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Lyapunov Function to Study the Stability of a Solution for Harmonic Stochastic Differential Equations

[1] Hekmat Sh. Mustafa [2] Zahra A. Edrees

[1,2]Department of Mathematics , College of Education for Pure Sciences , University of Al-Hamdaniya.

[1] hekmat78@uohamdaniya.edu.iq [2]zahramath@uohamdaniya.edu.iq

Abstract :

The stability of a Lyapunov function-based harmonic stochastic differential equation solution was investigated in this work. Two distinct first-order nonlinear (harmonic) stochastic differential equations were subjected to the aforementioned technique. In addition to being simple to find compared to previously established methods, our concept of studying the stability of the quadratic expectation convergent to the zero solution for harmonic stochastic differential equations can be applied to a wide range of applied problems, as demonstrated by our study of two examples. We came to the conclusion that, unlike other approaches, this one does not require an accurate solution to stochastic differential equations.

Keywords: stochastic, harmonic, lyapunov function, stability.

Introduction:

Stochastic differential equations have become very important in many fields, including branches of science as well as industry . We will use the random form of issues that come randomly or accidentally, etc... All basic definitions and proofs required in this paper are mentioned in [1-3]. We will focus on random stability in this paper. We will introduce stability theory [4]. In his work [5], Hasselman presented the idea of climatic stochastic models that appeared in 1976 to replace the complete deterministic system with a random differential equation. He suggested that when combining the effect of "weather" (rapid variables) in form of stochastic noise, there would be an improvement of deterministic models of "climate" (slow variables). Different ways of weather fluctuations have been determined by univariate linear systems. It was anticipated by researchers that the results of these models would be a valuable resource for examining the random effect of more complicated dynamics, such as in [6]. The Lyapunov function has been utilized to determine the prerequisites for the general system of the zero solution of the stochastic differential equation. Several examples were used to confirm the findings. In this paper, we employ a stochastic model.

Lyapunov's concept of stability states that the system's state is not sensitive to slight variations in the system's parameters or initial state . The paths that deviate must remain close to each other in relation to the continuous system as well as at a certain instant close to all another in all following stages [7] . The scientist Lyapunov created a technique that allows one to ascertain stability without having to solve an equation . We employ the second Lyapunov technique.

Stochastic differential equations: **[1]**

Let the probability space be (Q, F, P). and let $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be measurable functions. Then, let $W_s = (W_1(s), W_2(s), \dots, W_m(s))$ be an m-dimensional Wiener process. The stochastic differential equation is then solved by the process:

$y(s) = (y_1(s), y_2(s), \dots, y_m(s))$, $s \in [0, T]$.

$$dy(s) = f\big(s, y(s)\big)ds + g\big(s, y(s)\big)dW(s) \qquad (1)$$

$f(s, y(s)) \in \mathbb{R}, g(s, y(s)) \in \mathbb{R}$. We get the solution to stochastic differential equations from taking integral equation(I)

$$y(s) = y(0) + \int_0^s f(t, y(t))dt + \int_0^s g\big(t, y(t)\big)dW(t) \qquad (2)$$

Suppose each has an initial value $y_s(0) = y(0)$, $y(0) \in \mathbb{R}^n$, Since there is only one global solution, denoted by $y(s; s_0, y(0))$, the solution to the equation, $y(0) = 0$, is the same as the starting value, $y_s(0) = 0$. It is said that this solution is trivial.

In order to determine stability without having to solve the equation, Lyapunov created a method . The second Lyapunov approach is applied: Define H as the set of continuous non-decreasing functions $\mathcal{M}: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\mathcal{M}(0) = 0$ and $\mathcal{M}(\mathcal{L}) > 0$ if $\mathcal{L} > 0$; for $j > 0$, let $S_j = \{y \in \mathbb{R}^n : |y| < j\}$, an ongoing function positive-definiteness of $U(y, s)$ defined on $S_j \times [s_0, \infty]$ is indicated if $U(0, s) \equiv 0$ and, for some $\mathcal{M} \in H$, $U(y, s) \geq \mathcal{M}(|y|)$ for all $(y, s) \in S_j \times [s_0, \infty]$. If $-U$ is positive-definite, then a function $U(y, s)$ is considered negative-definite . A non-negative, continuous function If for some $\mathcal{M} \in H$, $U(y, s)$ is said to be decrescent, $U(y, s) \leq \mathcal{M}(|y|)$ for all $(y, s) \in S_j \times [s_0 , \infty]$. when a function $U(y, s)$ defined on $\mathbb{R}^n \in S_j \times [s_0, \infty]$ is radially unbounded, it means $\lim_{|y| \to \infty} (\inf_{s \geq s_0} U(y, s)) = \infty$. The set of all continuous functions $U(y, s)$ from $S_j \times [s_0, \infty]$ to $\mathbb{R}_+$ is denoted by $C^{1,1}\big(S_j \times [s_0, \infty]; \mathbb{R}_+\big)$. This set has continuous first order partial derivatives with respect to each component of y and to s . After that, a function of s with the derivative is represented by $u(s) = U(s, y_s)$. A function of t with the derivative is $u(s) = U(s, y_s)$

