On Divisibility In a Ring With Respect To a Fuzzy Subset

Hussein Hadi Abbas
AL-Kufa university
College of Education for Girls
Department of Mathematics
Email: msc_Hussien@yahoo.com

Abstract
In this paper , we study the divisibility in a ring with respect to a fuzzy subset . we also study the notions of units , associates elements ,prime elements with respect to a fuzzy subset .

الخلاصة
درسنا في هذا البحث القسمة في حلقة بالنسبة إلى مجموعة ضبابية وكذلك درسنا مفاهيم العناصر الواحدية والعناصر المرتبطة و العناصر الأولية بالنسبة إلى مجموعة ضبابية .

1.Introduction :-

The notation of fuzzy subset of a set was introduced by Zadeh[5] in 1965 . In 1999 Ray[3] introduced the concepts of translational invariant fuzzy subset .

The notations of units , associates , prime elements and divisibility in classical ring theory are generalized with respect to a fuzzy subset by Ray[4] in 2002 .

The main purpose of this paper , is to study some properties of divisibility in a ring with respect to a fuzzy subset .

Throught this paper R is an arbitrary ring with binary operations '+' and '.' .

2.Preliminaries :-

In this section , we review some definitions and some results which will be used in later section .

Definition (2.1) **:- [5]**

A fuzzy subset P of a set S is a mapping from S into [0,1].

Definition (2.2) **:- [3]**

Let P be a fuzzy subset of a set S and * be a binary operation on S .P is said to be left translational invariant with respect to '*' if $P(x)=P(y) \Rightarrow P(a*x)=P(a*y)$, $\forall$ x ,y ,a $\in$ S .

Definition (2.3) **:- [3]**

Let P be a fuzzy subset of a set S and '*' be a binary operation on S . P is said to right translational invariant with respect to '*' if $P(x)=P(y) \Rightarrow P(x*a)=P(y*a)$, $\forall$ x ,y ,a $\in$ S .

Definition (2.4) **:-[3]**

Let P be a fuzzy subset of a set S and '*' be a binary operation on S . P is said to translational invariant with respect to '*' if P is left and right translational invariant with respect to '*'.

Definition (2.5) **:- [1]**

A fuzzy subset P of a ring R is called

(i) Fuzzy multiplicatively left cancelative if $P(ax) = P(ay)$ implies that $P(x)=P(y)$.

(ii) Fuzzy multiplicatively right cancelative if $P(xa) = P(ya)$ implies that $P(x)=P(y)$.

Definition (2.6) **:- [2]**

A fuzzy subset P of a ring R is called fuzzy ideal of R if and only if , for all x ,y $\in$ R .

(i) $P(x-y) \geq \min \{ P(x) , P(y) \}$.

(ii) $P(x.y) \geq \max \{ P(x) , P(y) \}$.

Definition (2.7) **:- [4]**

Let P be a fuzzy subset of a ring R . An element $a \in R$ with $P(a) \neq P(0)$ is said to be a P-divisor of zero if there exists some $b \in R$ with $P(b) \neq P(0)$ such that $P(ab)=P(0)$.

Definition (2.8) ***:- [4]***

Let P be a fuzzy subset of a ring R and $a,b \in R$ with $P(a) \neq P(0)$.
The a is called divides b with respect to P or a is a P-divisor of b , written $(a \mid b)_P$ if there exists $c \in R$ such that $P(b)=P(ac)=P(ca)$.

Definition (2.9) ***:-* [4]**

Let P be a fuzzy subset of a ring R and $a,b \in R$ with $P(a) \neq p(0)$ and $P(b) \neq p(0)$ .Then a and b P-associates if $(b|a)_P$ and $(a|b)_P$.

Definition (2.10) ***:-* [4]**

Let R be a ring with identity e and P be a fuzzy subset of R .

An element $a \in R$ with $P(a) \neq P(0)$ is called a P-unit of R if there exists an element $u \in R$ such that $P(u) \neq P(0)$ and $P(au)=P(ua)=P(e)$.

Definition (2.11) ***:-* [4]**

Let R be a ring with identity and P be a fuzzy subset of R . Let $a \in R$ with $P(a) \neq P(0)$ and a not a P-unit of R . Then a is said to be p-prime if $(a|bc)_P$ implies $(a|b)_P$ or $(a|c)_P$.

3. The Main Results :-

In this section , we find some results about the divisibility with respect to a fuzzy subset .

Theorem (3.1) :-

Let R be a ring with identity e and P be a fuzzy subset of R satisfies $P(x) \neq P(0)$, for all $x \in R \setminus \{0\}$. Then a is a P-unit if and only if $(a|e)_P$.

Proof :-

Suppose a is a P-unit . The there exists $u \in R$ such that $P(u) \neq P(0)$ and $P(au)= P(ua)=P(e)$ this implies $(a|e)_P$.
Conversely ,

Let $a \in R$ such that $P(a) \neq P(0)$. Since $(a|e)_P$ then there exists u such that $P(au)= P(ua)=P(e)$. since $P(x) \neq P(0)$, for all $x \in R \setminus \{0\}$ this implies $u \neq 0 \Rightarrow P(u) \neq P(0)$ implies that a is a P-unit .

