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An Efficient Method for Solving Fractional Partial Differential Equations

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Abstract

This paper presents a new method for solving fractional partial differential equations (FPDE) which is called the polynomial approximation method based on the polynomial approximation $u_N(x,t)$ and on its general fractional derivative formula.By modifying the general fractional derivative formula of $u_N(x,t)$ and with the aid of the linear FPDE, another new formula can be found for the approximation $u_N(x,t)$. This is the basic idea of the proposed method. Furthermore, the mathematical proof of the convergence and stability of this method have been studied. Some numerical examples show that the proposed method exhibits a satisfactory results.

Introduction

Fractional ordinary differential equation (FODE) is an equation that contains fractional derivatives of an unknown function of a single variable, while fractional partial differential equation (FPDE) is an equation that contains fractional partial derivatives of an unknown function of several variables. Analytical solutions of FODEs and FPDEs are now an available in some special cases. But the solution to many FDEs (ordinary and partial) will have to relay on approximate and numerical methods, just like their integer-order counterparts. Fractional derivatives have been a round for centuries but recently they have found new applications in many fields of science and engineering. Applications of fractional ordinary derivatives in viscoelasticity may be found in (Diethelm, 1999). Also, some mechanical damping models have been presented in (Yuan & Agrawal,1998) as FODEs.Moreover, fractional ordinary time derivatives have been used in (Tseng, Pei & Hsia,2000) to compute the velocity and acceleration of some applications in signal processing, such as, radar and sonar applications.

Applications of FPDEs are found in physics (Shen & Liu, 2004), seismology (Hanyga, 2002), hydrology (Meerschaert,2005), and perhaps surprisingly, FPDEs have been linked with stable distributions, where a FPDE was introduced in (Lix,2003) whose solution gives nearly all the stable distributions.

This work is focused on solving the linear FPDEs with constant coefficients of the form:

$$D_t^{\alpha} u(x,t) + \beta \frac{\partial u(x,t)}{\partial x} = g(x,t) \qquad ,(x,t) \in D \qquad \ldots (1)$$

when the Riemann- Liouville integral operator is invertible, $\beta \in \Re$ and D $=\{(x,t): c \le x \le d,\ a \le t \le b\}$.

New Formulation of the Approximation $u_N(x,t)$

It is popular to use the approximation

$$u_N(x,t) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{ij}\, \phi_i(x) \phi_j(t) \qquad \ldots (2)$$

in the two-dimensional polynomial, orthogonal polynomial, and spline approximations. Many authors and researchers used the approximation (2), such as, Hopkins (Hopkins & Phillips, 1988) which used the above approximation in two-dimensional orthogonal polynomial, and spline approximations while Iglesias (Iglesias, 2001) used this approximation in two-dimensional spline approximation. It's popularity was due to it's simple shape, but this simple shape hide several disadvantages, such as, the difficulty in the matrix formulation of the used method (if it needed), and the number of additional terms in (2) that add worthless work. Here a new formulation for the approximation $u_N(x,t)$ will be derived. This formulation was constructed using some ideas given in (Davies, 1980) as it will be illustrated below. It was given in (Davies,1980) that each one of the approximations of the form

$$u(x,t) \approx a_0 + a_1 x + a_2 t = [1 \ \ x \ \ t]\{a_0 \ \ a_1 \ \ a_2\}$$

and

$$u(x,t) \approx a_0 + a_1 x + a_2 t + a_3 x^2 + a_4 xt + a_5 t^2 + a_6 x^2 t + a_7 xt^2$$
$$= [1 \ \ x \ \ t \ \ x^2 \ \ xt \ \ t^2 \ \ x^2 t \ \ xt^2]\{a_0 \ \ a_1, \ldots, a_7\}$$

has the form $\quad u(x,t) \approx p(x,t)\, \underline{a}$

where $p(x,t)$ is a row vector of linearly independent functions, and $\underline{a}$ is a column vector of constants. For example, if we want to approximate the unknown function $u(x,t)$ in the partial differential equation:

$$\frac{\partial u(x,t)}{\partial x} + 3u(x,t) = 2x + 3(t + x^2)$$

one may guess that the following approximation could be used

$$u(x,t) \cong [1 \ \ x \ \ t \ \ x^2]\{a_0 \ \ a_1 \ \ a_2 \ \ a_3\}$$
$$= a_0 + a_1 x + a_2 t + a_3 x^2$$

Particularly, it is not easy to give definite rules which are applicable in all cases. For this reason complete polynomials are often favorite. The necessary terms for all possible polynomials up to complete quintic are shown below (Davies, 1980):

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & x & & t & & & & \\ & & & x^2 & & xt & & t^2 & & & \\ & & x^3 & & x^2t & & xt^2 & & t^3 & & \\ & x^4 & & x^3t & & x^2t^2 & & xt^3 & & t^4 & \\ x^5 & & x^4t & & x^3t^2 & & x^2t^3 & & xt^4 & & t^5 \\ & & & & & \vdots & & & & & \end{array}$$

Thus a complete linear polynomial is of the form

$$a_0 + a_1 x + a_2 t$$

while a complete cubic polynomial is of the form

$$a_0 + a_1 x + a_2 t + a_3 x^2 + a_4 xt + a_5 t^2 + a_6 x^3 t + a_7 x^2 t + a_8 xt^2 + a_9 t^3$$

The above ideas are the outlines that we used to establish the following new formulation $u_N(x,t)$ of the function $u(x,t)$:

$$u_N(x,t) = \sum_{j=0}^{n_1} a_j x^j + \sum_{j=1}^{n_2} a_{n_1+j} t^j + \sum_{k=1}^{n_3} \sum_{j=1}^{m_k} a_{p_k+j} x^j t^k \qquad \ldots (3)$$

where $p_1=n_1 + n_2$, $p_{k+1}= p_k + m_k$;$k = 1, 2,\ldots,n_3$, such that n_1, n_2, n_3, and m_k, for $k =1, 2,\ldots,n_3$, are given nonnegative integers, and N is the number of terms in this approximation, i.e. N is the number of the unknowns coefficients a_j. Eq. (3) represents the general polynomial approximation that may be used to approximate $u(x,t)$. A special case is given when $n_1 = n_2 = n$, $n_3 = n - 1$, and $m_k = n - k$, as follows:

$$u_N(x,t) = \sum_{j=0}^{n} a_j x^j + \sum_{j=1}^{n} a_{n+j} t^j + \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} a_{p_k+j} x^j t^k \qquad \ldots (4)$$

where $p_1 = 2n$, $p_{k+1}= p_k + (n - k)$; $k = 1, 2,\ldots,n -1$, and n is a given nonnegative integer. It is obvious that eq.(4) represents the complete polynomial approximation for $u(x,t)$. To illustrate this let $n = 2$, then:

$$u_N(x,t) = \sum_{j=0}^{2} a_j x^j + \sum_{j=1}^{2} a_{j+2} t^j + \sum_{k=1}^{1} \sum_{j=1}^{2-k} a_{p_k+j} x^j t^k$$

Hence $\qquad u_N(x,t) = a_0 + a_1 x + a_2 x^2 + a_3 t + a_4 t^2 + a_5 xt$

which is the complete polynomial of order two.

