

An alternative approach for solving Kummer's differential equations

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ARTICLE INFO

Received: 16/09/2025
Revised: 20/12/2025
Accepted: 24/12/2025
Available online: 08/08/2026
August Issue
[10.37652/juaps.2025.165191.1692](https://doi.org/10.37652/juaps.2025.165191.1692)

 CITE @ JUAPS

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ABSTRACT

Kummer's equation has been used in many physical, engineering, and scientific applications, including the study of the Boltzmann energy distribution, the hydrogen atom through transformation of the radial equation into Kummer's equation, electrical-circuit modeling, and the solution of Laplace's equation in circular and cylindrical coordinates. This paper presents an alternative solution to Kummer's differential equation based on the Dirac delta distribution using the Fourier transform method (FTM). The proposed solution does not depend on Kummer's confluent hypergeometric function or Gordon's function. In addition, the Frobenius method was employed to solve the differential equation obtained after applying the Fourier transform method. The validity and effectiveness of the proposed method are demonstrated. However, in certain cases, classical solutions may be insufficient to describe singular phenomena or may fail to exist within standard function spaces. This motivates the study of generalized solutions formulated within the framework of distribution theory.

Keywords: *Fourier Transform Method, Inverse Fourier Transform, Kummer's differential equation, Dirac delta distribution*

1 INTRODUCTION

Differential equations are important in research because of their broad applications in medicine, science, and engineering. They can be used to study and simulate the emergence and spread of cancer, as well as to describe electric current flow, wave motion, and oscillatory systems [1–5].

Kummer's equation is a differential equation named after the mathematician Ernst Eduard Kummer. It is associated with the solution of Laplace's equation in spherical coordinates. Kummer polynomials form a special class of Kummer's confluent hypergeometric functions and are widely used in physics and engineering, particularly in problems involving Gaussian approximations, gamma distributions, and Poisson processes. The solution of Kummer's equation is important because it arises when the separation-of-variables method is applied in spherical coordinates to equations such as the Schrödinger equation, heat equation, and Poisson equation, which are widely used in quantum mechanics

and related fields [6–8]. Kummer's differential equation is also used in modeling physical phenomena such as heat transfer and the hydrogen atom [9].

Many researchers have used different methods to solve Kummer's equation. Prosser [10] studied the relationships among Kummer's solutions of the hypergeometric equation using a set of transformations. Ortner and Wagner [11] introduced a distributional solution for Kummer's differential equation. Jung [12] used Kummer functions to approximate analytic functions. In addition, Morita et al. [13] employed three theorems to obtain a particular solution of Kummer's differential equation.

This paper is organized as follows. Section 2 presents important definitions and properties of the Fourier transform and its relationship with the Dirac delta distribution. Section 3 applies the Fourier transform method to obtain a solution of Kummer's differential equation and verifies the validity of the solution. The conclusions are presented in the final section.

In this paper, the Fourier transform method is used to

solve Kummer's differential equation, given by:

$$xy'' + (\beta - x)y' - \alpha y = 0.$$

Where α, β are constants.

2 PRELIMINARIES

Definition 2.1: [14] The Fourier transform of $y(\tau)$ is given by

$$F(y(\tau)) = \int_{-\infty}^{\infty} y(\tau) e^{-i\omega\tau} d\tau = Y(\omega)$$

Definition 2.2: [14] The inverse Fourier transform of $Y(\omega)$ is given by

$$F^{-1}(Y(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Y(\omega) e^{i\omega\tau} d\omega = y(\tau)$$

Here, the Fourier transform of some functions [15]

$$F(1) = 2\pi\delta(\omega).$$

$$F(\tau^n) = 2\pi i^n \delta^{(n)}(\omega).$$

Theorem 2.1 [15] If $F(y(\tau)) = Y(\omega)$ then $F(y^{(n)}(\tau)) = (i\omega)^n F(y(\tau))$. **Theorem 2.2** [15] If $F(y(\tau)) = Y(\omega)$ then $F(\tau^n y(\tau)) = i^n \frac{d^n Y(\omega)}{d\omega^n}$. **Theorem 2.3** [15] If $F(y_1(\tau)) = Y_1(\omega)$, $F(y_2(\tau)) = Y_2(\omega)$ then

$$F(A_1 y_1(\tau) + A_2 y_2(\tau)) = A_1 Y_1(\omega) + A_2 Y_2(\omega).$$

Theorem 2.4 (Cauchy Integral Formula) [16]: Let f be an analytic function in C , and on a simple closed cycle C , which is oriented in the positive direction. If w_0 is any point in C , then

$$f^{(n)}(w_0) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - w_0)^{n+1}} dw$$

