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The mathematical applications of the constant shrinkage function of the state of a single observation of the normal distribution

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ABSTRACT

This study examines the mathematical applications of the constant shrinkage function of the state of a single observation for the normal distribution, and its important problems in the estimation of the parameter mean of the normal distribution if we have a single observation. One of the suggested methods to solve this problem is the shrinking method. This method depends on the shrinking function, we will also study in this research the constant shrinking function, which gives the shrinkage estimator, and we will study the Bayes estimator of a normal distribution and prove the existence of a relationship between the two methods and prove through the dissemination of this result the existence of a relationship between the Bayes estimator and any shrinking function.

Keywords: *Bayes estimator, Bayesian risk, Shrinkage function, Single observation, Loss function , Normal distribution.*

1 INTRODUCTION

Whenever the random sample is not sufficient to estimate any parameter, our need will increase to take measures to avoid this issue. If the random sample contains only one observation [1] the parameter estimation will become very difficult for such a sample [2, 3] The shrinking estimators [4, 5] try to find solutions for this problem in order to avoid a shortage in the sample [6] There are many shrinking functions in this regard [7, 8] which are divided in terms of dependence on the parameter to be estimated into two types [4, 9].

(a) Weighted fixed shrinking functions (not depend on the parameter to be estimated) [10] (b) Weighted non-fixed shrinking functions (depend on the parameter to be estimated) that is $\omega = \omega(\hat{\eta})$ [11]. The choice of any kind does not specify by terms, but it is the desire of the researcher. The first type was chosen by a number of researchers, the estimation of mean in the case of the problem in the case of $N(\eta, 1)$ for one observation [12] we note that there is a new result by searching on this subject when using the functions of the first kind. we will try to find a relationship between a shrinking estimator which takes the symbol $\varepsilon^{(1)}(x, \kappa)$ and Bayes estimator [13] which takes the symbol $\varepsilon^{(2)}(x, \kappa)$ and then generalize this relationship to get to the weakness of the fixed shrinking function and prove that it equals the Bayes

2 SHRINKING ESTIMATOR $\varepsilon^{(1)}(x, \kappa)$

Let X the random variable has a normal distribution with mean η and fixed variance one (that is $x \sim N(\eta, 1)$) and let

$$\varepsilon(x, \omega) = \omega(x)x$$

Estimator for the parameter η. We define (Shrinkage

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Function) as

$$\omega_\kappa^{(1)}(x) = \frac{1}{1+\kappa}$$

Where κ is a constant and $\kappa \neq -1$ as a shrinking function so in particular,

$$\omega_0^{(1)}(x) = {}_1, \omega_\infty^{(1)}(x) == {}_0$$

Let's take now the following estimator

$$\varepsilon^{(1)}(x,\kappa) = \varepsilon\left(x, \omega_\kappa^{(1)}\right) = \omega_\kappa^{(1)}(x)x = \frac{x}{1+\kappa}$$

as a shrinking estimator for the parameter η

3 THE RELATIONSHIP BETWEEN SHRINKING ESTIMATOR AND BAYES ESTIMATOR

Bayesian procedure in estimation is based on the idea that this method is that the parameter θ (a parameter to be estimated) random variable (not constant) has a prior distribution, and has the symbol $g(\theta)$ [14] There are two ways to estimate the Bayesian estimator [15, 16].

3.1 Posterior distribution method

this method relies on finding a conditional mean parameter when the random variable is given and this is done by finding a conditional function, first through the following relationship [17]

$$f(\theta \mid x) = \frac{f(\theta, x)}{f_1(x)} = \frac{f(x \mid \theta)g(\theta)}{\int_\theta f(x \mid \theta)g(\theta)}$$

And then we find Bayes estimator, which has the symbol $E(\theta \mid x)$ through the following relationship [18]

$$E(\theta \mid x) = \int_\theta \theta f(\theta \mid x)d(\theta) = \frac{\int_\theta \theta \cdot f(\theta \mid x)g(\theta)d\theta}{\int_\theta f(\theta \mid x)g(\theta)d\theta}$$

3.2 Loss function method

this method relies on the existence of the loss of function as defined by the following relationship [19]

$$l[\varepsilon(x), \eta] = [\varepsilon(x) - \eta]^2$$

The risk function of expectation function loss is [20]

$$R\left(\hat{\theta}_B \mid \theta\right) = E\left[l\left(\hat{\theta}_B \mid \theta\right)\right] = \int_\theta \left(\hat{\theta}_B - \theta\right)^2 g(\theta)d(\theta)$$

Bayesian Estimation, which is based on the loss function and prior distribution, is the Estimator which has Bayesian minimum risk [21]

$$R\left(\hat{\theta}_B \mid \theta\right) \leq R(\hat{\theta}) \quad \forall\theta$$

This relationship represents the amount of risk associated with the estimator $\hat{\theta}$. We can define the Bayesian risk for this estimator as [22]

$$R\left(\hat{\theta}_B \mid \theta\right) = \int_\theta R(\hat{\theta})g(\theta)d(\theta)$$