New General Formula of the Fractional Derivative of $u_N(x,t)$

In this section a new general formula of the fractional derivative of the approximation $u_N(x,t)$ was established.

Proposition(1):

Let $\alpha \geq 0$ and $u_N(x,t)$ be the two dimensional polynomial approximation which was given in eq.(4). The fractional derivative of $u_N(x,t)$ is given by:

$$D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} \left[n!\, u_N(x,t) + f(x,t) \right]$$

where

$$f(x,t) = \sum_{j=0}^{n} \gamma(0,n)\, x^j\, a_j + \sum_{j=1}^{n-1} j!\gamma(j,n)\, t^j\, a_{n+j} + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} k!\gamma(k,n)\, x^j t^k\, a_{p_k+j} \qquad \ldots(5)$$

and

$$\gamma(j,n) = \begin{cases} \prod_{\ell=j+1}^{n} (\ell-\alpha) - \prod_{\ell=j+1}^{n} \ell & , j=0,1,\ldots,n-1 \\ 0 & , otherwise \end{cases} \qquad \ldots(6)$$

Proof:

Recall eq.(4):

$$u_N(x,t) = \sum_{j=0}^{n} a_j\, x^j + \sum_{j=1}^{n} a_{n+j}\, t^j + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} a_{p_k+j} x^j\, t^k$$

where $p_1 = 2n$, $p_{k+1} = p_k + (n-k)$; for $k = 1, 2,\ldots,n-1$.

Then the fractional derivative of $u_N(x,t)$ is given by

$$D_t^{\alpha} u_N(x,t) = \sum_{j=0}^{n} a_j \frac{x^j}{\Gamma(1-\alpha)}\, t^{-\alpha} + \sum_{j=1}^{n} a_{n+j} \frac{j!}{\Gamma(j-\alpha+1)}\, t^{j-\alpha} + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} a_{p_k+j} \frac{k!\, x^j}{\Gamma(k-\alpha+1)} t^{k-\alpha}$$

Rearrangement the above equation to get

$$D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} \left[\sum_{j=0}^{n} a_j \frac{\Gamma(n-\alpha+1)}{\Gamma(1-\alpha)}\, x^j + \sum_{j=1}^{n} a_{n+j} \frac{j!\,\Gamma(n-\alpha+1)}{\Gamma(j-\alpha+1)}\, t^j \right. \\ \left. + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} a_{p_k+j} \frac{k!\,\Gamma(n-\alpha+1)}{\Gamma(k-\alpha+1)} x^j\, t^k \right] \qquad \ldots(7)$$

Since

$\Gamma(n+1-\alpha) = (n-\alpha)\,\Gamma(n-\alpha) = (n-\alpha)\,(n-1-\alpha) \cdots (3-\alpha)\,(2-\alpha)\,(1-\alpha)\,\Gamma(1-\alpha)$

then for any integer i, $0 \leq i < n$ we have:

$$\Gamma(n+1-\alpha) = \prod_{\ell=i+1}^{n} (\ell-\alpha)\, \Gamma(i+1-\alpha) \qquad \ldots(8)$$

Put eq.(8) into eq.(7) , then using eq. (6) to get:

$$D_t^{\alpha} u_N(x,t) = \frac{t^{-\alpha}}{\Gamma(n-\alpha+1)} \left[\sum_{j=0}^{n} a_j \left(\prod_{\ell=1}^{n} \ell + \gamma(0,n) \right) x^j + \sum_{j=1}^{n-1} a_{n+j}\, j! \left(\prod_{\ell=j+1}^{n} \ell + \gamma(j,n) \right) t^j + a_n\, n! t^n \right. \\ \left. + \sum_{k=1}^{n-1}\sum_{j=1}^{n-k} a_{p_k+j}\, k! \left(\prod_{\ell=k+1}^{n} \ell + \gamma(k,n) \right) x^j\, t^k \right]$$

Since $j!\prod_{\ell=j+1}^{n}\ell = n!$,therefore;

$$D_t^{\alpha}u_N(x,t)=\frac{t^{-\alpha}}{\Gamma(n-\alpha+1)}\left[\sum_{j=0}^{n}a_j\,(n!+\gamma(0,n))\;x^j+\sum_{j=1}^{n-1}a_{n+j}\,(n!+j!\gamma(j,n))\;t^j+a_n\,n!t^n\right.$$
$$\left.+\sum_{k=1}^{n-1}\sum_{j=1}^{n-k}a_{p_k+j}(n!+k!\gamma(k,n))\,x^j\;t^k\right]$$

$$D_t^{\alpha}u_N(x,t)=\frac{t^{-\alpha}}{\Gamma(n-\alpha+1)}\left[n!\left\{\sum_{j=0}^{n}a_j\,x^j+\sum_{j=1}^{n}a_{n+j}\,t^j+\sum_{k=1}^{n-1}\sum_{j=1}^{n-k}a_{p_k+j}\,x^j\;t^k\right\}\right.$$

$$\Rightarrow \qquad +\left\{\sum_{j=0}^{n}\gamma(0,n)\,x^j a_j+\sum_{j=1}^{n-1}j!\;\gamma(j,n)\;t^j a_{n+j}\right.$$
$$\left.\left.+\sum_{k=1}^{n-1}\sum_{j=1}^{n-k}k!\;\gamma(k,n)\,x^j\;t^k a_{p_k+j}\right\}\right]$$