Definition 2.3: The Dirac delta distribution is defined as follows

$$\delta(\tau) = \{0, \quad \tau \neq 0/\infty, \quad \tau = 0.$$

Moreover, [17] The Dirac delta distribution is the limit for $\varepsilon \rightarrow 0$ function defined by

$$\delta_\varepsilon(\tau) = \begin{cases} \frac{1}{\varepsilon}, & 0 < \tau < \varepsilon/2, \\ 0, & \tau < 0, \quad \tau > \varepsilon \end{cases}$$

That is $\delta(\tau) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(\tau)$.

Now, we present the important properties of the

- (1) Dirac delta distribution 1. Sifting Property [17, 18] $\int_{-\infty}^{\infty} \delta(\tau - \tau_0) y(\tau) d\tau = y(\tau_0)$. 2. Finite-Part (Moment) Identity For $n \in N$ [17, 18]

$$\int_{-\infty}^{\infty} \frac{\delta(\tau - \tau_0)}{(\tau - \tau_0)^n} y(\tau) d\tau = \frac{y^{(n)}(\tau_0)}{n!}$$

3. Product with Monomials For $n \in N$ [17, 18]

$$(\tau - \tau_0)^n \delta^{(n)}(\tau - \tau_0) = (-1)^n n! \delta(\tau - \tau_0).$$

4. Action of Derivatives of the Delta Distribution For $n \in N$ [17, 18]

$$\int_{-\infty}^{\infty} \delta^{(n)}(\tau - \tau_0) y(\tau) d\tau = (-1)^n y^{(n)}(\tau_0)$$

3 APPLYING THE FOURIER TRANSFORM METHOD (FTM) ON KUMMER'S DIFFERENTIAL EQUATION

In this section, we present a fundamental theory for solving Kummer's differential equation by Fourier Transform Method.

Theorem 3.1 Let us consider Kummer's differential equation,

$$xy'' + (\beta - x)y' - \alpha y = 0. \quad (2)$$

The weak solutions of equation (2) are

$$y(x) = \delta(x) \text{ for } \alpha = 1, \beta = 2$$

$$y(x) = \left(\frac{i}{(\mu - 1)!} \right) \sum_{k=0}^{\mu-1} x^k \binom{\mu-1}{k} (-1)^{\mu-1-k} (i)^{-\lambda-\mu+2k+1} \lambda(\lambda+1)$$

$$\cdots (\lambda + \mu - k - 2) + \left(\frac{i}{(\lambda - 1)!} \right) \sum_{k=0}^{\lambda-1} x^k e^x \binom{\lambda-1}{k} (-1)^{-\mu}$$

$$\times (i)^{-\lambda-\mu+2k+1} \mu(\mu+1)(\mu+2) \cdots (\lambda + \mu - k - 2).$$

$$\text{for } \alpha - 1 < 0, \beta - \alpha - 1 < 0, \mu = 1 - \alpha, \lambda = 1 + \alpha - \beta$$

$$y(x) = - \left(\frac{iH(-x)}{(P-1)!} \right) \sum_{k=0}^{P-1} \binom{P-1}{k} (ix)^k (-i)^{\alpha-P+k} e^x$$

$$\times (\alpha - 1)(\alpha - 2) \cdots (\alpha - P + k + 1).$$

$$\text{for } \alpha - 1 > 0, \beta - \alpha - 1 < 0, P = 1 + \alpha - \beta$$

$$y(x) = \left(\frac{i}{(q-1)!} \right) \sum_{k=0}^{q-1} \binom{q-1}{k} (ix)^k (i)^{\beta-\alpha-q+k} \\ \times (\beta-\alpha-1)(\beta-\alpha-2) \cdots (\beta-\alpha-q+k+1). \\ \text{for } \alpha-1 < 0, \beta-\alpha-1 > 0, q = 1-\alpha,$$

And

$$y(x) = a_0 \sum_{k=0}^{\beta-\alpha-1} \frac{(\alpha-\beta+1)(\alpha-\beta+2) \cdots (\alpha-\beta+k) \delta^{(k+\alpha-1)}}{i^{\alpha-1} k!} \\ \text{for } \alpha-1 > 0, \beta-\alpha-1 > 0$$