For the purpose of finding the estimator Bayes we're looking for, we generally take a random sample $\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_\mathrm{n}$ follows a normal distribution

$$f(x \mid \eta) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-1}{2}\left(\frac{x-\eta_2}{\sigma}\right)^2\right],$$
$$-\infty < \mathrm{x} < \infty, -\infty < \eta < \infty, \sigma^2 > 0$$

We know that $\bar{x}$ have the Maximum Likelihood Estimation (MLE) for parameter η and

$$\bar{x} \sim N\left(\eta, \frac{\sigma^2}{n}\right)$$
$$f(\bar{x} \mid \eta) = \frac{\sqrt{n}}{\sqrt{2\pi}\sigma} \exp\left[\frac{-\mathrm{n}}{2}\left(\frac{\bar{x}-\eta}{\sigma}\right)^2\right]$$

If the prior distribution of parameter η is normal, that is $\eta \sim N\left(\eta_0, \sigma_0^2\right)$ then

$$g(\eta) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left[\frac{-1}{2}\left(\frac{\eta-\eta_0}{\sigma_0}\right)^2\right]$$

We can prove after several mathematical operations : Using the likelihood $X \mid \eta \sim N\left(\eta, \sigma^2/n\right)$ and the prior $\eta \sim N\left(\eta_0, {\sigma_0}^2\right)$ Bayes' theorem yields the posterior density $\pi(\eta \mid \bar{x}) \propto \exp\left(\frac{-n(\bar{x}-\eta)^2}{(2\sigma^2)} - \frac{(\eta-\eta_0)^2}{(2{\sigma_0}^2)}\right)$ Completing the square in η shows that the posterior is normal with mean. Then Bayesian estimator takes the formula

$$E(\eta \mid \bar{\mathrm{x}}) = \frac{n\bar{\mathrm{x}}\sigma_0^2 + \eta_0\sigma^2}{n\sigma_0^2 + \sigma^2}$$

In the case of a single observation when $n = 1$, then $\bar{x} = x$ and if the parameter η has a prior distribution $N\left(0, \frac{1}{\kappa}\right)$ sense $\eta_0 = 0, \sigma_0^2 = \frac{1}{\kappa}$ then (11) takes the formula

$$E(\eta \mid \overline{\mathrm{x}}) = \frac{1(x)\frac{1}{\kappa} + 0(1)}{1\left(\frac{1}{\kappa}\right) + 1} = \frac{x\frac{1}{\kappa}}{\frac{1+\kappa}{\kappa}} = \frac{x}{1+\kappa}$$

This result represents the estimator $\varepsilon^{(1)}(x, \kappa)$ which is given in (4), since the estimator $\varepsilon^{(1)}(x, \kappa)$ is normal Bayesian estimator has prior distribution $N\left(0, \frac{1}{K}\right)$. This is proof of the weakness of shrinkage function $\omega_\kappa = \frac{1}{1+\kappa}$. This conclusion leads us to the possibility of generalization for any constant shrinkage function.

4 CONCLUSION

The results of the problem of $N(\eta, 1)$ show that the constant ω (for example $\omega = \frac{1}{2}$) is undesirable because it is symmetrical with a normal Bayesian estimator $\varepsilon^{(2)}(x, \kappa)$, and this symmetry was his proof in the section (3), where we prove that the estimator output from the shrinkage function $\omega_\kappa^{(1)}(x) = \frac{1}{1+\kappa}$ and normal Bayesian estimator, which has prior distribution $N\left(0, \frac{1}{K}\right)$, they have the same formula

$$\varepsilon^{(1)}(x, \kappa) = \varepsilon^{(2)}(x, \kappa) = \frac{x}{1+\kappa}$$

We can generalize the conclusion because the symmetry between the estimator $\varepsilon^{(2)}(x, k)$ and shrinkage functions is not only limited to the shrinkage function $\omega_\kappa^{(1)}(x) = \frac{1}{1+\kappa}$ but includes all constant functions, since of any constant function whatever value can be written as

$$\omega = \frac{a}{b}, \quad b > 0 \quad , 0 \leq a \leq b$$

This function satisfies the following conditions
- If $a = 0$ then $\omega = 0$
- If $a = b$ then $\omega = 1$
- If $0 \leq a \leq b$ then $0 \leq \frac{a}{b} \leq 1 \quad (0 \leq \omega \leq 1)$.

The shrinkage estimator resulting from the shrinkage function will be

$$\varepsilon(x, \omega) = \omega(x)x = \frac{ax}{b}$$

The estimator $\varepsilon^{(2)}(x, \kappa)$ corresponding to this case is the Bayesian estimator with prior distribution $N\left(0, \frac{1}{K}\right)$, in order to prove that

$$E(\eta \mid \overline{\mathrm{x}}) = \frac{n\bar{x}\sigma_0^2 + \eta_0\sigma^2}{n\sigma_0^2 + \sigma^2}$$

$$\bar{x} = x, n = 1, \eta_0 = 0, \sigma^2 = 1, \sigma_0^2 = \frac{a}{b-a}$$

$$E(\eta \mid \overline{\mathrm{x}}) = \frac{1(x)\frac{a}{b-a} + 0(1)}{(1)\frac{a}{b-a} + 1} = \frac{\frac{ax}{b-a}}{\frac{a+b-a}{b-a}} = \frac{ax}{b}$$

This proves that the constant shrinking function is generally undesirable and not only function $\omega_\kappa^{(1)}$ and the proof of the above means that the shrinking estimator resulting from any constant function is a Bayesian estimator with a specific prior distribution. Which is mean the weakness of the constant shrinkage function of the state of a single observation of the normal distribution. This highlights the inherent limitation of constant shrinkage functions: while they reduce variance when the parameter is close to the prior mean, they introduce bias when the parameter deviates, showing that the weakness applies to all constant shrinkage functions in the single-observation setting.