From equations (4) and (5) we conclude that:

$$D_t^{\alpha}u_N(x,t)=\frac{t^{-\alpha}}{\Gamma(n-\alpha+1)}\,[n!u_N(x,t)+f(x,t)]$$

Construction of the Polynomial Approximation Method

Our aim is to solve the linear FPDE with constant coefficients (1) when the R-L integral operator is invertible. Here, the approximated solution $u_N(x,t)$ will have the form

$$u_N(x,t)=\sum_{j=0}^{n-m}a_j\,t^{j+m}+\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}a_{p_k+j}x^j\,t^{k+m} \qquad \ldots(9)$$

Accordingly, the fractional derivative of $u_N(x,t)$ which have been given in proposition (1) will be:

$$D_t^{\alpha}u_N(x,t)=\frac{t^{-\alpha}}{\Gamma(n-\alpha+1)}\,[n!u_N(x,t)+f(x,t)] \qquad \ldots(10)$$

where

$$f(x,t)=\sum_{j=0}^{n-m-1}(j+m)!\gamma(j+m,n)t^{j+m}\,a_j+\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}(k+m)!\gamma(k+m,n)\;x^j\,t^{k+m}\,a_{p_k+j} \qquad \ldots(11)$$

and $u_N(x,t)$ is defined in eq.(9).

Now, recall eq. (1):

$$D_t^{\alpha}u(x,t)+\beta\,\frac{\partial u(x,t)}{\partial x}=g(x,t) \qquad ,\ (x,\,t)\in \mathrm{D}$$

where D = {(x,t): $0 \le x \le d$, $0 \le t \le b$}.

Differentiate eq.(9)with respect to x and put the result with eq.(10)into eq.(1) to get:

$$\frac{t^{-\alpha}}{\Gamma(n-\alpha+1)}[n!\,u_N(x,t)+f(x,t)]+\beta\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}a_{p_k+j}\;j\,x^{j-1}t^{k+m}=g(x,t) \qquad \ldots(12)$$

Simple arrangements in eq.(12) yield:

$$u_N(x,t)=G(x,t)-\frac{1}{n!}F(x,t) \qquad \ldots(13)$$

where

$$\left.\begin{aligned} G(x,t)&=\frac{\Gamma(n-\alpha+1)\,t^{\alpha}}{n\,!}\;g(x,t)\\ F(x,t)&=f(x,t)+\beta\,\Gamma(n-\alpha-1)\,t^{\alpha}\left[\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}a_{p_k+j}\;j\,x^{j-1}\,t^{k+m}\right]\end{aligned}\right\} \qquad \ldots(14)$$

and $f(x,t)$ is defined in eq.(11).

Now, equations (9) and (13) will be used to find the unknown coefficients a_j's. Let us first consider the unknowns a_{p_k+j}, for $k = 0,\ldots n-m-1$; $j=0,\ldots n-k-m$. Since when $n = m$ such terms do not exist in the approximated solution $u_N(x,t)$, we shall find equations for the unknowns a_{p_k+j} for all $n \geq m + 1$. It is clear that differentiating both sides of equations (9) and (13) with respect to t , r-times ,and with respect to x s-times, and equating them at a certain point in D will give the unknowns a_{p_k+j}. So, differentiate both sides of eq.(13) with respect to t, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}u_N(x,t)=\frac{\partial}{\partial t}G(x,t)-\frac{1}{n!}\Bigg\{&\sum_{j=0}^{n-m-1}(j+m)!\,\gamma(j+m,n)\,(j+m)\,t^{j+m-1}\,a_j\\ &+\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-m-1}\Big[(k+m)!\,\gamma(k+m,n)(k+m)\,t^{k+m-1}\,x^{j}\\ &+\beta\,\Gamma(n-\alpha-1)(\alpha+k+m)\,j\,t^{\alpha+k+m-1}x^{j-1}\Big]a_{p_k+j}\Bigg\}\end{aligned}$$

Repeat differentiation m-times to get:

$$\begin{aligned}\frac{\partial^m}{\partial t^m}u_N(x,t)=\frac{\partial^m}{\partial t^m}G(x,t)-\frac{1}{n!}\Bigg\{&\sum_{j=0}^{n-m-1}(j+m)!\,\gamma(j+m,n)\left(\prod_{\ell=j-m+1}^{j}(\ell+m)\right)t^{j}\,a_j\\ &+\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}(k+m)!\,\gamma(k+m,n)\left(\prod_{\ell=k-m+1}^{k}(\ell+m)\right)t^{k}\,x^{j}\,a_{p_k+j}\\ &+\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}\beta\,\Gamma(n-\alpha+1)\left(\prod_{\ell=0}^{m-1}(\alpha+k+m-\ell)\right)j\;t^{\alpha+k}x^{j-1}\,a_{p_k+j}\Bigg\}\end{aligned}$$

Hence, for $r \geq m$ we have:

$$\frac{\partial^r}{\partial t^r}u_N(x,t)=\frac{\partial^r}{\partial t^r}G(x,t)-\frac{1}{n!}\Bigg\{\sum_{j=r-m}^{n-m-1}(j+m)!\gamma(j+m,n)\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)t^{j+m-r}a_j \qquad \ldots(15)$$
$$+\sum_{k=r-m}^{n-m-1}\sum_{j=1}^{n-k-m}(k+m)!\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)t^{k+m-r}x^j a_{p_k+j}$$
$$+\beta\,\Gamma(n-\alpha+1)\sum_{k=0}^{n-m-1}\sum_{j=1}^{n-k-m}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)j\,t^{\alpha+k+m-r}x^{j-1}a_{p_k+j}\Bigg\}$$

Now, differentiate eq.(15) with respect to *x*, we get

$$\frac{\partial^{r+1}}{\partial x\partial t^r}u_N(x,t)=\frac{\partial^{r+1}}{\partial x\partial t^r}G(x,t)-\frac{1}{n!}\Bigg\{\sum_{k=r-m}^{n-m-1}s!(k+m)!\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)t^{k+m-r}a_{p_k+1}$$
$$+\sum_{k=r-m}^{n-m-2}\sum_{j=2}^{n-k-m}s!(k+m)!\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)j\,t^{k+m-r}x^{j-1}a_{p_k+j}$$
$$+\beta\,\Gamma(n-\alpha+1)\sum_{k=0}^{n-m-2}\sum_{j=2}^{n-k-m}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)j(j-1)\,t^{\alpha+k+m-r}x^{j-2}a_{p_k+j}\Bigg\}$$