$$\dot{u}(s) = U_s(s, y_s) + U_y(s, y_s)f(s, y_s) = \frac{\partial U}{\partial s}(s, y_s) + \sum_{i=1}^{n} \frac{\partial U}{\partial y_i}(s, y_s)\, f(s, y_s) .$$

The distance from equilibrium for y_s , as determined by $U(s, y_s)$, does not increase if $\dot{u}(s) \leq 0$, indicating that u(s) will not rise . The distance y_s from the equilibrium point, as measured by $U(s, y_s)$, does not increase if $\dot{u}(s) < 0$ since u(s) is not going to increase . If $\dot{u}(s) < 0$ is less than zero, then u(s) will become zero . Since the distance will also become zero, this indicates that y_s will approaching zero [1].

Theorem 1: [7]

The zero solution is said to be stable, providing a positive-definite function $U(s, y_s) \in C^{1,1}(S_j \times [s_0, \infty]; \mathbb{R}_+)$ exists, such that

$$\dot{U}(y, s) = U_s(s, y_s) + U_y(s, y_s) f(s, y_s) \leq 0$$

For all $(y, s) \in S_j \times [s_0, \infty]$.

Theorem 2 : [7]

The zero solution is said to be asymptotically stable , presuming a positive-definite decrease function exists $U(s, y_s) \in C^{1,1}(S_j \times [s_0, \infty]; \mathbb{R}_+)$ such that $\dot{U}(y, s)$ is negative - definite .

When every pair of $\mathcal{E} \in (0,1)$ and $\mathcal{L} > 0$ assuming it exists $\ell = \ell(\mathcal{E}, \mathcal{L}, s_0) > 0$ such that

$$P\{|y(s; s_0, y_0)| < \mathcal{L}, \text{ for all } s \geq s_0\} \geq 1 - \mathcal{E} \tag{3}$$

Every time $|y_0| < \ell_0$, Then The zero solution of equation (1) is called the stochastically stable solution or the stable probability solution . Otherwise, it is said to be stochastically unstable . If stochastic stability is present and, additionally, assuming it is, for each $\mathcal{E} \in (0,1)$, $\ell = \ell(\mathcal{E}, \mathcal{L}, s_0) > 0$ such that

$$P\left\{\lim_{n \to \infty} y(s; s_0, y_0) = 0\right\} \geq 1 - \mathcal{E} \tag{4}$$

Stochastically asymptotically stable is the term used to describe the zero solution of equation (1) when $|y_0| < \ell_0$. If the zero solution to equation (1) is stochastically stable, it is also stochastically asymptotically stable in the . Furthermore, for all $y_0 \in \mathbb{R}^d$ [1].

$$P\left\{\lim_{n \to \infty} y(s; s_0, y_0) = 0\right\} = 1 \tag{5}$$

Assuming we wanted to use the starting value as a random variable, we can now generalize Theorem (1) to the stochastic case . This definition can be reduced to the deterministic definitions that correspond to it . For each $g(s, y) = 0$. The family of all nonnegative functions $U(s, y)$ defined on $S_j \times \mathbb{R}_+$ is denoted by the notation $C^{2,1}(S_j \times \mathbb{R}_+, \mathbb{R}_+)$ according to the condition $0 < j \leq \infty$. Provide a definition of to the differential operator L associated with equation (1).

$$L = \frac{\partial}{\partial s} + \sum_{i=1}^{n} \frac{\partial}{\partial y_i}(s, y_s) f_i(s, y_s) + \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial y_i \, \partial y_j} [g(s, y_s) g^T(s, y_s)]_{ij} .$$

If L acts on a function $U \in C^{2,1}(S_j \times \mathbb{R}_+, \mathbb{R}_+)$, then:

$$LU(s,y) = U_t(s,y_s) + U_y(s,y_s)f(s,y_s) + \frac{1}{2}\text{trace}[g^T(s,y_s)U_{yy}(s,y_s)g(s,y_s)]$$

See [7] for the definition of U_s , U_y and U_{yy} , By Ito formula $y(s) \in S_j$, then

$$dU\big(s,y(s)\big) = LU(s,y(s)ds + U_y(s,y_s)g(s,y_s)dW(s)$$

We shall see that the inequality $\dot{U}(s,y) \leq 0$ will be substitute by $LU(s,y) \leq 0$ in order to obtain stochastic stability [7].

