Dynamics of meromorphic family λ $\frac{\cosh^m z}{z^{2m}}$ with non-critically finite

دراسة ديناميكية العائلة الميرومورفيكية λ$\frac{\cosh^m z}{z^{2m}}$ ذات عدد غير منته من القيم الشاذة

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Abstract

In this paper, a one-parameter family Ƭ = { $f_\lambda(z)$ = λf(z), λ > 0} with f(z) = $\frac{\cosh^m z}{z^{2m}}$, m is Natural number is considered, it is obtain that family has non-critically finite. The dynamics of the meromorphic transcendental functions f_λ ∊ Ƭ is studied in detailed. It was found that the bifurcations in the dynamics of the functions $f_\lambda(x)$ occur at parameter values λ_1 and λ_2 where λ_i = Φ (x_i), (i = 1, 2), and Φ(x) = $\frac{x}{f(x)}$ with x_1 is the solution of the equation coth x = $\frac{mx}{2m-1}$ while x_2 is the unique positive roots of the equation Φ′ (x) = 0.

الخلاصة

في هذا البحث سوف نتناول دراسة ديناميكية العائلة ذات المعلمة الواحدة $\frac{\cosh^m z}{z^{2m}}$ $f_\lambda(z)$ λ= والتي تضم دوالا" لها عدد غير منته من القيم الشاذة . لقد بينا أن التفرع لهذه العائلة يحصل عند قيم مختلفة للمعلمة λ وهي λ_1 و λ_2 حيث $\lambda_i=\Phi(x_i)$ و $\Phi(x)=x/f(x)$ وأن x_1 يمثل حل المعادلة coth x = $\frac{mx}{2m-1}$ بينما يمثل x_2 الجذر الحقيقي للمعادلة Φ′ (x) = 0.

1. Introduction

In complex analytic dynamics most of the work has centred around the dynamics of rational functions and entire functions. The behaviour of orbits of the critical values and asymptotic values of a function play an important role in determining the dynamics of a function. Devaney and Coworkers studied mainly the dynamics of critically finite function. Exploiting the critical finiteness of the function Devaney and Durkin [4], Devaney and krych [5] and Devaney and Tangerman [6] analyzed exhaustively the dynamics of some of the most interesting periodic functions like λe^z , λsinz and λcosz. The central objects studied in the complex analytic dynamics of a function are its Julia set and Fatou set. The Julia set or the set of non-normality of a function f, denoted by J(f), is defined to be the set of all complex numbers where the family of iterates { f^n } of f fails to be normal in the sense of Montel. The Fatou set or the set of normality, denoted by F(f), is the complement of the Julia set of f. Baker [2] obtained the well known characterization for the Julia set of entire functions as the closure of the set of repelling periodic points of f. Devaney and Tangerman [6] obtained another characterization that is found to be extremely helpful in computationally generating the Julia sets of critically finite entire transcendental functions.

Recently, Muslih [7] studied the dynamics of the family of critically finite transcendental meromorphic functions H = {$h_\lambda(z) = \lambda e^z / (z-1)$, $\lambda \in$ R} and the family of non-critically finite transcendental meromorphic functions G ={$g_\lambda(z) = \lambda\cosh(z) / z^2$, $\lambda > 0$}, Al-Husseiny [6] studied the dynamics of the family of non-critically finite transcendental H ={ $f_\lambda(z) = \lambda\frac{\sinh^m z}{z^{2m}}$, $\lambda > 0$ m$\in$ N, and m is even, z $\in \mathbb{C}$}. In this paper, the dynamical behavior of functions in the family Ƭ is now described. In section 3, the fixed points of the function $f_\lambda(x)$, x $\in$ R\{0} are obtained and their nature is investigated. The dynamics of the function $f_\lambda(x)$ is described in section 4. The dynamics of the function $f_\lambda(z) \in$ Ƭ for z $\in \hat{\mathbb{C}}$ is investigated in section 5.

Theorem 1.1([3])
Let f(z) be a transcendental meromorphic function. Suppose z_o lies on an attracting cycle or a parabolic cycle f(z). Then, the orbit of at least one critical value or asymptotic value is attracted to a point in the orbit of z_o.

2. One parameter family Ƭ of non-critically finite functions

Let Ƭ = { $f_\lambda(z) = \lambda\frac{\cosh^m z}{z^{2m}}$, $\lambda > 0$ and m $\in$ N, z $\in \mathbb{C}$} be one parameter family of even transcendental meromorphic functions. The following proposition shows that the functions in the family Ƭ are indeed non-critically finite.

Proposition 2.1 Let $f_\lambda \in$ Ƭ. Then, the function $f_\lambda(z)$ is non-critically finite.

Proof The derivative of the function $f_\lambda(z)$ for z $\neq$ 0 is given by

$$f_\lambda' = \lambda\frac{m\cosh^{m-1} z(z\sinh z - 2\cosh z)}{z^{2m+1}}$$

The critical points of the function $f_\lambda(z)$ are solutions of the equation $f_\lambda'(z) = 0$. and these solutions are z = kπi, where k is a non-zero integer and solutions of the equation

$$z\sinh z - 2\cosh z = 0 \tag{2.1}$$

are critical points of $f_\lambda(z)$. The solutions of equation (2.1) are the same as the solutions of the equation coth z =$\frac{z}{2}$. This equation has a solution z_o if and only if the equation cot w =$\frac{w}{2}$ has a solution iz_o.Now equating real and imaginary parts of the equation cot w =$\frac{w}{2}$, for a non-zero z = x+iy,

$\frac{\sin 2x}{\cos 2x + \cosh 2y} = \frac{1}{2}x$ and $\frac{\sinh 2y}{\cos 2x + \cosh 2y} = \frac{1}{2}y$. This implies that

$$\frac{\sin 2x}{x} = \frac{\sinh 2y}{y} \tag{2.2}$$

It is easily to show that, for x, y$\neq$ 0, the | $\frac{\sin 2x}{x}$| < 2 and | $\frac{\sinh 2y}{y}$| > 2 so that (2.2) is not possible in this case. Therefore, at least one of x, y must be zero. Hence (2.2) has only real or purely imaginary roots. If x = 0, then cot w = $\frac{w}{2}$ implies that coth y = $\frac{1}{2}$y and this equation has two zeros.

