p^n , p is a prim, $n > 1$.

The order of Z_{p^n} is p^n , now, to compose a subset of Z_{p^n} without zero element in it we must cancel zero element because that $0 \cdot x = x \cdot 0 = 0 \quad \forall x \in Z_{p^n}$, [6],

and all elements give us zero element if multiplication modulo p^n with each other. Since among the first positive integer, those which are divisible by p are

$p, 2p, 3p, \dots, kp$, where k is the largest integer such that $kp \leq p^n$, if $k = p^{n-1}$ then

$kp = p^{n-1}p = p^n = 0$ since we multiplication modulo p^n so the number of those elements is p^{n-1} . We must cancel all those elements of Z_{p^n} since

$tp^{n-1} = 0$ where $t \leq k$, $1 \leq t$ and $t \in Z_{p^n}$,

$tp^2 p^{n-2} = 0$ where $t \leq k$, $1 \leq t$ and $t \in Z_{p^n}$,

and so on,

$tp^n p = 0$ where $t \leq k$, $1 \leq t$. and $t \in Z_{p^n}$,

therefore $p^n - p^{n-1}$ is the number of the remainder element of Z_{p^n} , which are relatively prime with p^n

where $p^n - p^{n-1} = \phi(p^n)$, [3].

Now let H is the set of those elements, i.e,

$H = \{1, \dots, p-1, p+1, \dots, 2p-1, 2p+1, \dots, kp-1\}$ so H is a group of order

$p^n - p^{n-1}$, to prove this,

if $x, y \in H \subset Z_{p^n}$ then $xy \in Z_{p^n}$

and since xy didn't divisible by p because that x, y didn't divisible by p , since H contains each element of Z_{p^n} which didn't divisible by p so $xy \in H$,

it is clear that H is associative under multiplication modulo p^n and the identity element of H is 1, now to prove that all $x \in H$ has inverse element ,

let $x, y \in H$ then $yx = xy \equiv 1 \pmod{p^n}$ has a unique solution modulo p^n

iff $\gcd(x, p^n) = 1$, [5], so x has y as inverse . In fact y may be obtained direct from

$$y \equiv x^{\varphi(p^n)-1} \pmod{p^n} .$$

Since an application of Euler's theorem leads immediately to

$$xy \equiv x^{\varphi(p^n)} \equiv 1 \pmod{p^n} .$$

so $(H, \cdot, p^n)$ is a group $\blacksquare$

One can check that by using Mathlap programming after determine the value of n and p so we can present all examples about this subject .

Proposition 4.2: Let Z_{p^n} be the semigroup (p is a prime , $n > 1$) under multiplication modulo p^n , p is an odd prime then Z_{p^n} is a Smarandache weakly Lagrange semigroup.

Proof: Let $(Z_{p^n}, \cdot, p^n)$ be the semigroup under multiplication modulo p^n , p is an odd prime , $n > 1$.

By proposition 4.1 , Z_{p^n} has the subgroup H of order $p^n - p^{n-1}$, $O(Z_{p^n}) = p^n$ and $o(H) = p^n - p^{n-1} = p^{n-1}(p -$

$1)$, $O(Z_{p^n})$ didn't divisible by $o(H)$ since $\frac{p^n}{p^n - p^{n-1}} = \frac{p}{p - 1}$,

also Z_{p^n} has a subgroup of order 2 which is $\{1, p^{n-1}\}$ and $O(Z_{p^n})$ didn't divisible by 2 since p is an

odd prime , but by Sylow theorem in group theory, [5], H has K

as a p - Sylow subgroup of order p^{n-1} where $p-1$ didn't divisible by p , since K is a subgroup of $H \subset Z_{p^n}$ so K is a subgroup of Z_{p^n} and

$$\frac{o(Z_{p^n})}{o(K)} = \frac{p^n}{p^{n-1}} = p , \text{ so by definition 2.11, } Z_{p^n} \text{ is a Smarandache Weakly Lagrange semigroup } \blacksquare$$

Example 4.1 : Let $Z_{3^3} = \{0, 1, \dots, 26\}$ be a S- semigroup under multiplication modulo 27 , it is clear that the order of Z_{3^3} is 27 .