Continue this process to get:

$$\frac{\partial^{r+s}}{\partial x^s\partial t^r}u_N(x,t)=\frac{\partial^{r+s}}{\partial x^s\partial t^r}G(x,t)-\frac{1}{n!}\Bigg\{\sum_{k=r-m}^{n-m-s}(k+m)!\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)t^{k+m-r}a_{p_k+s} \qquad \ldots(16)$$
$$+\sum_{k=r-m}^{n-m-s-1}\sum_{j=s+1}^{n-k-m}(k+m)!\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)\left(\prod_{\ell=j-s+1}^{j}\ell\right)t^{k+m-r}x^{j-s}a_{p_k+j}$$
$$+\beta\,\Gamma(n-\alpha+1)\sum_{k=0}^{n-m-s-1}\sum_{j=s+1}^{n-k-m}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)\left(\prod_{\ell=j-s}^{j}\ell\right)t^{\alpha+k+m-r}x^{j-s-1}a_{p_k+j}\Bigg\}$$

In the same manner we can differentiate eq.(9) with respect to t , r-times, $r \geq m$, to get:

$$\frac{\partial^r}{\partial t^r}u_N(x,t)=\sum_{j=r-m}^{n-m}\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)t^{j+m-r}a_j+\sum_{k=r-m}^{n-m-1}\sum_{j=1}^{n-k-m}\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)x^j t^{k+m-r}a_{p_k+j} \qquad \ldots(17)$$

Then differentiate eq.(17) with respect to x , s-times:

$$\frac{\partial^{r+s}}{\partial x^s\partial t^r}u_N(x,t)=\sum_{k=r-m}^{n-m-s}s!\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)t^{k+m-r}a_{p_k+s}+\sum_{k=r-m}^{n-m-s-1}\sum_{j=s+1}^{n-k-m}\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)\left(\prod_{\ell=j-s+1}^{j}\ell\right)t^{k+m-r}x^{j-s}a_{p_k+j} \qquad \ldots(18)$$

Now, let $(0,\Delta t)$ be any point in D which satisfies:

$$0<\Delta t\leq T; \qquad T=\min\left\{(2(n-m+1)R)^{m-n}\ ,1\right\} \qquad \ldots(19)$$

where

$$\left.\begin{array}{l} R=\max\{\,Q_1\,,\,|\,\beta\,\Gamma(n-\alpha+1)\,|\,Q_2\,\}\\ Q_1=\max\limits_{\substack{m\le r\le n\\ r-m+1\le k\le n-m}}\left|\dfrac{\prod\limits_{\ell=k-r+1}^{k}(\ell+m)\;\;[n!+(k+m)!\,\gamma(k+m,n)]}{r!\,[n!+r!\,\gamma(r,n)]}\right|\\ Q_2=\max\limits_{\substack{0\le k\le n-m-1\\ 0\le s\le n-r\\ m\le r\le n}}\left|\dfrac{(s+1)\left(\prod\limits_{\ell=0}^{r-1}(\alpha+k+m-\ell\,)\right)}{r!\,[n!+r!\,\gamma(r,n)]}\right| \end{array}\right\}\quad\ldots(20)$$

Condition (19) insure the convergence of this method as it will be illustrated later.Equate eq.(16) with eq. (18) at the point $(0,\Delta t)$ to get:

$$\sum_{k=r-m}^{n-m-s} s!\left[\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}+\frac{1}{n!}\,(k+m)!\,\gamma(k+m,n)\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}\right]a_{p_k+s}$$

$$+\frac{1}{n!}\,\beta\,\Gamma(n-\alpha+1)\sum_{k=0}^{n-m-s-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell\,)\right)\left(\prod_{\ell=1}^{s+1}\ell\right)(\Delta t)^{\alpha+k+m-r}\,a_{p_k+s+1}=\frac{\partial^{r+s}}{\partial x^s\,\partial t^r}G(x,t)\bigg|_{(0,\Delta t)}$$

This implies that:

$$\begin{aligned}&\left[s!\,r!+s!\frac{(r!)^2}{n!}\gamma(r,n)\right]a_{p_{r-m}+s}+\frac{s!}{n!}\sum_{k=r-m+1}^{n-m-s}\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}[n!+(k+m)!\,\gamma(k+m,n)]\,a_{p_k+s}\\ &+\frac{(s+1)!}{n!}\,\beta\,\Gamma(n-\alpha+1)\sum_{k=0}^{n-m-s-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell\,)\right)(\Delta t)^{\alpha+k+m-r}\,a_{p_k+s+1}=\frac{\partial^{r+s}}{\partial x^s\,\partial t^r}G(x,t)\bigg|_{(0,\Delta t)}\end{aligned}\quad\ldots(21)$$

Dividing both sides of eq.(21) by the coefficient of $a_{p_{r-m}+s}$ yield:

$$\begin{aligned}&a_{p_{r-m}+s}+\frac{1}{r!\,[n!+r!\,\gamma(r,n)]}\sum_{k=r-m+1}^{n-m-s}\left(\prod_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}[n!+(k+m)!\,\gamma(k+m,n)]\,a_{p_k+s}\\ &\quad+\frac{(s+1)\beta\,\Gamma(n-\alpha+1)}{r!\,[n!+r!\,\gamma(r,n)]}\sum_{k=0}^{n-m-s-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell\,)\right)(\Delta t)^{\alpha+k+m-r}\,a_{p_k+s+1}\\ &\quad=\frac{n!}{s!\,r!\,[n!+r!\,\gamma(r,n)]}\,\frac{\partial^{r+s}}{\partial x^s\,\partial t^r}G(x,t)\bigg|_{(0,\Delta t)}\end{aligned}\quad\ldots(22)$$

$$for\quad r=m,m+1,\ldots,n-1,\; s=1,2,\ldots,n-r$$

Eq. (22) represents M equations with M unknowns a_{p_k+j} ,where $\mathrm{M}=\sum\limits_{r=m}^{n-1}(n-r)$.