Therefore, cot w = $\frac{w}{2}$ has two purely imaginary solutions. If y = 0, then cot w = $\frac{w}{2}$ implies that cot x = $\frac{1}{2}$ x and this equation has infinitely many purely imaginary solutions. Thus, the function $f_\lambda(z)$ has two real and infinitely many imaginary critical points. To find the critical values of the function $f_\lambda(z)$, we note that $f_\lambda(k\pi i) = 0$, where k is a non-zero integer. Let $\{i\, y_L\}_{L=-\infty}^{\infty}$, y_L be critical points of $f_\lambda(z)$ other than the critical points kπi, k = ± 1, ± 2… since

$$f_\lambda(i\, y_L) = \lambda \frac{\cos^m y_L}{y_L^{2m}}$$

and the values $\frac{\cos^m y_L}{y_L^{2m}}$ are real and distinct L, it follows that the values in the set $\{ f_\lambda(i\, y_L)\}_{L=-\infty}^{\infty}$ are real and distinct. Therefore, the function $f_\lambda(z)$ possesses infinitely many real critical values. □

3. Fixed points and their nature for function in T

In this section, the existence and nature of fixed points of the function $f_\lambda(x) \in$ T is described in the following. Let f(x) = $\frac{\cosh^m x}{x^{2m}}$ and

$$\Phi(x) = \frac{x}{f(x)} = \frac{x^{2m+1}}{\cosh^m x} \tag{3.1}$$

Then it is easily seen that the function Φ(x) satisfied the following properties:-

Proposition (3.1)

1.Φ(x) is continuous in R.

2. $\Phi'(x) = \frac{(2m+1)x^{2m}\cosh x - mx^{2m+1}\sinh x}{\cosh^{m+1} x}$ is continuous in R.

3. Φ(x) → 0 as x→ -∞ and Φ(x) → 0 as x→ ∞.

4. Φ(x) is positive in (0, ∞), is negative in (-∞, 0).

5.Φ'(x) has a unique positive real zero at x = x_2, where x_2 is a real positive solution of coth x $= \frac{mx}{2m+1}$; since Φ'(x) =0 gives coth x $= \frac{mx}{2m+1}$ and by Newton-Raphson's method x_2 is a real positive solution of $\Psi(x)$= coth x - $\frac{mx}{2m+1}$ = 0. (Fig.1 (a)), the property (5) follows.

6.$$\Phi''(x) = \begin{bmatrix} \cosh x[(2m+1)x^{2m}\sinh x + 2m(2m+1)x^{2m-1}\cosh x - mx^{2m+1}\cosh x \\ -m(2m+1)\sinh x] - (m+1)\sinh x[(2m+1)x^{2m}\cosh x - mx^{2m+1}\sinh x] \end{bmatrix} \Big/ \left[\cosh^{m+2} x\right]$$

and hence Φ''(x_2) < 0. It is follows that x_2 is a maximum point for the Φ(x) in (0, ∞). Accordingly, Φ(x) increases in (0, x_2) and it decreases in (x_2, ∞).

Proof clear and is supported by the figure (1.a)

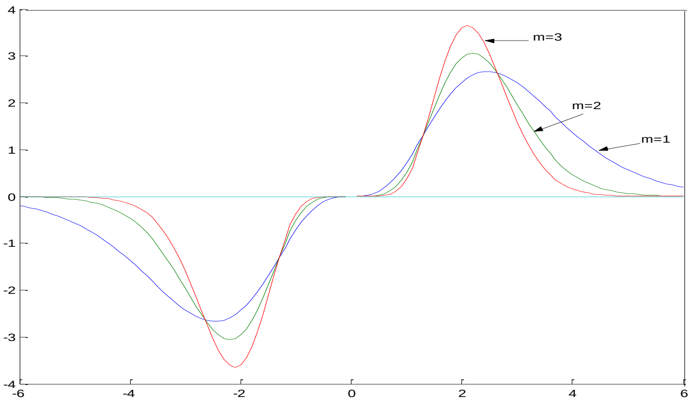

Fig.1.a the graph of Φ(x)

Now, let

$$\lambda_2 = \Phi(x_2) \qquad (3.2)$$

be the critical parameter value corresponding to the critical value x_2.
In the following proposition, the existence of the real fixed points of $f_\lambda(x)$ for $\lambda > 0$ is described.

Proposition 3.2 Let $f_\lambda(x) = \lambda \dfrac{\cosh^m x}{x^{2m}}$, then the locations of the fixed points of $f_\lambda(x)$ are given as follows: -

1- For $0 < \lambda < \lambda_2$, $f_\lambda(x)$ has two fixed points, one of them is in the interval $(0, x_2)$ and the other in the interval (x_2, ∞).

2- For $\lambda = \lambda_2$, $f_\lambda(x)$ has exactly one fixed point at $x = x_2$.

3- For $\lambda > \lambda_2$, $f_\lambda(x)$ has no fixed points.

Proof The fixed points of the function $f_\lambda(x)$ are the solutions of the equation $\lambda = \Phi(x)$, where $\Phi(x)$ is given by (3.1). We have the following cases:-

1- For $0 < \lambda < \lambda_2$, we have $x = x_2$ is a maximum value in $(0, \infty)$. From proposition (3.1-6), $\Phi(x)$ is strictly decreasing in (x_2, ∞) and it is strictly increasing in $(0, x_2)$. Thus the line $\lambda = c$ intersects the graph of $\Phi(x)$ at exactly one point in each of the intervals $(0, x_2)$ and (x_2, ∞).

2- For $\lambda = \lambda_2$, since $\lambda_2 = \Phi(x_2)$ is the maximum value of $\Phi(x)$ in $(0, \infty)$, then the line $\lambda = \lambda_2$ intersects the graph of $\Phi(x)$ at only one point which is x_2. Thus $f_\lambda(x)$ has only one fixed point at $x = x_2$.