Now by proposition 4.1 it has H_1 as a subgroup of order $3^3-3^2=18$ where $H_1=\{1,2,4,5,7,8,10,11,13,14,16,17,19,20,22,23,25,26\}$ and the table of H_1 is given by

.27	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26
1	1	2	4	5	7	8	10	11	13	14	16	17	19	20	22	23	25	26
2	2	4	8	10	14	16	20	22	26	1	5	7	11	13	17	19	23	25
4	4	8	16	20	1	5	13	17	25	2	10	14	22	26	7	11	19	23
5	5	10	20	25	8	13	23	1	11	16	26	4	14	19	2	7	17	22
7	7	14	1	8	22	2	16	23	10	17	4	11	25	5	19	26	13	20
8	8	16	5	13	2	10	26	7	23	4	20	1	17	25	14	22	11	19
10	10	20	13	23	16	26	19	2	22	5	25	8	1	11	4	14	7	17
11	11	22	17	1	23	7	2	13	8	19	14	25	20	4	26	10	5	16
13	13	26	25	11	10	23	22	8	7	20	19	5	4	17	16	2	1	14
14	14	1	2	16	17	4	5	19	20	7	8	22	23	10	11	25	26	13
16	16	5	10	26	4	20	25	14	19	8	13	2	7	23	1	17	22	11
17	17	7	14	4	11	1	8	25	5	22	2	19	26	16	23	13	20	10
19	19	11	22	14	25	17	1	20	4	23	7	26	10	2	13	5	16	8
20	20	13	26	19	5	25	11	4	17	10	23	16	2	22	8	1	14	7
22	22	17	7	2	19	14	4	26	16	11	1	23	13	8	25	20	10	5
23	23	19	11	7	26	22	14	10	2	25	17	13	5	1	20	16	8	4
25	25	23	19	17	13	11	7	5	1	26	22	20	16	14	10	8	4	2
26	26	25	23	22	20	19	17	16	14	13	11	10	8	7	5	4	2	1

and 27 didn't divisible by 18 , Z_{3^3} has H_2 as a subgroup of order 2 where $H_2=\{1,26\}$ and the table of H_2 is given by

.27	1	26
1	1	26
26	26	1

and 27 didn't divisible by 2 but Z_{3^3} has H_3 as a subgroup of order 9 where H_3 is a subset of Z_{27} and satisfy the definition of the group and 27 divisible by 9 where $H_3=\{1,4,7,10,13,16,19,22,25\}$ and the table of H_3 is given by

.27	1	4	7	10	13	16	19	22	25
1	1	4	7	10	13	16	19	22	25
4	4	16	1	13	25	10	22	7	19
7	7	1	22	16	10	4	25	19	13
10	10	13	16	19	22	25	1	4	7
13	13	25	10	22	7	19	4	16	1
16	16	10	4	25	19	13	7	1	22
19	19	22	25	1	4	7	10	13	16
22	22	7	19	4	16	1	13	25	10
25	25	19	13	7	1	22	16	10	4

So Z_{3^3} is Smarandache weakly Lagrange semigroup ■

Proposition 4.3: Let Z_{p^n} be the semigroup (p is a prime , $n > 1$) under multiplication modulo p^n , p is an even prime then Z_{p^n} is a Smarandache Lagrange semigroup.

Proof: Let (Z_{p^n}, p^n) be the semigroup under multiplication modulo p^n , p is an even prime, $n > 1$.

By proposition 4.1, Z_{p^n} has the subgroup H of order $p^n - p^{n-1}$,

$$O(Z_{p^n}) = p^n, \text{ since } p \text{ is an even so } p=2 \text{ and } \frac{p^n}{p^n - p^{n-1}} = \frac{p}{p-1} = p.$$

So Z_{p^n} divisible by $o(H)$, also every other subgroups of Z_{p^n} must be a subgroup of H since as we see in proposition 4.1 H contains all elements which are not zero and the multiplication of one with each other not zero so any other subgroup of Z_{p^n} must be subset of H , by Lagrange theorem in the group theory, [8][9], the order of H divisible by the order of those subgroups.