Proof: Equation (2) can be written as

$$xy'' + \beta y' - xy' - \alpha y = 0.$$

Applying the Fourier transform to the previous equation yields.

$$F[xy''] + \beta F[y'] - F[xy'] - \alpha F[y] = 0 \\ i \frac{d}{dw} \left((iw)^2 Y \right) + \beta (iwY) - i \frac{d}{dw} (iwY) - \alpha Y = 0 \\ (-iw^2 + w) \frac{dY}{dw} = -(-2iw + i\beta w + 1 - \alpha)Y$$

$$\frac{dY}{Y} = \frac{(2iw - i\beta w - 1 + \alpha)}{(-iw^2 + w)} \\ \int \frac{dY}{Y} = \int \frac{(2iw - i\beta w - 1 + \alpha)}{(-iw^2 + w)} dw \\ Y = e^{\int \frac{(2iw - i\beta w - 1 + \alpha)}{-iw(w+i)} dw} \\ \frac{-i\beta w - 1 + \alpha}{iw(w+i)} dw = \int \frac{(\beta-2)w + (\alpha-1)i}{w(w+i)} dw \\ \frac{(\beta-2)w + (\alpha-1)i}{w(w+i)} = \frac{A}{w} + \frac{C}{w+i} \\ (\beta-2)w + (\alpha-1)i = A(w+i) + Cw$$

If $w = 0$; $(\alpha-1)i = iA$, then $A = \alpha-1$

If $w = -i$; $-i(\beta-2) + i(\alpha-1) = -iC$, then $C = \beta - \alpha - 1$

Thus

$$\int \frac{2iw - i\beta w - 1 + \alpha}{-iw(w+i)} dw \\ = (\alpha-1) \ln(w) + (\beta-\alpha-1) \ln(w+i) + D$$

Therefore

$$Y = Dw^{\alpha-1} (w+i)^{\beta-\alpha-1}.$$

By taking inverse Fourier transform to the previous equation, we obtain.

$$y = \frac{1}{2\pi} \int_{-\infty}^{\infty} w^{\alpha-1} (w+i)^{\beta-\alpha-1} e^{iwx} dw$$

Case 1: If $\alpha-1 = 0, \beta-\alpha-1 = 0$ that means $\alpha = 1, \beta = 2$. Then

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{iwx} dw = \delta(x)$$

Now we will show the correctness of the resulting solution.

$$xy'' + (2-x)y' - y \\ = x\delta''(x) + (2-x)\delta'(x) - \delta(x) \\ \text{by the property } x\delta^{(n)}(x) = -n\delta^{(n-1)}(x), \text{ we get} \\ = -2\delta'(x) + 2\delta'(x) + \delta(x) - \delta(x) \\ = 0.$$

Case 2: If μ, λ are positive integers such that $\alpha-1 = -\mu$ and $\beta-\alpha-1 = -\lambda$, then

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w^{-\mu} (w+i)^{-\lambda} e^{iwx} dw$$

If the value of μ here is greater than 1, this integral does not exist in the classical sense. The integral is divergent. For $\mu > 1$, the inverse Fourier transform can be calculated in a distributional sense.

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w^{-\mu} (w+i)^{-\lambda} e^{iwx} dw$$

If $x < 0$, then

$$y(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iwx}}{w^{\mu} (w+i)^{\lambda}} dw \\ = i(Res(f, 0_{\mu}) - Res(f, -i_{\lambda})) \\ = i \left(\frac{1}{(\mu-1)!} \right) \lim_{w \rightarrow 0} \frac{d^{\mu-1}}{dw^{\mu-1}} \left(\frac{e^{iwx}}{(w+i)^{\lambda}} \right)$$

$$-i \left(\frac{1}{(\lambda-1)!} \right) \lim_{w \rightarrow -i} \frac{d^{\lambda-1}}{dw^{\lambda-1}} \left(\frac{e^{iwx}}{w^\mu} \right)$$

Therefore

$$\begin{aligned} y(x) &= \left(\frac{i}{(\mu-1)!} \right) \sum_{k=0}^{\mu-1} x^k \binom{\mu-1}{k} (-1)^{\mu-1-k} \\ &\quad \times (i)^{-\lambda-\mu+2k+1} \lambda(\lambda+1) \cdots (\lambda+\mu-k-2) + \\ &\quad \times \left(\frac{i}{(\lambda-1)!} \right) \sum_{k=0}^{\lambda-1} x^k e^x \binom{\lambda-1}{k} (-1)^{-\mu} \\ &\quad \times (i)^{-\lambda-\mu+2k+1} \mu(\mu+1)(\mu+2) \cdots (\lambda+\mu-k-2). \end{aligned}$$