Our next aim is to find $(n-m+1)$ equations for the $(n-m+1)$ unknowns a_j, $j=0, 1, \ldots, n-m$. To this end, equate equations (15) and (17) at the points $(0,\Delta t)$ to get:

$$\sum_{j=r-m}^{n-m}\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)(\Delta t)^{j+m-r}\,a_j+\frac{1}{n!}\sum_{j=r-m}^{n-m-1}(j+m)!\,\gamma(j+m,n)\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)(\Delta t)^{j+m-r}\,a_j$$
$$+\frac{\beta\,\Gamma(n-\alpha+1)}{n!}\sum_{k=0}^{n-m-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r}\,a_{p_k+1}=\frac{\partial^r}{\partial t^r}G(x,t)\bigg|_{(0,\Delta t)}\quad\ldots(23)$$

for $\quad r=m,m+1,\ldots,n$

Put $r = n$ in eq. (23) to get:

$$a_{n-m}+\frac{\beta\,\Gamma(n-\alpha+1)}{(n!)^2}\sum_{k=0}^{n-m-1}\left(\prod_{\ell=0}^{n-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-n}\,a_{p_k+1}=\frac{1}{n!}\frac{\partial^n}{\partial t^n}G(x,t)\bigg|_{(0,\Delta t)}\quad\ldots(24)$$

Hence, for $r \le n-1$ we get:

$$\left[r!+\frac{(r!)^2\,\gamma(r,n)}{n!}\right]a_{r-m}+\left(\prod_{\ell=n-m-r+1}^{n-m}(\ell+m)\right)(\Delta t)^{n-r}\,a_{n-m}$$
$$+\sum_{j=r-m+1}^{n-m-1}\left(\prod_{\ell=j-r+1}^{j}(\ell+m)\right)(\Delta t)^{j+m-r}\left[1+\frac{1}{n!}(j+m)!\,\gamma(j+m,n)\right]a_j$$
$$\frac{\beta\,\Gamma(n-\alpha+1)}{n!}\sum_{k=0}^{n-m-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r}\,a_{p_k+1}=\frac{\partial^r}{\partial t^r}G(x,t)\bigg|_{(0,\Delta t)}$$

This implies:

$$a_{r-m}+\frac{n!\left(\prod_{\ell=n-m-r+1}^{n-m}(\ell+m)\right)(\Delta t)^{n-r}}{r!\,[n!+r!\,\gamma(r,n)]}\;a_{n-m}$$
$$+\sum_{j=r-m+1}^{n-m-1}\left(\prod_{\ell=j+r+1}^{j}(\ell+m)\right)(\Delta t)^{j+m-r}\left[\frac{n!+(j+m)!\,\gamma(j+m,n)}{r!\,[n!+r!\,\gamma(r,n)]}\right]a_j$$
$$+\frac{\beta\,\Gamma(n-\alpha+1)}{r!\,[n!+r!\,\gamma(r,n)]}\sum_{k=0}^{n-m-1}\left(\prod_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r}\,a_{p_k+1}\quad\ldots(25)$$
$$=\frac{n!}{r!\,[n!+r!\,\gamma(r,n)]}\,\frac{\partial^r}{\partial t^r}G(x,t)\bigg|_{(0,\Delta t)}$$

for $\quad r=m,m+1,\ldots,n-1$

Eqs. (22), (24) and (25) are the N equations needed for the evaluation of the N unknowns $a_0, a_1, \ldots, a_{N-1}$. These equations may be written in matrix form as: $\qquad H\,\underline{a} = B \qquad \ldots(26)$

where

$H = [h_{ij}]_{N\times N}$, $H = [b_i]_N$; $i, j = 0, 1, \ldots, N-1$, $\qquad \underline{a} = (a_0, a_1, \ldots, a_{N-1})^T$ and

$$h_{ij}=\begin{cases} 1 & , i=j; i,j=0,1,\ldots,\mathrm{N}-1 \\ \left.\begin{array}{ll} \dfrac{\left(\prod\limits_{\ell=j-i-m+1}^{j}(\ell+m)\right)(\Delta t)^{j-i}\,[n!+(j+m)!\gamma(j+m,n)]}{(m+i)!\,[n!+(m+i)!\gamma(m+i,n)]} & , j=i+1,\ldots,n-m \\ \dfrac{\beta\Gamma(n-\alpha+1)\left(\prod\limits_{\ell=0}^{m+i-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k-i}}{(m+i)!\,[n!+(m+i)!\gamma(m+i,n)]} & , j=p_k+1;\ k=0,1,\ldots,n-m-1 \end{array}\right\}, \\ for \quad i=0,\ldots,n-m \\ \left.\begin{array}{ll} \dfrac{\left(\prod\limits_{\ell=k-r+1}^{k}(\ell+m)\right)(\Delta t)^{k+m-r}\,[n!+(k+m)!\gamma(k+m,n)]}{r!\,[n!+r!\gamma(r,n)]} & , j=p_k+s;\ k=r-m+1,\ldots,n-m-s \\ \dfrac{(s+1)\beta\Gamma(n-\alpha+1)\left(\prod\limits_{\ell=0}^{r-1}(\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k+m-r}}{r!\,[n!+r!\gamma(r,n)]} & , j=p_k+s+1;\ k=0,\ldots,n-m-s-1 \end{array}\right\}, \\ for \quad i=p_{r-m}+s;\ r=m,\ldots,n-1; s=1,\ldots,n-r \\ 0 & , otherwise \end{cases} \qquad \ldots(27)$$

Journal of Kirkuk University – Scientific Studies , vol.3, No.1,2008

$$b_i=\begin{cases} \dfrac{n!}{(m+i)!\,[n!+(m+i)!\gamma(m+i,n)]}\,\dfrac{\partial^{m+i}}{\partial t^{m+i}}G(x,t)\Big|_{(0,\Delta t)}, \\ \quad for \quad i=0,1,\ldots,n-m \\ \dfrac{n!}{s!\,r!\,[n!+r!\gamma(r,n)]}\,\dfrac{\partial^{r+s}}{\partial t^{r}\partial x^{s}}G(x,t)\Big|_{(0,\Delta t)}, \\ \quad for \quad i=p_{r-m}+s; r=m,m+1,\ldots,n-1; s=1,2,\ldots,n-r \end{cases}$$

Finally, the approximate solution $u_N(x,t)$ in eq.(9) can be obtained by solving system (26) for the unknowns a_0, a_1, …, a_{N-1} using the Jacobi or Gauss-Seidel methods.The next two sections are concerned with the conditions that must be satisfied if the approximate solution (9) is to be reasonably accurate approximation to the solution of the FPDE, eq.(1). These conditions are associated with two problems, stability and convergence of the approximate solution to the solution of the FPDE