Corollary (3.2) :-

Let R be a ring with identity e and P be a fuzzy subset of R satisfies $P(x) \neq P(0)$, for all $x \in R \setminus \{0\}$. Then a is a P-unit if and only if a and e are P-associates .

Proof :-

Let a be a P-unit . Then by theorem(3.1) we have $(a|e)_P$. Since
$P(a)= P(ea)=P(ae)$, then $(e|a)_P$ .So we have $(a|e)_P$ and $(e|a)_P$ implies that a and e are P-associates .
Conversely ,

Since a and e are P-associates , then $(a|e)_P$ .From theorem(3.1) yields a is a P-unit .

Lemma (3.3) ***:-***

Let P be a fuzzy translational invariant with respect to '.' of the commutative ring R and $a,b \in R$ such that $(a|b)_P$ then $(ac|bc)_P$ for all $c \in R$, $c \neq 0$.

Proof :-

Since $(a|b)_P$. Then there exists $u \in R$ such that $P(au)= P(ua)=P(b)$.
Since P translational invariant with respect to '.' , then
$P(bc)= P((ua)c)= P(u(ac))=P(ac(u))$ for all $c \in R$, $c \neq 0$. Then $(ac|bc)_P$.

Remark (3.4) ***:-***

In general if $(a|b)_P$ not implies $(ac|bc)_P$ as in the following example .
Consider Z_{12} the ring of integers modulo 12 . Let P be a fuzzy subset of Z_{12} such that P(0)=0 ,P(1)=P(2)=P(3)=P(5)=P(7)=P(9) =0.5 ,p(4)=p(8)=0.25 and P(6)=0.125 . Since $P(3) \neq P(0)$ and $P(3.1)=p(1 . 3)=P(2)$. Then $(3|2)_P$ but $(3.2 \nmid 2.2)_p$ i.e. $(6 \nmid 4)_p$ [since P(6a)=P(6) or P(6a)=P(0) , for all $a \in Z_{12}$. Then $P(6a) \neq P(4)$ for all $u \in Z_6$] .

Lemma (3.5) ***:-***

Let p be a fuzzy subset of a field F and $a,b \in F$ such that $P(a) \neq p(0)$ and $P(b) \neq p(0)$. Then a and b are P-associates .

Proof :-

Since $P(a) \neq P(0)$. Then $a \neq 0 \Rightarrow a^{-1} \in F$ then we have $a(a^{-1}b)=b \Rightarrow P(a(a^{-1}b))= P((a^{-1}b)a)=P(b)$ $\Rightarrow (a|b)_P$ similary $(b|a)_P \Rightarrow$ a and b are P-associates .

Proposition (3.6) :-

Let P be a fuzzy translational invariant with respect to '.' of a field F and $a \in F$ is not a P-unit with $P(a) \neq P(0)$, then a is a P-prime .

Proof :-

Assume $(a|bc)_P$, $b,c \in F$, then there exists $u \in F$ such that $P(au)=P(ua)=P(bc)$.
If b=0 or c=0 then $P(au)=P(0) \Rightarrow (a|b)_P$ or $(a|c)_P \Rightarrow$ a is a P-prime .
If $b \neq 0$ and $c \neq 0$. Since P is a fuzzy translational invariant with respect to '.' , then $P((au)c^{-1}))= P((bc)c^{-1}) \Rightarrow P(a(uc^{-1}))=P((uc^{-1})a)=P(b) \Rightarrow (a|b)_P$.
Similarly $(a|c)_P$. Thus a is a P-prime .

Proposition (3.7) :-

Let P be a fuzzy ideal of a ring R , then

1- If $(a|b)_P$ then $P(b) \geq P(a)$.
2- If a,b are P-associates then $P(a)=P(b)$.

Proof :-

1- Since $(a|b)_P$, then there exists $c \in R$ such that $P(ac)=P(ca)=P(b)$. Since P is a fuzzy ideal of R , then $P(b)=P(ac) \geq \max\{P(a),P(c)\} \geq P(a)$.
2- Since a and b are P-associates ,we have $(a|b)_P$ and $(b|a)_P$. From (1) we get $P(b) \geq P(a)$ and $P(a) \geq P(b) \Rightarrow P(a)=P(b)$.

References :-

[1] Kavikumar. J. ,"Fuzzy Ideals and Fuzzy Quasi – ideals in Ternary Semi Rings ", IAENG International Journal of Applied Mathematics ,37:2 , IJAM , 2006 .
[2] Liu . W.J. , "Fuzzy Invariant Subgroups and Fuzzy Ideals ", Fuzzy sets and System 8(1982),133-139 .
[3] Ray . A.K.,"Quotient Group Of a Group Generated By a Subgroup and Fuzzy Subset ",The Journal of Fuzzy Mathematics Vol.7 , No2(1999),459-463 .
[4] Ray . A.K. ,"Ideals and Divisibility in a Ring With Respect to a Fuzzy Subset ",NOVISADJ.MATH Vol.32 ,No.2 ,2002,67-75 .
[5] Zadeh . L. A. ,"Fuzzy Sets", Information and Control ,vol.8,pp 338-353, 1965 .