If $x > 0$, then

$$\begin{aligned} y(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iwx}}{w^\mu (w+i)^\lambda} dw = i (Res(f, 0_\mu)) \\ &= i \left[\left(\frac{1}{(\mu-1)!} \right) \lim_{w \rightarrow 0} \frac{d^{\mu-1}}{dw^{\mu-1}} \left(\frac{e^{iwx}}{(w+i)^\lambda} \right) \right] \end{aligned}$$

Therefore,

$$\begin{aligned} y(x) &= \left(\frac{i}{(\mu-1)!} \right) \sum_{k=0}^{\mu-1} x^k \binom{\mu-1}{k} (-1)^{\mu-1-k} \\ &\quad \times (i)^{-\lambda-\mu+2k+1} \lambda(\lambda+1) \cdots (\lambda+\mu-k-2) \end{aligned}$$

If we take $\mu=3$, $\lambda=2$, then

$$y(x) = i \left[3 + 2x + \frac{x^2}{2} + 3e^x - xe^x \right].$$

Case 3: If P is a positive integer such that $\alpha-1 > 0$ and $\beta-\alpha-1 = -P$, then

$$\begin{aligned} y(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} w^{\alpha-1} (w+i)^{-P} e^{iwx} dw \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{w^{\alpha-1} e^{iwx}}{(w+i)^P} dw \\ &= \frac{-H(-x)}{2\pi} (2\pi i) \operatorname{Res} \left(\frac{w^{\alpha-1} e^{iwx}}{(w+i)^P}, -i \right) \\ &= \frac{-iH(-x)}{(P-1)!} \lim_{w \rightarrow -i} \frac{d^{P-1}}{dw^{P-1}} \left(w^{\alpha-1} e^{iwx} \right) \\ &= \frac{-iH(-x)}{(P-1)!} \lim_{w \rightarrow -i} \sum_{k=0}^{P-1} \binom{P-1}{k} \left(\frac{d^k (e^{iwx})}{dw^k} \right) \\ &\quad \times \left(\frac{d^{P-1-k} (w^{\alpha-1})}{dw^{P-1-k}} \right) \\ &= \frac{-iH(-x)}{(P-1)!} \sum_{k=0}^{P-1} \binom{P-1}{k} (ix)^k (-i)^{\alpha-P+k} e^x \\ &\quad \times (\alpha-1)(\alpha-2) \cdots (\alpha-P+k+1). \end{aligned}$$

This result holds provided that

$$\alpha-1 \geq \alpha-P+k+1.$$

Now, we will show the correctness of the resulting solution. If $x < 0$ then

$$\begin{aligned} y &= \left(\frac{i}{(P-1)!} \right) \sum_{k=0}^{P-1} \binom{P-1}{k} (ix)^k (-i)^{\alpha-P+k} e^x (\alpha-1)(\alpha-2) \\ &\quad \cdots (\alpha-P+k+1) \end{aligned}$$

$$\begin{aligned} y &= \left(\frac{ie^x}{(\alpha-\beta)!} \right) [(-i)^{\beta-1} (\alpha-1)(\alpha-2) \cdots \beta \\ &\quad + (-i)^\beta ix(\alpha-\beta)(\alpha-1)(\alpha-2) \cdots (\beta+1) \\ &\quad + \cdots + (ix)^{\alpha-\beta} (-i)^{\alpha-1}] \end{aligned}$$

$$\begin{aligned} y' &= \left(\frac{ie^x}{(\alpha-\beta)!} \right) [(-i)^{\beta-1} (\alpha-1)(\alpha-2) \cdots \beta \\ &\quad + (-i)^\beta ix(\alpha-\beta)(\alpha-1)(\alpha-2) \cdots (\beta+1) \\ &\quad + \cdots + (ix)^{\alpha-\beta} (-i)^{\alpha-1}] + \left(\frac{ie^x}{(\alpha-\beta)!} \right) \\ &\quad [(-i)^\beta i + \cdots + (\alpha-\beta)(ix)^{\alpha-\beta-1} i (-i)^{\alpha-1}] \end{aligned}$$