3- For $\lambda > \lambda_2$, according to proposition (3.1), $\Phi(x)$ has maximum value at $x = x_2$ in the interval $(0, \infty)$ with $\lambda_2 = \Phi(x_2)$. Therefore the graph of $\Phi(x)$ does not intersect the line $\lambda = c$ for all $c > \lambda_2$. Hence $\Phi(x) = \lambda$ has no solutions for all $\lambda > \lambda_2$. Thus $f_\lambda(x)$ has no fixed points in this case.

Let

$$\lambda_1 = \Phi(x_1) \qquad (3.3)$$

Where x_1 is a positive solution of $\coth x = \dfrac{mx}{2m-1}$.

The fixed points of the function $f_\lambda(x)$ found in proposition (3.2) are denoted by $r_1 \in (0, x_1)$, $r_2 \in (x_3, \infty)$, $a_\lambda \in (x_1, x_2)$ and $r_\lambda \in (x_2, x_3)$, where x_2 is a positive solution of $\coth x = \dfrac{mx}{2m-1}$ and x_1, x_3 be solutions of $\lambda_1 = \Phi(x)$ lying in the intervals $(0, x_2)$ and (x_2, ∞) respectively.

The following theorem gives the nature of fixed points of the function $f_\lambda(x)$ on the real axis for different values of parameter λ.

Theorem 3.3 Let $f_\lambda(x) = \lambda \dfrac{\cosh^m x}{x^{2m}}$ for $x \in R \setminus \{0\}$ with x_1, x_2 and x_3 as given before. Then

1- For $0 < \lambda < \lambda_1$, $f_\lambda(x)$ has two fixed points $r_1 \in (0, x_1)$ and $r_2 \in (x_3, \infty)$ which are repelling.

2- For $\lambda = \lambda_1$, we have two fixed points x_1 and x_3, x_1 is indifferent and x_3 is repelling.

3- For $\lambda_1 < \lambda < \lambda_2$, then the fixed point $a_\lambda \in (x_1, x_2)$ is attracting and the fixed point $r_\lambda \in (x_2, x_3)$ is repelling.

4- For $\lambda = \lambda_2$, $f_\lambda(x)$ has only one fixed point at $x = x_2$ which is indifferent.

Proof: since the derivative of the function $f_\lambda(x)$ is given by

$$f'_\lambda(x) = \lambda \left[m(x\cosh^{m-1} x \sinh x - 2\cosh^m x \right] / x^{2m+1}$$

and the fixed points of the function $f_\lambda(x)$ are solution of $\lambda = \dfrac{x^{2m+1}}{\cosh^m x}$, it follows that the multiplier $f'_\lambda(x_f)$ of the fixed point x_f is given by

$$\left| f'_\lambda(x_f) \right| = m \left| x_f \tanh(x_f) - 2 \right| \qquad (3.4)$$

Let

$$G(x) = \begin{cases} m\,(x \tanh x - 2) & \text{for } x \neq 0 \\ -2 & \text{for } x = 0 \end{cases}$$

The function G(x) is differentiable and its derivative is given by

$$G'(x) = \begin{cases} m\,(\tanh x + x\,\mathrm{sec}h^2 x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Since, $G'(x) \neq 0$, $G'(0) = 0$ and $G''(0) > 0$, the function G(x) has exactly one minima at x = 0 and the minimum value is (-2). Since $G'(x) > 0$ for $x \in (0, \infty)$ and $G'(x) < 0$ for $x \in (-\infty, 0)$, thus G(x) increases for $(0, \infty)$ and decreases for $(-\infty, 0)$. Therefore, it follows that the function $|G(x)|$ (Fig. 2) satisfies

$$|G(x)| \begin{cases} <1 & \text{for } x \in (-x_2, -x_1) \cup (x_1, x_2) \\ =1 & \text{for } x = \pm x_1,\ \pm x_2 \\ >1 & \text{for } x \in (-\infty, -x_2) \cup (-x_1, 0) \cup (0, x_1) \cup (x_2, \infty) \end{cases}$$

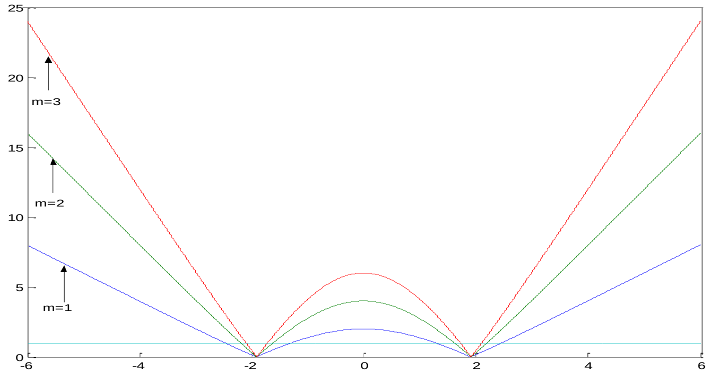

Fig.2.Graph of $|G(x)|$

Consequently, by (3.4), we get that the multiplier $f_\lambda'(x_f)$ of the fixed point x_f satisfies

$$|f_\lambda'(x_f)| \begin{cases} <1 & \text{for } x \in (-x_2, -x_1) \cup (x_1, x_2) \quad (3.5) \\ =1 & \text{for } x = \pm x_1, \pm x_2 \quad (3.6) \\ >1 & \text{for } x \in (-\infty, -x_2) \cup (-x_1, 0) \cup (0, x_1) \cup (x_2, \infty) \quad (3.7) \end{cases}$$

Now, we can get the proof of the theorem as follows:-

1- For $0 < \lambda < \lambda_1$, the fixed points of $f_\lambda(x)$ are given by $r_1 \in (0, x_1)$ and $r_2 \in (x_3, \infty)$. Hence by inequality (3.7), we have $|f_\lambda'(r_i)| > 1$(i =1, 2). Thus r_1 and r_2 are repelling fixed points.