So p^n divisible by the order of all subgroups of Z_{p^n} so and by definition of a Smarandache

Lagrange semigroup Z_{p^n} is a Smarandache Lagrange semigroup, where $p=2$ ■

Example 4.2: Let $Z_{2^5} = Z_{32} = \{0, 1, \dots, 31\}$ be the S-semigroup of order 32 under multiplication modulo 32, Z_{32} have the following subgroups:

Three subgroup of order 2 which are $\{1, 15\}, \{1, 17\}, \{1, 31\}$ and the table of those subgroups as following:

.32	1	15
1	1	15
15	15	1

.32	1	17
1	1	17
17	17	1

.32	1	31
1	1	31
31	31	1

It is clear that each one of them is a subset of Z_{32} and satisfy the definition of the group so it is a subgroup of Z_{32} .

Two subgroups of order 4 which are $\{1, 7, 17, 23\}, \{1, 9, 17, 25\}, \{1, 15, 17, 31\}$ and the table of those subgroups as following:

.32	1	7	17	23
1	1	7	17	23
7	7	17	23	1
17	17	23	1	7
23	23	1	7	17

.32	1	9	17	25
1	1	9	17	25
9	9	17	25	1
17	17	25	1	9
25	25	1	9	17

.32	1	15	17	31
1	1	15	17	31
15	15	1	31	17
17	17	31	1	15
31	31	17	15	1

Two subgroups of order 8 which are $\{1, 3, 9, 11, 17, 19, 25, 27\}, \{1, 5, 9, 13, 17, 21, 25, 29\}$ and the table of those subgroups as following:

.32	1	5	9	13	17	21	25	29
1	1	5	9	13	17	21	25	29

5	5	25	13	1	21	9	29	17
9	9	13	17	21	25	29	1	5
13	13	1	21	9	29	17	5	25
17	17	21	25	29	1	5	9	13
21	21	9	29	17	5	25	13	1
25	25	29	1	5	9	13	17	21
29	29	17	5	25	13	1	21	9

.32	1	3	9	11	17	19	25	27
1	1	3	9	11	17	19	25	27
3	3	9	27	1	19	25	11	17
9	9	27	17	3	25	11	1	19
11	11	1	3	25	27	17	19	9
17	17	19	25	27	1	3	9	11
19	19	25	11	17	3	9	27	1
25	25	11	1	19	9	27	17	3
27	27	17	19	9	11	1	3	25

It is clear that each one of them is a subset of Z_{32} and satisfy the definition of the group so it is a subgroup of Z_{32} .

Also by proposition 4.1 has H as the subgroup of order 16 where

$H = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$ and the table of H is given by the following table :

.32	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
1	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31
3	3	9	15	21	27	1	7	13	19	25	31	5	11	17	23	29
5	5	15	25	3	13	23	1	11	21	31	9	19	29	7	17	27
7	7	21	3	17	31	13	27	9	23	5	19	1	15	29	11	25
9	9	27	13	31	17	3	21	7	25	11	29	15	1	19	5	23
11	11	1	23	13	3	25	15	5	27	17	7	29	19	9	31	21
13	13	7	1	27	21	15	9	3	29	23	17	11	5	31	25	19
15	15	13	11	9	7	5	3	1	31	29	27	25	23	21	19	17
17	17	19	21	23	25	27	29	31	1	3	5	7	9	11	13	15
19	19	25	31	5	11	17	23	29	3	9	15	21	27	1	7	13
21	21	31	9	19	29	7	17	27	5	15	25	3	13	23	1	11
23	23	5	19	1	15	29	11	25	7	21	3	17	31	13	27	9
25	25	11	29	15	1	19	5	23	9	27	13	31	17	3	21	7
27	27	17	7	29	19	9	31	21	11	1	23	13	3	25	15	5
29	29	23	17	11	5	31	25	19	13	7	1	27	21	15	9	3
31	31	29	27	25	23	21	19	17	15	13	11	9	7	5	3	1

We note that the order of all those subgroups of Z_{32} divides the order of Z_{32} so it is Lagrange semigroup.

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