Stability Analysis

The stability of the polynomial approximation method will be discussed by investigating the stability of methods used to solve a system of equations. The stability of these methods are examined by determining the condition number of the matrix H, which is defined by:

$$cond\,(\mathrm{H})=\frac{\max\limits_{\lambda\in\sigma(\mathrm{H})}|\lambda|}{\min\limits_{\lambda\in\sigma(\mathrm{H})}|\lambda|}$$

where σ (H) denotes the set of all eigenvalues of H. The stability of such methods insured when this number is nearly one. Notice that the condition number of any matrix is always bounded below by one, i.e. we always have: cond (H) ≥ 1

Theorem (1):

The polynomial approximation method is unconditionally stable when it used to the FPDE, equ. (1).

Proof:

Our aim is to find the condition number of the matrix $H_{N\times N}$ which is represented by eq. (27), i.e. finding the eigenvalues of $H_{N\times N}$. Since there is no direct procedure that we could follow to find $|(H-\lambda I)_{N\times N}|$, and when $n = m$ there will be only one unknown in the approximate solution (9) which is a_0, so we shall find $|(H-\lambda I)_{N\times N}|$ for the special cases $n = m + 1$, $m + 2$, …, and stop when we are able to construct a procedure that could be followed

to find this determinate for any $n > m$. To this end, we shall use the column expansion method. Consider first $H_{2\times 2}$ for $n = m + 1$

$$H_{2\times 2}=\begin{bmatrix}1 & h_{01}\\ 0 & 1\end{bmatrix} \quad \Rightarrow \quad |(H-\lambda I)_{2\times 2}| = (1-\lambda)^2 = (1-\lambda)^N$$

Next, consider $H_{6\times 6}$ for $n = m + 2$

$$H_{6\times 6}=\begin{bmatrix}1 & h_{01} & h_{02} & h_{03} & 0 & h_{05}\\ 0 & 1 & h_{12} & h_{13} & 0 & h_{15}\\ 0 & 0 & 1 & h_{23} & 0 & h_{25}\\ 0 & 0 & 0 & 1 & h_{34} & h_{35}\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & h_{54} & 1\end{bmatrix}$$

Hence $|(H-\lambda I)_{6\times 6}|$ can be found by using columns 0, 1 and 2 respectively, where $n - m = 2$, then using columns 4 and 6 respectively, where $p_0 + 1 = 4$ and $p_1 + 1 = 6$. This procedure yields:

$$|(H-\lambda I)_{6\times 6}| = (1-\lambda)^6 = (1-\lambda)^N$$

Also, for $n = m + 3$ we have:

$$(\mathrm{H}-\lambda \mathrm{I})_{10\times 10}=\begin{bmatrix} 1-\lambda & h_{01} & h_{02} & h_{03} & h_{04} & 0 & 0 & h_{07} & 0 & h_{09} \\ 0 & 1-\lambda & h_{12} & h_{13} & h_{14} & 0 & 0 & h_{17} & 0 & h_{19} \\ 0 & 0 & 1-\lambda & h_{23} & h_{24} & 0 & 0 & h_{27} & 0 & h_{29} \\ 0 & 0 & 0 & 1-\lambda & h_{34} & 0 & 0 & h_{37} & 0 & h_{39} \\ 0 & 0 & 0 & 0 & 1-\lambda & h_{45} & 0 & h_{47} & h_{48} & h_{49} \\ 0 & 0 & 0 & 0 & 0 & 1-\lambda & h_{56} & 0 & h_{58} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_{75} & 0 & 1-\lambda & h_{78} & h_{79} \\ 0 & 0 & 0 & 0 & 0 & 0 & h_{86} & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & h_{95} & 0 & 0 & h_{98} & 1-\lambda \end{bmatrix}$$

Thus using columns 0, 1, 2 and 3 respectively yield:

$$|(\mathrm{H}-\lambda\mathrm{I})_{10\times 10}| = (1-\lambda)^{n-m+1}\,|(\mathrm{p}-\lambda\mathrm{I})_{\mathrm{M}\times\mathrm{M}}|$$

where n–m+1 = 4 and $\mathrm{M}=\sum_{r=m}^{n-1}(n-r)$ = 6 and

$$\mathrm{P}_{6\times 6}=\begin{bmatrix} 1 & h_{45} & 0 & h_{47} & h_{48} & h_{49} \\ 0 & 1 & h_{56} & 0 & h_{58} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & h_{75} & 0 & 1 & h_{78} & h_{79} \\ 0 & 0 & h_{86} & 0 & 1 & 0 \\ 0 & h_{95} & 0 & 0 & h_{98} & 1 \end{bmatrix}$$

Now, by using columns 4 = p_0 + 1, 7 = p_1 + 1, 9 = p_2 + 1, 5 = p_0 + 2 and 8 = p_1 + 2 respectively we get:

$$|(\mathrm{H}-\lambda\mathrm{I})_{10\times 10}| = (1-\lambda)^{n-m+1}(1-\lambda)^{\mathrm{M}} = (1-\lambda)^{\mathrm{N}}.$$

Now, it is clear that for any $n > m$ we have:

$$|(\mathrm{H}-\lambda\mathrm{I})_{\mathrm{N}\times\mathrm{N}}| = (1-\lambda)^{n-m+1}\,|(\mathrm{p}-\lambda\mathrm{I})_{\mathrm{M}\times\mathrm{M}}| \qquad \ldots(28)$$

where

$$\mathrm{P}_{M\times M}=\begin{bmatrix} 1 & h_{p_0+1,\,p_0+2} & \cdots & h_{p_0+1,\,p_{n-m-1}+1} \\ h_{p_0+2,\,p_0+1} & 1 & \cdots & h_{p_0+2,\,p_{n-m-1}+1} \\ \vdots & & & \\ h_{p_{n-m-1}+1,\,p_0+1} & h_{p_{n-m-1}+1,\,p_0+2} & \cdots & 1 \end{bmatrix}$$

Accordingly, we can find $|(\mathrm{H}-\lambda\mathrm{I})_{\mathrm{N}\times\mathrm{N}}|$ if we find $|(\mathrm{p}-\lambda\mathrm{I})_{\mathrm{M}\times\mathrm{M}}|$. Now, we are able to construct a procedure which may be used to find $|(\mathrm{H}-\lambda\mathrm{I})_{\mathrm{N}\times\mathrm{N}}|$ for any $n > m$ as follows:

1. Use columns 0, 1, …, $n-m$ respectively.
2. Use columns p_k+1, $k = 0, 1, \ldots, n-m-1$ respectively.
3. Use columns p_k+2, $k = 0, 1, \ldots, n-m-2$ respectively.