$$\begin{aligned}
y'' = & \left(\frac{ie^x}{(\alpha - \beta)!} \right) [(-i)^{\beta-1}(\alpha - 1)(\alpha - 2) \cdots \beta \\
& + (-i)^{\beta}ix(\alpha - \beta)(\alpha - 1)(\alpha - 2) \cdots (\beta + 1) \\
& + \cdots + (ix)^{\alpha-\beta}(-i)^{\alpha-1}] + \left(\frac{2ie^x}{(\alpha - \beta)!} \right) \\
& [(-i)^{\beta}i(\alpha - \beta)(\alpha - 1)(\alpha - 2) \cdots (\beta + 1) \\
& + \cdots + (\alpha - \beta)(ix)^{\alpha-\beta-1}i(-i)^{\alpha-1}] \\
& + \left(\frac{ie^x}{(\alpha - \beta)!} \right) [(-i)^{\beta+1}2i(i) \frac{(\alpha - \beta)(\alpha - \beta - 1)}{2} \\
& (\alpha - 1)(\alpha - 2) \cdots (\beta + 2) + \\
& \cdots + (\alpha - \beta)(\alpha - \beta - 1)(ix)^{\alpha-\beta-2}i^2(-i)^{\alpha-1}]
\end{aligned}$$

By substituting the functions y , y' and y'' into Equation (2), we get

$$\begin{aligned}
& x(-i)^{\beta-1} \frac{(\alpha - 1)!}{(\alpha - \beta)!} + (-i)^{\beta}ix^2(\alpha - \beta) \frac{(\alpha - 1)!}{\beta!} + \cdots \\
& + x(ix)^{\alpha-\beta}(-i)^{\alpha-1} + 2x(-i)^{\beta}i(\alpha - \beta) \frac{(\alpha - 1)!}{\beta!} + \cdots + \\
& (\alpha - \beta)2x(ix)^{\alpha-\beta-1}i(-i)^{\alpha-1} + x(-i)^{\beta+1}2i(i) \\
& \frac{(\alpha - \beta)(\alpha - \beta - 1)}{2} \frac{(\alpha - 1)!}{(\beta + 1)!} \\
& + \cdots + (\alpha - \beta)(\alpha - \beta - 1)x(ix)^{\alpha-\beta-2}i^2(-i)^{\alpha-1} + \beta(-i)^{\beta-1} \\
& \frac{(\alpha - 1)!}{(\beta - 1)!} + \beta(-i)^{\beta}ix(\alpha - \beta) \frac{(\alpha - 1)!}{(\beta)!} + \cdots \\
& + \beta(ix)^{\alpha-\beta}(-i)^{\alpha-1} + \beta(-i)^{\beta}i(\alpha - \beta) \frac{(\alpha - 1)!}{\beta!} \\
& + \cdots + \beta(\alpha - \beta)(ix)^{\alpha-\beta-1}i(-i)^{\alpha-1} - x(-i)^{\beta-1} \\
& \frac{(\alpha - 1)!}{(\beta - 1)!} - x(-i)^{\beta}ix(\alpha - \beta) \frac{(\alpha - 1)!}{(\beta)!} \\
& - \cdots - x(ix)^{\alpha-\beta}(-i)^{\alpha-1} - x(-i)^{\beta}i(\alpha - \beta) \\
& \frac{(\alpha - 1)!}{\beta!} + \cdots - x(\alpha - \beta)(ix)^{\alpha-\beta-1} \\
& i(-i)^{\alpha-1} - \alpha(-i)^{\beta-1} \frac{(\alpha - 1)!}{(\beta - 1)!} - \alpha(-i)^{\beta}ix(\alpha - \beta) \\
& \frac{(\alpha - 1)!}{\beta!} - \cdots - \alpha(ix)^{\alpha-\beta}(-i)^{\alpha-1}
\end{aligned}$$