2- For $\lambda = \lambda_1$, we have $|f_\lambda'(x_1)| = 1$, where x_1 is a fixed point of $f_\lambda(x)$.Hence x_1 is an indifferent fixed point for $f_\lambda(x)$. Similarly, since $|f_\lambda'(x_3)| > 1$, where $x_3 \in (x_2, \infty)$ is a fixed point of $f_\lambda(x)$, then x_3 is repelling fixed point.

3- For $\lambda_1 < \lambda < \lambda_2$, the fixed point $a_\lambda \in (x_1, x_2)$. Thus by inequality (3.5), $|f_\lambda'(a_\lambda)| < 1$. and hence a_λ is an attracting fixed point. But, the fixed point $r_\lambda \in (x_2, x_3)$, therefore by (3.7), $|f_\lambda'(r_\lambda)| > 1$. Thus r_λ is repelling fixed point for $f_\lambda(x)$.

4- For $\lambda = \lambda_2$, by equation (3.6), $|f_\lambda'(x_2)| = 1$. Consequently, $x = x_2$ is an indifferent fixed point of $f_\lambda(x)$.

4. Bifurcations in Dynamics on R\ {0}

In this section, the dynamics of functions $f_\lambda \in T$ on the real line is described. It is proved in the following theorem that there exist parameter values $\lambda_1, \lambda_2 > 0$ such that bifurcations in the dynamics of the function $f_\lambda(x)$, $x \in R\backslash T_o$ occur at $\lambda = \lambda_1$ and $\lambda = \lambda_2$, where T_o is the set of the points that are backward orbits of the pole 0 of the function $f_\lambda(x)$.

Theorem 4.1. Let $f_\lambda(x) = \lambda \dfrac{\cosh^m x}{x^{2m}}$ for $x \in R\backslash \{0\}$.

a.If $0 < \lambda < \lambda_1$, then

1) $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in [(-\infty, -r_2) \cup (-\alpha_1, 0) \cup (0, \alpha_1) \cup (r_2, \infty)]\backslash T_o$.

2) The orbits $\{ f_\lambda^n(x) \}$ is periodic or qusi-periodic or chaotic for $x \in [(-r_2,-r_1) \cup (-r_1,-\alpha_1) \cup (\alpha_1, r_1) \cup (r_1, r_2)]\backslash T_o$, where α_1 is a positive solution of $f_\lambda(x) = r_2$.

b. If $\lambda = \lambda_1$, then

1) $f_\lambda^n(x) \rightarrow x_1$ as $n \rightarrow \infty$ for $x \in [(-x_3,-\alpha_2) \cup (\alpha_2, x_3)]\backslash T_o$.

2) $f_\lambda^n(x) \rightarrow \infty$ as $n \rightarrow \infty$ for $x \in [(-\infty, -x_3) \cup (-\alpha_2, 0) \cup (0, \alpha_2) \cup (x_3, \infty)]\backslash T_o$, where α_2 is a positive solution of $f_\lambda(x) = x_3$.

c. If $\lambda_1 < \lambda < \lambda_2$, then

1) $f_\lambda^n(x) \rightarrow a_\lambda$ as $n \rightarrow \infty$ for $x \in [(-r_\lambda, -\alpha_3) \cup (\alpha_3, r_\lambda)]\backslash T_o$.

2) $f_\lambda^n(x) \rightarrow \infty$ as $n \rightarrow \infty$ for $x \in [(-\infty, -r_\lambda) \cup (-\alpha_3, 0) \cup (0, \alpha_3) \cup (r_\lambda, \infty)]\backslash T_o$, where α_3 is a positive solution of $f_\lambda(x) = r_\lambda$.

d. If $\lambda = \lambda_2$, then

1) $f_\lambda^n(x) \rightarrow x_2$ as $n \rightarrow \infty$ for $x \in [(-x_2, -\alpha_4) \cup (\alpha_4, x_2)]\backslash T_o$.

2) $f_\lambda^n(x) \rightarrow \infty$ as $n \rightarrow \infty$ for $x \in [(-\infty, -x_2) \cup (-\alpha_4, 0) \cup (0, \alpha_4) \cup (x_2, \infty)]\backslash T_o$, where α_4 is a positive solution of $f_\lambda(x) = x_2$.

e. If $\lambda > \lambda_2$, then $f_\lambda^n(x) \rightarrow \infty$ as $n \rightarrow \infty$ for all $x \in R\backslash T_o$.

Proof. Define the function $t_\lambda(x) = f_\lambda(x) - x$ for $x \in R\backslash \{0\}$. It is easily seen that the function $t_\lambda(x)$ is continuously differentiable for $x \in R\backslash \{0\}$. Note that the fixed points of the function $f_\lambda(x)$ are zeros of the function $t_\lambda(x)$.

a. For $0 < \lambda < \lambda_1$, Theorem (3.3) implies that $f_\lambda(x)$ has two fixed points $r_1 \in (0, x_1)$ and $r_2 \in (x_3, \infty)$ which are repelling. Since $t'_\lambda(r_1) < 0$ and $t'_\lambda(x)$ is continuous in some neighborhood of r_1, then $t'_\lambda(x) < 0$; i.e. $t_\lambda(x)$ is decreasing in this neighborhood. But $t_\lambda(x)$ is continuous, so for sufficiently small $\delta_1 > 0$, $t_\lambda(x) > 0$ in $(r_1 - \delta_1, r_1)$ and $t_\lambda(x) < 0$ in $(r_1, r_1+\delta_1)$. On the other hand, since $t'_\lambda(r_2) > 0$ and in a neighborhood of r_2, $t'_\lambda(x)$ is continuous then $t'_\lambda(x) > 0$, hence $t_\lambda(x)$ is increasing in this neighborhood. Thus for sufficiently small $\delta_2 > 0$, $t_\lambda(x) < 0$ in $(r_2-\delta_2, r_2)$ and $t_\lambda(x) > 0$, for $(r_2, r_2+\delta_2)$. Since $t_\lambda(x) \neq 0$ in $(0, r_1) \cup (r_1, r_2) \cup (r_2, \infty)$, then it follows that

$$t_\lambda(x) \begin{cases} > 0 & , x \in (0, r_1) \cup (r_2, \infty) \\ < 0 & , x \in (r_1, r_2) \end{cases} \tag{4.1}$$