Theorem3 : [7]

1. It is argued that if a positive-definite function exists, the zero solution of equation (1) is stochastically stable $U(s,y_s) \in C^{2,1}(S_j \times [s_0,\infty]; \mathbb{R}_+)$ such that $LU(s,y) \leq 0$ for all $(s,y) \in S_j \times [s_0,\infty]$ (proof see [5]) .
2. If $LU(s,y)$ is negative-definite and $U(s,y_s) \in C^{2,1}(S_j \times [s_0,\infty]; \mathbb{R}_+)$ has a positive-definite decrescent function, then the zero solution of equation (1) is considered stochastically asymptotically stable. (See [5] for proof).
3. The zero solution of equation (1) is said to be stochastically asymptotically stable in the large $U(s,y_s) \in C^{2,1}(\mathbb{R}^d \times [s_0,\infty]; \mathbb{R}_+)$ such that if a positive-definite decrescent radially unbounded function exists, then $LU(s,y)$ is negative-definite. (See [5] for proof).

Theorem 4:[1, 8]

Lyapunov function U is given

$$U(y_s) = y_s^T Q y_s \tag{6}$$

Q is a symmetric matrix with positive definiteness with dimensions of $m \times m$. The function LU is:

$$LU(y_s) = 2y_s^T Qf(s,y_s) + g^T(s,y_s)Q_d g(s,y_s) \tag{7}$$

If $LU(y_s)$ is negative-definite in some neighbourhood of $y_s = 0$ for $s \geq s_0$ with regard to equation (I), the zero solution of equation (1) is said to be asymptotically mean square stable on the interval $[0,\infty)$.

proof:

We first compute the Lyapunov function using equation (6) and equation (1).

$$dU(y_s) = U(y_s + dy_s) - U(y_s) = (y_s^T + dy_s^T)Q(y_s + dy_s) - y_s^T Q y_s$$

$$= (y_s^T + f^T(s,y_s)ds + g^T(s,y_s)dW_s)Q(y_s + f(s,y_s)ds + g(s,y_s)dW_s) - y_s^T Q y_s$$

Using the guidelines $ds \cdot ds = ds \cdot dW_s = dW_s \cdot ds = 0$, $dW_s \cdot dW_s = ds$ and we have

$$dU(y_s) = y_s^T Qf(s, y_s)ds + y_s^T Qg(s, y_s)dW_s + f^T(s, y_s)dsQy_s$$
$$+g^T(s, y_s)dW_s Qy_s + g^T(s, y_s)dW_s Qg(s, y_s)ds$$

After applying expectation $E\{dU(y_s)\}$, we get

$$E\{dU(y_s)\} = y_s^T Qf(s, y_s)ds + f^T(s, y_s)Qy_s ds + g^T(s, y_s)Q_d g(s, y_s)ds = LU(y_s)ds$$

$-LU(y_s) \geq kU(y_s)$, k is constant

$\frac{d}{dt}E\{U(y_s)\} \leq -kE\{U(y_s)\}$,

$E\{U(y_s)\} \leq \exp(-ks)$.

Therefore,

$\lim_{t\to\infty} E^2\{y_s\} = \lim_{s\to\infty} E\{y_s y_s^T\} = \Theta$

The method used to obtain the result of theorem (5) can also be used to obtain an analogous result of instability**.** It suggests that asymptotically stable in the large. If, with regard to equation (1), $LU(y_s)$ is positive-definite in some neighborhood, then $y_s = 0$. So, the zero-valued answer to equation (1) is precarious, as theorem (4) states:

The last portion derives the linear stochastic system of differential equations:

$$dy_s = \lambda y_s ds + c y_s dW_s \tag{8}$$

Where λ, c are $m \times m$ constant matrices.

corollary 1: [8]

We define:

$$LU(y_s) = y_s^T Q\lambda y_s + (\lambda y_s)^T Qy_s + (cy_s)^T Q_d cy_s$$

It is, with regard to system (8), negative definite in the neighborhood of $y_s = 0$ for $s \geq s_0$. Based on the results of Theorem (4), the solution to Equation (8) is stochastically asymptotically stable.

Proof:

$$dU(y_s) = U(y_s + dy_s) - U(y_s) = (y_s^T + dy_s^T)Q(y_s + dy_s) - y_s^T Qy_s$$
$$= y_s^T Q\lambda y_s ds + y_s^T Qcy_s dW_s + (\lambda y_s)^T dsQy_s + (cy_s)^T dW_s Qy_s + (cy_s)^T Qcy_s ds$$

$$E\{dU(y_s)\} = y_s^T Q\lambda y_s ds + (\lambda y_s)^T Qy_s ds + (cy_s)^T Q_d cy_s ds = LU(y_s)ds$$

Remark 1: [1, 6]

We define:

$$LU(y_s) = y_s^T Q \lambda y_s + (\lambda y_s)^T Q y_s + (cy_s)^T Q_d c y_s$$

The zero solution of equation (8) is deemed unstable on the interval $[0, \infty)$ if $LU(y_s)$ is positive definite with respect to it.