$$\begin{aligned}
& = \beta(-i)^{\beta-1} \frac{(\alpha - 1)!}{(\beta - 1)!} + \beta(-i)^{\beta}ix(\alpha - \beta) \\
& \frac{(\alpha - 1)!}{(\beta)!} - \alpha(-i)^{\beta-1} \frac{(\alpha - 1)!}{(\beta - 1)!} + x[(-i)^{\beta-1} \\
& \frac{(\alpha - 1)!}{(\alpha - \beta)!} + 2(-i)^{\beta}i(\alpha - \beta) \\
& \frac{(\alpha - 1)!}{\beta!} + (-i)^{\beta+1}2i(i) \\
& \frac{(\alpha - \beta)(\alpha - \beta - 1)}{2} \\
& \frac{(\alpha - 1)!}{(\beta + 1)!} + \beta(-i)^{\beta}i(\alpha - \beta) \frac{(\alpha - 1)!}{(\beta)!} + \\
& (-i)^{\beta+1}2i(i) \frac{(\alpha - \beta)(\alpha - \beta - 1)}{2} \\
& \frac{(\alpha - 1)!}{(\beta + 1)!} - (-i)^{\beta-1} \\
& \frac{(\alpha - 1)!}{(\beta - 1)!} - \alpha(-i)^{\beta}i(\alpha - \beta) \frac{(\alpha - 1)!}{\beta!} + \cdots
\end{aligned}$$

$$\begin{aligned}
& = (-i)^{\beta-1}(\alpha - 1)! \left[\frac{\beta}{(\beta - 1)!} + \beta(-i) \right. \\
& \left. \frac{i(\alpha - \beta)}{\beta!} - \frac{\alpha}{(\beta - 1)!} \right] + x(-i)^{\beta-1} \\
& \frac{(\alpha - 1)!}{(\beta - 1)!} \left[1 - 2i^2 \frac{(\alpha - \beta)}{\beta} - 2i^2 \right. \\
& \left. \frac{(\alpha - \beta)(\alpha - \beta - 1)}{2} \frac{1}{\beta(\beta + 1)} + \alpha - \beta + 2 \right. \\
& \left. \beta \frac{(\alpha - \beta)(\alpha - \beta - 1)}{2} \frac{1}{\beta(\beta + 1)} - \right. \\
& \left. 1 - \frac{(\alpha - \beta)}{\beta} - \alpha(-i) \frac{i(\alpha - \beta)}{\beta} \right] \\
& = (-i)^{\beta-1} \frac{(\alpha - 1)!}{(\beta - 1)!} \left(\beta + \frac{\beta(\alpha - \beta)}{\beta} \right. \\
& \left. - \alpha \right) + x(-i)^{\beta-1} \frac{(\alpha - 1)!}{(\beta - 1)!} \frac{\beta}{\beta} \\
& (1 + 2\alpha - 2\beta + \frac{(\alpha - \beta)(\alpha - \beta - 1)}{\beta + 1} \\
& + \beta(\alpha - \beta) + \frac{\beta(\alpha - \beta)(\alpha - \beta - 1)}{\beta + 1} \\
& - 1 - \alpha + \beta)
\end{aligned}$$

$$= 0$$

If we take $\alpha = 3, \beta = 3, P = 1$ then

$$y(x) = ie^x.$$

If we take $\alpha = 4, \beta = 2, P = 1$ then

$$y(x) = (x^2 + 6x + 6)e^x.$$

Case 4: If q is a positive integer, such that $\alpha - 1 = -q$ and $\beta - \alpha - 1 > 0$, then

$$\begin{aligned} y(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} w^{-q} (w+i)^{\beta-\alpha-1} e^{iwx} dw \\ y(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(w+i)^{\beta-\alpha-1} e^{iwx}}{w^q} dw \\ &= \frac{1}{2\pi} (2\pi) i \operatorname{Res}(f, 0_q) \\ &= \left(\frac{i}{(q-1)!} \right) \lim_{w \rightarrow 0} \frac{d^{q-1}}{dw^{q-1}} \left((w+i)^{\beta-\alpha-1} e^{iwx} \right) \\ y(x) &= \left(\frac{i}{(q-1)!} \right) \sum_{k=0}^{q-1} \binom{q-1}{k} (ix)^k (i)^{\beta-\alpha-q+k} \\ &\quad \times (\beta-\alpha-1)(\beta-\alpha-2) \cdots (\beta-\alpha-q+k+1). \end{aligned}$$

If we take $\alpha = -1, \beta = 1, q = 2$, then $y(x) = i(1-x)$

Case 5: for $\alpha - 1 > 0, \beta - \alpha - 1 > 0$

$$\begin{aligned} (-iw^2 + w)Y' + (-2iw + i\beta w + 1 - \alpha)Y &= 0 \\ -iw^2Y' + wY' + (\beta - 2)iwY + (1 - \alpha)Y &= 0 \end{aligned} \quad (3)$$