To discuss the dynamics of $f_\lambda(x)$ we have two cases:

Case (1) ($x \in [(-\infty, -r_2) \cup (-\alpha_1, 0) \cup (0, \alpha_1) \cup (r_2, \infty)]\backslash T_o$):

By (4.1), it follows $f_\lambda(x) > x$. since $f_\lambda(x)$ is strictly increasing in (r_2, ∞) and $r_2 > 0$, then for all $x \in (r_2, \infty)$

$$0 < r_2 < x < f_\lambda(x) < f_\lambda^2(x) < \ldots < f_\lambda^n(x) < \ldots$$

Thus the sequence $\{f_\lambda^n(x)\}$ is increasing sequence which is not bounded above. Thus $f_\lambda^n(x) \rightarrow \infty$ as $n \rightarrow \infty$, for $x \in (r_2, \infty)$. Since $f_\lambda(x)$ is increasing in $(0, \alpha_1)$ and $f_\lambda(\alpha_1) = r_2$, then $f_\lambda(x)$ maps the interval $(0, \alpha_1)$ into (r_2, ∞). Thus $f_\lambda^n(x) \rightarrow \infty$ as $n \rightarrow \infty$ for $x \in \{(0, \alpha_1) \cup (r_2, \infty)\}$.

But $f_\lambda(x)$ is an even map, therefore $f_\lambda^n(x) \rightarrow \infty$ as $n \rightarrow \infty$, for $x \in [(-\infty, -r_2) \cup (-\alpha_1, 0)]\backslash T_o$.

Case (2) ($x \in [(-r_2, -r_1) \cup (-r_1,-\alpha_1) \cup (\alpha_1, r_1) \cup (r_1, r_2)]\backslash T_o$):

The system of dynamics of $f_\lambda(x)$ has no point attractors. Thus the dynamical system will move indefinitely, and the orbit $f_\lambda(x)$ will be periodic or qusi-periodic or even chaotic for $x \in (\alpha_1, r_1) \cup (r_1, r_2)$. Since $f_\lambda(x)$ is even, then the orbit has the same dynamical behavior in $\{(-r_2, -r_1) \cup (-r_1,-\alpha_1)\}$. This completes the proof of this case.

b. Assume that $\lambda = \lambda_1$, by Theorem (3.3), $f_\lambda(x)$ have an indifferent fixed point x_1 and a repelling fixed point x_3. Since $t'_\lambda(x_3) > 0$ and $t'_\lambda(x)$ is continues in a neighborhood of x_3, then $t'_\lambda(x) > 0$ in some neighborhood of x_3. Thus $t_\lambda(x)$ is increasing in a neighborhood of x_3. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_1 > 0$, $t_\lambda(x) < 0$ in $(x_3-\delta_1, x_3)$ and $t_\lambda(x) > 0$ in $(x_3, x_3+\delta_1)$. Further, since $t'_\lambda(x_1) < 0$ and is continuous is some neighborhood of x_1, then $t'_\lambda(x) < 0$ in this neighborhood. Hence $t_\lambda(x)$ is decreasing in this neighborhood. By the continuity of $t_\lambda(x)$, there exists $\delta_2 > 0$ such that $t_\lambda(x) > 0$ in $(x_1-\delta_2, x_1)$ and $t_\lambda(x) < 0$ in $(x_1, x_1+\delta_2)$. Since $t_\lambda(x) \neq 0$ in $(0, x_1) \cup (x_1, x_3) \cup (x_3, \infty)$, it now follows that

$$t_\lambda(x) \begin{cases} > 0 & , x \in (0, x_1) \cup (x_3, \infty) \\ < 0 & , x \in (x_1, x_3) \end{cases} \qquad (4.2)$$

Now to describe the dynamics of $f_\lambda(x)$, we can consider the following cases:

Case (1) ($x \in [(-x_3,-\alpha_2) \cup (\alpha_2, x_3)]\backslash T_o$):

By (4.2), it follows that $f'_\lambda(x) < 1$ for $x \in (\alpha_2, x_3)$, $f'_\lambda(x_1) = 1$ and $f'_\lambda(x) > 1$ for $x > x_3$, it follows that, using Mean Value Theorem, $|f'_\lambda(x) - x_1| < |x - x_1|$ for $x \in (\alpha_2, x_3)$. Therefore, $f^n_\lambda(x) \to x_1$ as $n \to \infty$ for $x \in (\alpha_2, x_1)$. Further, since $f_\lambda(x)$ is an even function, using the above arguments again, $f^n_\lambda(x) \to x_1$ as $n \to \infty$ for $x \in [(-x_3,-\alpha_2)\backslash T_o$.

Case (2) ($x \in [(-\infty, -x_3) \cup (-\alpha_2, 0) \cup (0, \alpha_2) \cup (x_3, \infty)]\backslash T_o$):

By (4.2), $f_\lambda(x) > x$ for $x \in (x_3, \infty)$. Moreover $f_\lambda(x)$ strictly increasing in this interval, then

$$0 < x < f_\lambda(x) < f^2_\lambda(x) < \ldots < f^n_\lambda(x) < \ldots$$

Thus, the sequence $\{ f^n_\lambda(x) \}$ is increasing and it is not bounded above.