⋮

(M-1). Use columns $p_k+(n-m-1)$, $k = 0$, respectively.

This procedure will always result:

$$|(H - \lambda I)_{N\times N}| = (1-\lambda)^N \quad , \text{for any } n > m \qquad \ldots(29)$$

Eq. (29) implies that $\lambda_i = 1$, $i = 0, 1,\ldots, N-1$, i.e. for all $n > m$ we have

$\sigma(H) = \{1\}$, which means that $\max\limits_{\lambda\in\sigma(H)} |\lambda| = \min\limits_{\lambda\in\sigma(H)} |\lambda| = 1$

Therefore; cond (H) = 1

Hence, the polynomial approximation method is unconditionally stable for all $n \geq m$, where n is the order of the polynomial that used to approximate the exact solution of the FPDE (1).

Convergence Analysis

From the stability analysis of this method we conclude that we may use any method for solving system (26) and the method will be unconditionally stable. But to insure the convergence of this method, we shall use either Jacobi or Gauss-Seidel method. These two methods converge if

$$\sum_{\substack{j=0\\ j\neq i}}^{N-1} |h_{ij}| < |h_{ii}| \quad , \text{for } i = 0, 1,\ldots, N-1 \qquad \ldots(30)$$

From eq. (27) we have $h_{ii} = 1$, for all, $i = 0, 1,\ldots, N-1$, so eq.(30) becomes:

$$\sum_{\substack{j=0\\ j\neq i}}^{N-1} |h_{ij}| < 1 \quad , \text{for } i = 0, 1, \ldots, N-1 \qquad \ldots(31)$$

Eq. (31) will be used in the prove the following convergence result:

Theorem (2):

The condition $0 < \Delta t \leq T$ is true then the polynomial approximation method converges to the unique solution of the FPDE (1).

Proof:

Based on the range of i in eq. (27), this proof was divided into two parts:

Part 1: For $i = 0, 1,\ldots\ n-m$

Since $$\sum_{j=i+1}^{n-m} |h_{ij}| = \sum_{j=i+1}^{n-m} \left| \frac{\left(\prod\limits_{\ell=j-i-m+1}^{j} (\ell+m)\right) (\Delta t)^{j-i} \left[n! + (j+m)!\gamma(j+m,n)\right]}{(m+i)!\left[n!+(m+i)!\gamma(m+i,n)\right]} \right|$$

and $$\sum_{k=0}^{n-m-1} \left|h_{i,p_k+1}\right| = \sum_{k=0}^{n-m-1} \left| \frac{\beta\Gamma(n-\alpha+1)\left(\prod\limits_{\ell=0}^{m+i-1} (\alpha+k+m-\ell)\right)(\Delta t)^{\alpha+k-i}}{(m+i)!\left[n!+(m+i)!\gamma(m+i,n)\right]} \right|$$

therefore $$\sum_{\substack{j=0\\ j\neq i}}^{N-1} \left| h_{ij} \right| \le \sum_{j=i+1}^{n-m} Q_1 (\Delta t)^{j-i} + \sum_{k=0}^{n-m-1} \left| \beta \Gamma(n-\alpha+1) \right| Q_2 (\Delta t)^{\alpha+k-i}$$
$$\le R \left\{ \sum_{j=i+1}^{n-m} (\Delta t)^{j-i} + \sum_{k=0}^{n-m-1} (\Delta t)^{\alpha+k-i} \right\}$$
where Q_1, Q_2 and R are defined in eq. (20).
Now, since $\Delta t \le 1$, then $(\Delta t)^{j-i} \le (\Delta t)^{-i}$ and $(\Delta t)^{\alpha+k-i} < (\Delta t)^{-i}$
Put this into eq. (32) to get:
$$\sum_{\substack{j=0\\ j\neq i}}^{N-1} \left| h_{ij} \right| \le (\Delta t)^{-i} R \left\{ \sum_{j=i+1}^{n-m} 1 + \sum_{k=0}^{n-m-1} 1 \right\} < 2(n-m+1)\,(\Delta t)^{-i} R,\ i = 0, 1, \ldots, n-m \qquad \ldots(33)$$
Part 2: For $i = p_{r-m} + s$; $r = m, \ldots n-1$, $s = 1, \ldots n-r$
Since $$\sum_{k=r-m+1}^{n-m-s} \left| h_{i,p_k+s} \right| = \sum_{k=r-m+1}^{n-m-s} \frac{\left(\prod_{\ell=k-r+1}^{k} (\ell+m) \right) (\Delta t)^{k+m-r} \left[n! + (k+m)!\gamma(k+m,n) \right]}{r! \left[n! + r!\gamma(r,n) \right]}$$
and $$\sum_{k=0}^{n-m-s-1} \left| h_{i,p_k+s+1} \right| = \sum_{k=0}^{n-m-s-1} \frac{(s+1)\beta \Gamma(n-\alpha+1) \left(\prod_{\ell=0}^{r-1} (\alpha+k+m-\ell) \right) (\Delta t)^{\alpha+k+m-r}}{r! \left[n! + r!\gamma(r,n) \right]}$$
therefore; $$\sum_{\substack{j=0\\ j\neq i}}^{N-1} \left| h_{ij} \right| = \sum_{k=r-m+1}^{n-m-s} \left| h_{i,p_k+s} \right| + \sum_{k=0}^{n-m-s-1} \left| h_{i,p_k+s+1} \right|$$

$$< R \left\{ \sum_{k=0}^{n-m} (\Delta t)^{k+m-r} + \sum_{k=0}^{n-m-1} (\Delta t)^{\alpha+k+m-r} \right\} < 2(n-m+1)\,(\Delta t)^{m-r} R \qquad \ldots(34)$$
$i = p_{r-m} + s$; $r = m, \ldots, n-1$, $s = 1, \ldots, n-r$ Now, since $\Delta t \le 1$ and $i = 0, 1, \ldots n-m$, $r = m, \ldots n-1$ then:
$$(\Delta t)^{m-n} = \max_{\substack{0 \le i \le n-m\\ m \le r \le n-1}} \left\{ (\Delta t)^{-i}, (\Delta t)^{m-r} \right\} \qquad \ldots(35)$$
Combining inequalities (33)-(35) yield
$$\sum_{\substack{j=0\\ j\neq i}}^{N-1} \left| h_{ij} \right| < 2(n-m+1)\,(\Delta t)^{m-n} R \quad , i = 0, 1, \ldots, N-1 \qquad \ldots(36)$$