Example 1:

We look at stochastic differential equations

$$dy_s = (-1 - \cos y_s)ds + (\cos y_s - 1)dW_s$$

We will use theorem (5) to determine stability. The Lyapunov function is defined as follows: Q=1 in form (7)

$$dU(y_s) = y_s^T Q(-1 - \cos y_s)ds + y_s^T Q(\cos y_s - 1)dW_s + (-1 - \cos y_s)^T ds Q y_s$$

$$+(\cos y_s - 1)^T dW_s Q y_s + (\cos y_s - 1)^T Q(\cos y_s - 1)ds$$

Then:

$$E\{dU(y_s)\} = y_s^T Q(-1 - \cos y_s)ds + (-1 - \cos y_s)^T Q y_s ds$$

$$+(\cos y_s - 1)^T Q_d(\cos y_s - 1)ds$$

$$= (2y_s(\cos y_s - 1) + (\cos y_s - 1)^2)ds = LU(y_s)ds$$

The $LU(y_t)$ function is negative definite. It can be observed that the inequality $(2y_s(\cos y_s - 1) + (\cos y_s - 1)^2) < 0$ is satisfied. Stable is the trivial solution. We now establish a limit:

$$\lim_{t\to\infty} E^2\{y_s\} = \lim_{i\to\infty} E\{(2y_s(\cos y_s - 1) + (\cos y_s - 1)^2)\} \neq \Theta$$

Thus, while stable, the trivial solution is not asymptotically stable.

Example 2:

We examine the following type of two-dimensional stochastic differential equation:

$$dy_1(s) = \sin(y_2(s))ds + \cos(y_1(s))dW_s$$

$$dy_2(s) = -\sin(y_1(s))ds + dW_s$$

Where $y = (y_1, y_2)^T$ is a two-dimensional vector state function vector state function, f, g vector functions, $f = (\sin(y_2(s)), -\sin(y_1(s)))^T$, $g = (\cos(y_2(s)), 1)^T$ on a neighborhood $\mathcal{O}(0,0)$ with the exception of the zero point (0,0),

Conclusion:

We show that our approach to studying asymptotic mean square stability of the zero solution of stochastic differential equations performs better than known methods for stability analysis of stochastic differential equations. As previously stated, the exact solution to stochastic differential equations is not necessary for this method to work . These kinds of equations can be applied in a wide range of fields, including biomedical research [5, 9], epidemic modeling [3], animal motion description [10], signal reception [11, 12], and many more. Nevertheless, they employ estimation techniques that diverge from ours and are based on numerical or statistical methods.

References:

1. Klimešová, M. *Stability of the stochastic differential equations*. in *Proceedings of the 21st Conference Student EEICT*. 2015.
2. Oksendal, B., *Stochastic differential equations: an introduction with applications*. 2013: Springer Science & Business Media.
3. Rao, F. *Dynamics analysis of a stochastic SIR epidemic model*. in *Abstract and applied analysis*. 2014. Hindawi.
4. Khasminskii, R., *Stochastic stability of differential equations*. 2012: Springer.
5. Hasselmann, K., *Stochastic climate models part I. Theory.* tellus, 1976. **28**(6): p. 473-485.
6. Arnold, L., *Hasselmann's program revisited: The analysis of stochasticity in deterministic climate models*. 2001: Springer.
7. Mao, X., *Stochastic differential equations and applications*. 2007: Elsevier.
8. Dzhalladova, I., M. Růžičková, and V. Štoudková Růžičková, *Stability of the zero solution of nonlinear differential equations under the influence of white noise.* Advances in Difference Equations, 2015. **2015**: p. 1-11.
9. Donnet, S. and A. Samson, *A review on estimation of stochastic differential equations for pharmacokinetic/pharmacodynamic models.* Advanced drug delivery reviews, 2013. **65**(7): p. 929-939.
10. Brillinger, D.R., et al., *Employing stochastic differential equations to model wildlife motion.* Bulletin of the Brazilian Mathematical Society, 2002. **33**: p. 385-408.
11. Picchini, U., et al., *Parameters of the diffusion leaky integrate-and-fire neuronal model for a slowly fluctuating signal.* Neural computation, 2008. **20**(11): p. 2696-2714.
12. Swain, S., *Handbook of stochastic methods for physics, chemistry and the natural sciences.* Optica Acta: International Journal of Optics, 1984. **31**(9): p. 977-978.