Hence $f^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x \in (x_3, \infty)$.Further, since $f_\lambda(\alpha_2) = x_3$,and $f_\lambda(x)$ is decreasing in $(0, \alpha_2)$, therefore $f_\lambda(x)$ maps the interval $(0, \alpha_2)$ into (x_3, ∞). Thus $f^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x \in \{(0, \alpha_2) \cup (x_3, \infty)\}$. Since $f_\lambda(x)$ is even, then $f^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x \in \{(-\infty, -x_3) \cup (-\alpha_2, 0)\}\backslash T_o$.

c. For $\lambda_1 < \lambda < \lambda_2$, Theorem (3.3) implies that $f_\lambda(x)$ has an attracting fixed point $a_\lambda \in (x_1, x_2)$ and repelling fixed point $r_\lambda \in (x_2, x_3)$. Since $t'_\lambda(a_\lambda) < 0$ and $t'_\lambda(x)$ is continuous in some neighborhood of a_λ, then $t'_\lambda(x) < 0$ in some neighborhood of a_λ. Hence $t_\lambda(x)$ is decreasing in a neighborhood of a_λ. Since $t_\lambda(x)$ is continuous, then for sufficiently small $\delta_1 > 0$, $t_\lambda(x) > 0$ in $(a_\lambda -\delta_1, a_\lambda)$ and $t_\lambda(x) < 0$ in $(a_\lambda, a_\lambda+\delta_1)$. Further, since $t'_\lambda(r_\lambda) > 0$ and $t'_\lambda(x)$ is continuous in some neighborhood of r_λ, therefore $t'_\lambda(x) > 0$ in some neighborhood of r_λ. Thus $t_\lambda(x)$ is increasing in this neighborhood. By continuity of $t_\lambda(x)$, for sufficiently small $\delta_2 > 0$, $t_\lambda(x) < 0$ in $(r_\lambda - \delta_2, r_\lambda)$ and $t_\lambda(x) > 0$ in $(r_\lambda, r_\lambda + \delta_2)$. But $t_\lambda(x) \neq 0$ in $(0, a_\lambda) \cup (a_\lambda, r_\lambda)$, so

$$t_\lambda(x) \begin{cases} > 0 & , x \in (0, a_\lambda) \cup (r_\lambda, \infty) \\ < 0 & , x \in (a_\lambda, r_\lambda) \end{cases} \qquad (4.3)$$

To discuss the dynamics of $f_\lambda(x)$ for $\lambda_1 < \lambda < \lambda_2$, we have two cases:

Case (1) **($x \in [(-r_\lambda, -\alpha_3) \cup (\alpha_3, r_\lambda)]\backslash T_o$):**

By (4.3), it follows that, for $x \in (a_\lambda, r_\lambda)$, $f_\lambda(x) < x$. Since $f_\lambda(x)$ is decreasing, then by continuing forward iteration process, it gives that

$$a_\lambda < \ldots < f_\lambda^n(x) < \ldots < f_\lambda^2(x) < f_\lambda(x) < x$$

Therefore, the sequence $\{ f_\lambda^n(x) \}$ is decreasing and bounded below by a_λ, and there is no fixed point larger than a_λ. Hence $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in (a_\lambda, r_\lambda)$. Further, since $f_\lambda(\alpha_3) = r_\lambda$ and is increasing in (α_3, a_λ), $f_\lambda(x)$ maps the intervals (α_3, a_λ) into (a_λ, r_λ). It follows that, using the above arguments, $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in (\alpha_3, a_\lambda)$. Thus $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in (\alpha_3, r_\lambda)$. But $f_\lambda(x)$ is even function, so that, $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in (-r_\lambda, -\alpha_3)$. Thus $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$, for $x \in [(-r_\lambda, -\alpha_3) \cup (\alpha_3, r_\lambda)]\backslash T_o$.

Case (2) ($x \in [(-\infty, -r_\lambda) \cup (-\alpha_3, 0) \cup (0, \alpha_3) \cup (r_\lambda, \infty)]\backslash T_o$):

For $x \in (r_\lambda, \infty)$, $f_\lambda(x) > x$. Since $f_\lambda(x)$ is increasing, by continuing forward iteration process, it follows that

$$0 < r_\lambda < x < f_\lambda(x) < f_\lambda^2(x) < \ldots\ldots < f_\lambda^n(x) < \ldots$$

Therefore, the sequence $\{ f_\lambda^n(x) \}$ is increasing, unbounded above, and there is no fixed point larger than r_λ, the orbit must go to ∞ as $n \to \infty$. Hence $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (r_\lambda, \infty)$. Next, since $f_\lambda(\alpha_3) = r_\lambda$, $f_\lambda(x)$ maps the interval $(0, \alpha_3)$ into (r_λ, ∞), then using the above arguments, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (0, \alpha_3)$. Further, $f_\lambda(x)$ is even, therefore $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in [(-\infty, -r_\lambda) \cup (-\alpha_3, 0)]\backslash T_o$.

d. If $\lambda = \lambda_2$, by Theorem (3.3), $f_\lambda(x)$ has an indifferent fixed point x_2. Since $t'_\lambda(x_2) = 0$ and $t''_\lambda(x_2) > 0$, then $t_\lambda(x)$ has minimum at x_2. But $t_\lambda(x_2) = 0$, so $t_\lambda(x) > 0$, for all x in a neighborhood of x_2. Thus by continuity of $t_\lambda(x)$, for sufficiently small $\delta > 0$, $t_\lambda(x) > 0$ in $(x_2 - \delta, x_2) \cup (x_2, x_2 + \delta)$. Now, since $t_\lambda(x) \neq 0$ in $(0, x_2) \cup (x_2, \infty)$, it now follows that

$$t_\lambda(x) > 0 \qquad \text{for } x \in (0, x_2) \cup (x_2, \infty) \qquad (4.4)$$

The dynamics of $f_\lambda(x)$ is described as follows:

Case (1) ($x \in [(-x_2, -\alpha_4) \cup (\alpha_4, x_2)]\backslash T_o$):

Since x_2 is a minimum point for $t_\lambda(x)$ and $t_\lambda(x) > 0$ in $(0, x_2) \cup (x_2, \infty)$, then for $x \in (\alpha_4, x_2)$ we have $t'_\lambda(x) < 0$. So $f'_\lambda(x) - 1 < 0$, thus $f'_\lambda(x) < 1$. However $f'_\lambda(x) > 1$, for $x > x_2$, therefore by the mean value theorem $|f_\lambda(x) - f_\lambda(x_2)| = f'_\lambda(c)|x - x_2|$ for $c \in (\alpha_4, x_2)$. But $f_\lambda(x_2) = x_2$ and $f'_\lambda(c) < 1$, so $|f_\lambda(x) - x_2| < |x - x_2|$, for $x \in (\alpha_4, x_2)$. Hence

$f_\lambda^n(x) \to x_2$ as n → ∞ for x ∈ (α_4, x_2). Since $f_\lambda(x)$ is even function, then $f_\lambda^n(x) \to x_2$ as n→ ∞ for x ∈ (-x_2, -α_4). Thus $f_\lambda^n(x) \to x_2$ as n → ∞ for x ∈ [(-x_2, -α_4) ∪ (α_4, x_2)]\T_o.