Notice that, inequality (36) is true for all $i = 0, 1, \ldots N-1$ because $h_{ij} = 0$, for all i not in the range of the two previous parts. Accordingly, if the right hand side of inequality (36) is less than or equal to one, then inequality (31) is satisfied for all $i = 0, 1, \ldots N-1$. This happens if:

$$\Delta t \le \frac{1}{(2(n-m+1)R)^{n-m}}$$

Which is true from condition (19).

Remark:

In fact theorem (2) insures the convergence and stability of the polynomial approximation method when the Jacobi iterative method is used to solve system (26).

Numerical Examples

Example (1):

Consider the FPDE $D_t^{1.1} u(x,t) - \frac{\partial u(x,t)}{\partial t} = g(x,t)$, $0 \le x \le 2,\ 0 \le t \le 4$

where $g(x,t) = \frac{10}{\Gamma(1.9)} x t^{0.9} + \frac{18}{\Gamma(2.9)} t^{1.9} - 5t^2$

while the exact solution is $u(x,t) = 5\,x\,t^2 + 3\,t^3$

Let $n = 3$, then $u_3(x,t) = a_0 t^2 + a_1 t^3 + a_2 x t^2$, since we may take any value of Δt in the interval (0, 0.528), so let $\Delta t = 1/3$. The results of the polynomial approximation method are obtained. These results are given by $a_0 = 0$, $a_1 = 3$ and $a_2 = 5$.

Example (2):

Consider the FPDE

$$D_x^{5/2} u(x,t) - 3\frac{\partial u(x,t)}{\partial t} = g(x,t) \qquad , 0 \le x \le 1,\ 0 \le t \le 1$$

where $g(x,t) = (\frac{6\sqrt{t}}{\Gamma(1.5)} x t^{0.9} - 3t)e^x$,and the exact solution is $u(x,t) = t^3 e^x$.

let $n = 7$. Since $m = 3$, then we get:

$$\begin{aligned} u_{15}(x,t) = {} & a_0 t^3 + a_1 t^4 + a_2 t^5 + a_3 t^6 + a_4 t^7 + a_5 x t^3 \\ & + a_6 x^2 t^3 + a_7 x^3 t^3 + a_8 x^4 t^3 + a_9 x t^4 + a_{10} x^2 t^4 \\ & + a_{11} x^3 t^4 + a_{12} x t^5 + a_{13} x^2 t^5 + a_{14} x t^6 \end{aligned}$$

Let $\Delta t = 1 * 10^{-15}$, or any value in $(0, 1.6288 * 10^{-15})$. The results of the polynomial approximation method with the least square error and the running time are listed in table (1):

Table(1)

x	t	$u(x,t) = t^3e^x$	Poly. Approx
0	0	0.00000000	0.00000000
0.1	0.1	0.00110517	0.00110517
0.2	0.2	0.00977122	0.00977121
0.3	0.3	0.03644619	0.03644616
0.4	0.4	0.09547678	0.09547675
0.5	0.5	0.20609016	0.20609023
0.6	0.6	0.39357766	0.39357838
0.7	0.7	0.69071718	0.69072035
0.8	0.8	1.13947696	1.13948777
0.9	0.9	1.79305067	1.79308224
LSE			0.00000000

Also, more accurate results may be obtained by increasing the number of the parameters a_j's. Depending on the least square error and running time, a comparison has been made in table (2) between the exact and approximate solutions, where the approximate solution was obtained with n=10 and $\Delta t = 5* 10^{-35}$ $(\Delta t \in (0,8.773*10^{-35})$.

x	t	$u(x,t) = t^3e^x$	Poly. Approx
0	0	0.00000000	0.00000000
0.1	0.1	0.00110517	0.00110517
0.2	0.2	0.00977122	0.00977119
0.3	0.3	0.03644619	0.03644558
0.4	0.4	0.09547678	0.09547085
0.5	0.5	0.20609016	0.20605452
0.6	0.6	0.39357766	0.39342210
0.7	0.7	0.69071718	0.69017411
0.8	0.8	1.13947696	1.13786809
0.9	0.9	1.79305067	1.78884654
LSE			0.00002058
Running Time			0:0:3:14

Table (2)

Running Time	0:0:6:27

Discussion

A new efficient method, which is called the polynomial approximation method, was introduced to find the approximate solution of FPDEs. Several examples were included for illustration. The following points have been identified:

1. This method gives the exact solution when the unknown function is a polynomial of degree *n*, while for other types of functions, the accuracy of the solution depends on the degree of the used approximation.
2. A disadvantage of this method is the hand evaluation of the partial derivatives of the function G(*x t*) .
3. An advantage of this method is the few number of computations which is clear from its short running time.
4. The convergence condition of this method gives us a range of values from which the value of Δt may be chosen. This range depends on the given values of *n* and *m*.

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طريقة كفوءة لحل المعادلات التفاضلية الجزئية الكسرية

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الخلاصة

يقدم هذا البحث طريقة جديدة لحل المعادلات التفاضلية الجزئية الكسرية ، هي طريقة التقريب بمتعـــددات الحدود و التي تعتمد على التقريب بمتعددات الحدود $u_N(x,t)$ و على الصيغة العامة لمشتقتها الكسرية.تم إيجاد صيغة جديدة أخرى لتقريب $u_N(x,t)$ بعد تطوير صيغة المشتقة الكسرية لها. علاوة على ذلـــك تـــم دراســـة البرهان الرياضي لتقارب وأستقرارية الطريقة المقترحة. بعض الأمثلة العددية ترينـــا أن الطريقـــة المقترحـــة تعرض نتائج مقنعة.