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REFERENCES

[1] Ungar SG, Pearlman JS, Mendenhall JA, Reuter D. Overview of the earth observing one (eo-1) mission. IEEE Transactions on Geoscience

and Remote Sensing. 2003;41(6):1149–1159. 10.1109/tgrs.2003.815999

[2] Reinhart DR, Townsend TG. Landfill Bioreactor Design and Operation. Routledge; 2018. 10.1201/9780203749555

[3] Eykhoff P. Process parameter and state estimation. Automatica. 1968;4(4):205–233. 10.1016/0005-1098(68)90015-0

[4] Casella G. Shrinkage Estimation. Wiley; 2005. 10.1002/0470011815.b2a15144

[5] Thompson JR. Some Shrinkage Techniques for Estimating the Mean. Journal of the American Statistical Association. 1968;63(321):113–122. 10.1080/01621459.1968.11009226

[6] Morris CN, Lysy M. Shrinkage Estimation in Multilevel Normal Models. Statistical Science. 2012;27(1). 10.1214/11-sts363

[7] Claessen K. Shrinking and showing functions: (functional pearl). In: Proceedings of the 2012 Haskell Symposium. ICFP'12. ACM; 2012. p. 73–80. 10.1145/2364506.2364516

[8] Bruce AG, Gao HY. WaveShrink: Shrinkage functions and thresholds. In: wavelet applications in signal and image processing III. vol. 2569. SPIE; 1995. p. 270-81

[9] Davì F, et al. Input-shrinking functions: theory and application. 2012

[10] Saewan S, Kumam P. The shrinking projection method for solving generalized equilibrium problems and common fixed points for asymptotically quasi-ϕ-nonexpansive mappings. Fixed Point Theory and Applications. 2011;2011(1). 10.1186/1687-1812-2011-9

[11] Shirafkan MT, Seidgar H, Javadian N. A new mathematical model for non-fixed destination multi-depot multiple travelling salesmen with time window problem. International Journal of Services and Operations Management. 2018;31(4):530. 10.1504/ijsom.2018.096167

[12] Baker DG. Effect of Observation Time on Mean Temperature Estimation. Journal of Applied Meteorology. 1975;14(4):471–476. 10.1175/1520-0450(1975)014<0471:eootom>2.0.co;2

[13] Stephens M, Phil D. Bayesian methods for mixtures of normal distributions. 1997

[14] Hassan AS, Abdelghaffar AM. Bayesian and E-Bayesian Estimation of Gompertz Distribution in Stress-Strength Reliability Model under Partially Accelerated Life Testing. Computational Journal of Mathematical and Statistical Sciences. 2025;4(1):348–378. 10.21608/cjmss.2025.353923.1107

[15] Asparouhov T, Muthén B. Bayesian estimation of single and multilevel models with latent variable interactions. Structural Equation Modeling: A Multidisciplinary Journal. 2020;28(2):314–328. 10.1080/10705511.2020.1761808

[16] Eleftheriadis P, Giazitzis S, Kowal J, Leva S, Ogliari E. Joint State of Charge and State of Health Estimation Using Bidirectional LSTM and Bayesian Hyperparameter Optimization. IEEE Access. 2024;12:80244–80254. 10.1109/access.2024.3410675

[17] Campbell H, Gustafson P. Bayes Factors and Posterior Estimation: Two Sides of the Very Same Coin. The American Statistician. 2022;77(3):248–258. 10.1080/00031305.2022.2139293

[18] Shi H, Drton M. On universal inference in Gaussian mixture models. arXiv; 2024. 10.48550/ARXIV.2407.19361

[19] Jewson J, Rossell D. General Bayesian Loss Function Selection and the use of Improper Models. Journal of the Royal Statistical Society Series B: Statistical Methodology. 2022;84(5):1640–1665. 10.1111/rssb.12553

[20] Steinwart I. How to Compare Different Loss Functions and Their Risks. Constructive Approximation. 2007;26(2):225–287. 10.1007/s00365-006-0662-3

[21] González-Rubio J, Juan A, Casacuberta F. Minimum bayes-risk system combination. In: Proceedings of the 49th annual meeting of the association for computational linguistics: human language technologies; 2011. p. 1268-77

[22] Czado C, Brechmann EC. In: Bayesian Risk Analysis. Springer International Publishing; 2014. p. 207–240. 10.1007/978-3-319-04486-6_8

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