Let's solve this resulting differential equation using Frobenius method Let $Y = \sum_{k=0}^{\infty} a_k w^{k+r}, Y' = \sum_{k=0}^{\infty} (k+r)a_k w^{k+r-1}$. By substituting the functions Y and Y' into Equation (3), we get

$$\begin{aligned} -iw^2 \sum_{k=0}^{\infty} (k+r)a_k w^{k+r-1} + w \sum_{k=0}^{\infty} (k+r)a_k w^{k+r-1} + \\ (\beta - 2)iw \sum_{k=0}^{\infty} a_k w^{k+r} + (1 - \alpha) \sum_{k=0}^{\infty} a_k w^{k+r} &= 0 \\ \sum_{k=0}^{\infty} -i(k+r)a_k w^{k+r+1} + \sum_{k=0}^{\infty} (k+r)a_k w^{k+r} \\ + (\beta - 2)i \sum_{k=0}^{\infty} a_k w^{k+r+1} + (1 - \alpha) \sum_{k=0}^{\infty} a_k w^{k+r} &= 0 \end{aligned} \quad (4)$$

Equation (4) can be written as

$$\begin{aligned} [r + (1 - \alpha)]a_0 w^r + \sum_{k=0}^{\infty} [(-i(k+r) + (\beta - 2)i)a_k \\ + ((k+r+1) + (1 - \alpha))a_{k+1}] w^{k+r+1} = 0 \end{aligned} \quad (5)$$

This leads to $[r + (1 - \alpha)]a_0 = 0$

If $a_0 \neq 0$, then the indicial root is $r = \alpha - 1$.

Equating the coefficient of w^{k+r+1} Equal to zero, we get

$$(-i(k+r) + (\beta - 2)i)a_k + ((k+r+1) + (1 - \alpha))a_{k+1}$$

Or

$$a_{k+1} = \frac{i(k+r-\beta+2)}{(k+r-\alpha+2)} a_k, \quad \forall k \geq 0 \quad (6)$$

for $r = \alpha - 1$ in the formula (6), we obtain

$$\begin{aligned} a_{k+1} &= \frac{i(k+\alpha-\beta+1)}{(k+1)} a_k, \quad \forall k \geq 0 \\ k=0: \quad a_1 &= i(\alpha-\beta+1)a_0 \\ k=1: \quad a_2 &= a_1 \frac{i(\alpha-\beta+2)}{2} = \frac{i(\alpha-\beta+1)i(\alpha-\beta+2)}{2} a_0 \\ k=2: \quad a_3 &= a_2 \frac{i(\alpha-\beta+3)}{3} = \frac{i(\alpha-\beta+1)i(\alpha-\beta+2)i(\alpha-\beta+3)}{2 \cdot 3} a_0 \\ k=3: \quad a_4 &= a_3 \frac{i(\alpha-\beta+4)}{4} = \frac{(i)^4(\alpha-\beta+1)(\alpha-\beta+2)(\alpha-\beta+3)(\alpha-\beta+4)}{4!} a_0 \\ &\vdots \\ a_k &= \frac{(i)^k(\alpha-\beta+1)(\alpha-\beta+2)(\alpha-\beta+3) \cdots (\alpha-\beta+k)}{k!} a_0 \end{aligned}$$

Therefore

$$\begin{aligned} Y &= a_0 \sum_{k=0}^{\beta-\alpha-1} \frac{(-i)^k (\beta-\alpha-1)!}{(\beta-\alpha-k-1)!k!} w^{k+\alpha-1} \\ &= w^{\alpha-1} + w^{\alpha}(\alpha-\beta+1)i + i^2 \frac{(\alpha-\beta+1)(\alpha-\beta+2)}{2!} w^{\alpha+1} + \dots \end{aligned}$$

By taking Inverse Fourier transform to the previous equation, we have the solution $y(x)$ corresponding to $r = \alpha - 1$.

$$y(x) = \frac{\delta^{(\alpha-1)}}{i^{\alpha-1}} + (\alpha - \beta + 1)i \frac{\delta^{(\alpha)}}{i^\alpha} + \frac{i^2(\alpha - \beta + 1)(\alpha - \beta + 2)\delta^{(\alpha+1)}}{i^{\alpha+1}2!} + \dots$$