Case (2) (x ∈ [(-∞, -x_2) ∪ (-α_4, 0) ∪ (0, α_4) ∪ (x_2, ∞)]\T_o):

According to equation (4.4) for x ∈ (x_2, ∞) we have $f_\lambda(x) > x$,

But $f_\lambda(x)$ is strictly increasing in this interval, therefore

$$0 < x < f_\lambda(x) < f_\lambda^2(x) < \ldots < f_\lambda^n(x) < \ldots$$

Thus { $f_\lambda^n(x)$ } is increasing sequence, which is not bounded above. Thus $f_\lambda^n(x) \to \infty$ as n → ∞ for x ∈ (x_2, ∞). Next, since f_λ(α_4) = x_2, $f_\lambda(x)$ maps the interval (0, α_4) into (x_2, ∞), then using the above arguments, $f_\lambda^n(x) \to \infty$ as n→ ∞ for x ∈ (0, α_4). But $f_\lambda(x)$ is even, so $f_\lambda^n(x) \to \infty$ as n → ∞ for x ∈ [(-∞, -x_2) ∪ (-α_4, 0)]\$T_o$.

e. For λ > λ_2, by proposition 3.2, $f_\lambda(x)$ has no fixed points. Since $f_\lambda(x) > x$ for x > 0, then $t_\lambda(x)$ > 0. But $f_\lambda(x)$ is strictly increasing, therefore

$$0 < x < f_\lambda(x) < f_\lambda^2(x) < \ldots\ldots < f_\lambda^n(x) < \ldots$$

Hence the sequence { $f_\lambda^n(x)$ } is increasing and it is not bounded above. Thus $f_\lambda^n(x) \to \infty$, for x ∈ (0, ∞). Since $f_\lambda(x)$ is an even function, $f_\lambda^n(x) \to \infty$ as n → ∞ for x ∈ (-∞, 0)\$T_o$. Thus, $f_\lambda^n(x) \to \infty$ as n → ∞, x ∈ [(-∞, 0) ∪ (0, ∞)]\$T_o$.

It follows by Theorem 4.1 that bifurcation in the dynamics of the function $f_\lambda(x)$ for all x ∈ R\{0} occur at the two critical parameter values λ = λ_1, λ_2, where $\lambda_1 = \dfrac{x_1^{2m+1}}{\cosh^m x_1}$ and $\lambda_2 = \dfrac{x_2^{2m+1}}{\cosh^m x_2}$ such that x_1 and x_2 are the unique positive real roots of the equations coth x = $\dfrac{mx}{2m-1}$ and coth x = $\dfrac{mx}{2m+1}$ respectively.

5. Dynamics on $\hat{C}$

The dynamics of functions from the one– parameter family $\mathcal{T}$ is indicated by describing the dynamics of $f_\lambda \in \mathcal{T}$ for z ∈ $\mathbb{C}$ in the present section. This includes the study of Julia set of f_λ in the extended complex plane $\hat{C}$ for different values of λ ∈ R. If the singular values of transcendental function is bounded, then the Julia set is a closure of escaping points of function under iterations [7]; i.e., J(f) =I (f), where I(f) = { z ∈ $\mathbb{C}$: $f^n(z) \to \infty$ as n→ ∞ and $f^n(z) \neq \infty$}

Proposition 5.1 let $f_\lambda \in \mathcal{T}$ and 0 < λ< λ_1. Then the Julia set J(f_λ) contains both real and imaginary axes.

Proof By Theorem 4.1(a), $f_\lambda^n(x) \to \infty$ for x ∈ [(-∞, -r_2) ∪ (-α_1, 0) ∪ (0, α_1) ∪ (r_2, ∞)]\$T_o$ and the orbits { $f_\lambda^n(x)$ } are chaotic for x ∈ [(-r_2,-r_1) ∪ (-r_1,-α_1) ∪ (α_1, r_1) ∪ (r_1, r_2)]\$T_o$, it follows that R\$T_o$ ⊂ J(f). In addition, since the poles and their preimages are contained in the Julia set, thus R⊂ J(f).

Now, we have f_λ maps the real interval into iR. Then R ∪ iR ⊂ J(f). Hence the forward orbits of all singular values tend to ∞. Therefore, J(f) contains both real and imaginary axes.

Proposition 5.2 let $f_\lambda \in \mathcal{T}$ and $\lambda = \lambda_1$. Then

1) The Fatou set F(f) contains a unique parabolic domain.

2) The Julia set contains the intervals $(-\infty, -x_3)$, $(-x_1, 0)$, $(0, x_1)$ and (x_3, ∞) and the Fatou set contains the intervals $(-x_3, -x_1)\setminus T_o$ and (x_1, x_3), where x_1 is indifferent fixed point, x_3 is a repelling fixed point of $f_\lambda(x)$.

Proof

1) Let $U_1 = \{z \in \mathbb{C}\ ;\ f_\lambda^n(x) \to x_1 \text{ as } n \to \infty\}$. By Theorem (3.3-2), it follows that $f_\lambda(z)$ has an indifferent fixed point at $x = x_1$. Since, by Theorem 4.1(b), $f_\lambda^n(x) \to x_1$ as $n \to \infty$ for $x \in (x_1, x_3)\setminus T_o$ and $f_\lambda^n(x) \to \infty$ for $x \in (0, x_1)$, the indifferent fixed point x_1 lies on the boundary of U_1. Thus, U_1 is a parabolic domain in the Fatou set of $f_\lambda(z)$. Again, by Theorem 4.1(b), it follows that the forward orbits of all singular values either tend to x_1 or tend to ∞. Therefore, by Theorem 1.1, F(f) does not contain any parabolic domain other than U_1.

2) By Theorem 4.1(b), for $\lambda = \lambda_1$. $f_\lambda^n(x) \to x_1$ for $x \in [(-x_3, -x_1) \cup (x_1, x_3)]\setminus T_o$. Therefore, the intervals $(-x_3, -x_1)\setminus T_o$ and (x_1, x_3) are contained in the parabolic domain U_1. Since Fatou set contains parabolic domains, the intervals $(-x_3, -x_1)\setminus T_o$ and (x_1, x_3) belong to the Fatou set. Further, $f_\lambda^n(x) \to \infty$ for $x \in [(-\infty, -x_3) \cup (-x_1, 0) \cup (0, x_1) \cup (x_3, \infty)]\setminus T_o$. Thus, these intervals are contained in the Julia set of $f_\lambda(z)$. Since pole and preimages of the pole also belong to the Julia set.

Proposition 5.3 let $f_\lambda \in \mathcal{T}$ and $\lambda_1 < \lambda < \lambda_2$. Then

1) The Fatou set F(f) does not contain any basin or parabolic domain except the basin of attraction of the real attracting fixed point a_λ of $f_\lambda(x)$.

2) The intervals $[(-r_\lambda, -\alpha_3) \cup (\alpha_3, r_\lambda)]\setminus T_o$ are contained in F(f) and the intervals $(-\infty, -r_\lambda)$, $(-\alpha_3, 0)$, $(0, \alpha_3)$, (r_λ, ∞) are contained in J(f).

Proof

1) By Theorem (3.3-3), $f_\lambda(z)$ has a real attracting fixed point a_λ. Let

$$A(a_\lambda) = \{z \in \mathbb{C}\ ;\ f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}$$

be the basin of attraction of the attracting fixed point a_λ. For any point $z \in A(a_\lambda)$, the sequence of iterates $\{f_\lambda^n(z)\}$ tends to a_λ as $n \to \infty$ so that the sequence of iterates $\{f_\lambda^n(z)\}$ forms a normal family at z. Consequently, z ∈ F(f). Thus, $A(a_\lambda) \subset$ F(f). Further, by Theorem 4.1(c), it follows that the forward orbits of all singular values either tend to a_λ or tend to ∞. Therefore, by Theorem 1.1, F(f) does not contain the basin of attractions other than $A(a_\lambda)$. That F(f) does not contain any parabolic domains follows similarly using Theorem 1.1.

2) By Theorem 4.1(c), $f_\lambda^n(x) \to \infty$ for $x \in [(-\infty, -r_\lambda) \cup (-\alpha_3, 0) \cup (0, \alpha_3) \cup (r_\lambda, \infty)]\setminus T_o$. Therefore, the intervals $[(-\infty, -r_\lambda) \cup (-\alpha_3, 0)]\setminus T_o$, $(0, \alpha_3)$ and (r_λ, ∞) belong to the Julia set of $f_\lambda(z)$. Since pole and preimages of the pole lie in the Julia set, it now follows that the Julia set contains the intervals $(-\infty, -r_\lambda)$, $(-\alpha_3, 0)$, $(0, \alpha_3)$ and (r_λ, ∞). Again, by Theorem 4.1(c), $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in [(-r_\lambda, -\alpha_3) \cup (\alpha_3, r_\lambda)]\setminus T_o$. Therefore, it follows that the intervals $(-r_\lambda, -\alpha_3)\setminus T_o$ and (α_3, r_λ) are contained in the basin of attraction $A(a_\lambda)$. Since the Fatou set contains basin of attractions, the intervals $(-r_\lambda, -\alpha_3)\setminus T_o$ and (α_3, r_λ) belong to the Fatou set of $f_\lambda(z)$.

Proposition 5. 4 let $f_\lambda \in \mathbf{T}$ and $\lambda = \lambda_2$. Then

1) The Fatou set F(f) contains a unique parabolic domain.
2) The Julia set J(f) contains the intervals $(-\infty, -x_2)$, $(-\alpha_4, 0)$, $(0, \alpha_4)$ and (x_2, ∞) and the Fatou set F(f) contains the intervals $(-x_2, -\alpha_4)\backslash T_o$ and (α_4, x_2)

Proof The proof of proposition is analogous to that of proposition 5.2 for the case $\lambda = \lambda_1$ and is hence omitted.

Proposition 5.5 let $f_\lambda \in \mathbf{T}$ and $\lambda > \lambda_2$. Then, the Julia set J(f) contains both real and imaginary axes.

Proof By Theorem 4.1(e), $f_\lambda^n(x) \to \infty$ for all $x \in R\backslash T_o$, it follows that $R\backslash T_o \subset J(f)$. Since $f_\lambda(x)$ maps imaginary axis on real axis and $f_\lambda^n(x) \to \infty$ for all $x \in R\backslash T_o$, it gives that $iR\backslash iT_o \subset J(f)$. Since the asymptotic value 0 is also a pole of $f_\lambda(z)$ $f_\lambda(z)$. $0 \in J(f)$ and since preimages of pole are contained in Julia set, $T_o \subset J(f)$. Therefore, J(f) contains both real and imaginary axes.

Proposition 5.6 For $\lambda > \lambda_2$, Fatou set cannot have any basin of attraction, parabolic domain.

Proof Since $f_\lambda^n(x) \to \infty$ for all $x \in R\backslash T_o$, the forward orbit of critical values on real axis tend to ∞. Further, the asymptotic value 0 is also a pole of $f_\lambda(z)$ so that orbit of 0 termintes. Therefore, Fatou set cannot have any basin of attraction, parabolic domain.

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