Therefore

$$y(x) = \frac{1}{i^{\alpha-1}} \left[\delta^{(\alpha-1)} + (\alpha - \beta + 1)\delta^{(\alpha)} + \frac{(\alpha - \beta + 1)(\alpha - \beta + 2)}{2}\delta^{(\alpha+1)} + \dots \right]$$

$$y(x) = a_0 \sum_{k=0}^{\beta-\alpha-1} \frac{(\alpha - \beta + 1)(\alpha - \beta + 2)(\alpha - \beta + 3) \cdots (\alpha - \beta + k)\delta^{(k+\alpha-1)}}{i^{\alpha-1}k!}.$$

If $\alpha = \lambda + 1$ and $\beta \notin \mathbb{Z}$, then

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(\alpha - \beta + 1)(\alpha - \beta + 2)(\alpha - \beta + 3) \cdots (\alpha - \beta + k)\delta^{(k+\alpha-1)}}{i^{\alpha-1}k!}.$$

Now, we will show the correctness of the resulting solution.

$$\begin{aligned} & xy'' + \beta y' - xy' - \alpha y \\ &= x\delta^{(\alpha+1)} + (\alpha - \beta + 1)x\delta^{(\alpha+2)} + \frac{(\alpha - \beta + 1)(\alpha - \beta + 2)}{2}x\delta^{(\alpha+3)} + \dots \\ &+ \beta\delta^{(\alpha)} + \beta(\alpha - \beta + 1)\delta^{(\alpha+1)} + \frac{\beta(\alpha - \beta + 1)(\alpha - \beta + 2)}{2}\delta^{(\alpha+2)} + \dots \\ &- x\delta^{(\alpha)} - (\alpha - \beta + 1)x\delta^{(\alpha+1)} - \frac{(\alpha - \beta + 1)(\alpha - \beta + 2)}{2}x\delta^{(\alpha+2)} - \dots \\ &- \alpha\delta^{(\alpha-1)} - \alpha(\alpha - \beta + 1)\delta^{(\alpha)} - \alpha\frac{(\alpha - \beta + 1)(\alpha - \beta + 2)}{2}\delta^{(\alpha+1)} - \dots \\ &= -x\delta^{(\alpha)} - \alpha\delta^{(\alpha-1)} + x\delta^{(\alpha+1)} + \beta\delta^{(\alpha)} - (\alpha - \beta + 1)x\delta^{(\alpha+1)} - \alpha(\alpha - \beta + 1)\delta^{(\alpha)} + \dots \\ &= -\frac{x(-1)^\alpha \alpha! \delta(x)}{x^\alpha} - \frac{\alpha(-1)^{\alpha-1}(\alpha-1)! \delta(x)}{x^{\alpha-1}} + \frac{x(-1)^{\alpha+1}(\alpha+1)! \delta(x)}{x^{\alpha+1}} + \frac{\beta(-1)^\alpha \alpha! \delta(x)}{x^\alpha} \\ &- \frac{(\alpha - \beta + 1)x(-1)^{\alpha+1}(\alpha+1)! \delta(x)}{x^{\alpha+1}} - \frac{\alpha(\alpha - \beta + 1)(-1)^\alpha \alpha! \delta(x)}{x^\alpha} + \dots \\ &= \frac{\alpha! \delta(x)}{x^{\alpha-1}} (-(-1)^\alpha + (-1)^\alpha)_{\sim \mathcal{D}} \\ &+ \frac{\delta(x)(-1)^\alpha}{x^\alpha} (-(\alpha+1)! + \beta\alpha! + (\alpha+1)!(\alpha - \beta + 1) - \alpha(\alpha - \beta + 1)\alpha!)_{\omega 0} \\ &: \\ &= 0. \end{aligned}$$

4 CONCLUSION

In this paper, an alternative solution to Kummer's differential equation was introduced. Kummer's equation is important in mathematics because of its applications in the physical sciences. The proposed solution is based on the Dirac delta distribution and does not depend on Kummer's confluent hypergeometric function or Gordon's function. In addition, several properties and theorems related to the proposed method were presented. The correctness of the solution was also verified.

Author contributions

Authors equally contributed to this work.

Funding source

No funds received.

Data availability

N/A

DECLARATIONS

Conflict of interest

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Consent to publish

N/A

Ethical approval

There are no ethical issues with the publication of this manuscript.

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How to cite this article

Issa A, Duz M. An alternative approach for solving Kummer's differential equations. *Journal of University of Anbar for Pure Science*. 2026; 20(2):395-403. doi:[10.37652/juaps.2025.165191.1692](https://doi.org/10.37652/juaps.2025.165191.1692)