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A New Stiffness Matrix Derivation of Arch Beams Using Finite Elements

اشتقاق مصفوفة صلادة جديدة للعتبات المنحنية بطريقة العناصر المحددة

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Abstract

In the present study, a new stiffness matrix for arch beams had been derived using finite element method. Transverse shear deformation was included in the derivation.The present stiffness matrix could be used in both linear and nonlinear analysis of concrete and steel arch beams. The new stiffness matrix had been used in some applications of both curved thick and thin arch beams. The conclusion shows a very good agreement between the present study and the results of exact solution of other researchers. The maximum difference in deflection was found to be (9%). Also the influence of shear deformations was found to be significant in the thick arch example with a ratio of (10%).

الخلاصة

تم في هذه الدراسة اشتقاق مصفوفة صلادة جديدة للعتبات المنحية باستخدام طريقة العناصر المحددة مع الأخذ بنظر الاعتبار تأثير تشوهات القص. اجري الاشتقاق بحيث يمكن استخدام هذه المصفوفة في التحليل الخطي واللاخطي للأعضاء الحديدية والكونكريتية. كما وتم مقارنة هذه الطريقة مع الدراسات السابقة للأعضاء المنحنية ولكلا النوعين السميكة والرقيقة حيث أظهرت النتائج توافقا جيدا مع هذه الأمثلة. حيث كان الفرق الأكبر للإزاحات العمودية (9%). كذلك لوحظ إن تأثير تشوهات القص يكون واضحا في مثال العتبات المقوسة السميكة وبنسبة (10%).

Notations

R= radius of arch
M_i, P_i, V_i = moment ,axial force , shear force at node i
β= angle of arch member
b,h = cross section dimensions
Zi= distance from neutral axis of the section to the center of the layer i.
u(x)= axial displacement equation .
w(x)= vertical displacement equation.
θ(x)= normal rotation equation.
N_1, N_2= shape functions of axial displacement.
$\overline{N}_1, \overline{N}_2, \overline{N}_3, \overline{N}_4$ = shape functions of vertical displacement.
$\overline{\overline{N}}_1, \overline{\overline{N}}_2, \overline{\overline{N}}_3, \overline{\overline{N}}_4$ = shape functions of normal rotation.
u_i, w_i, θ_i = axial, vertical, normal rotation deformations at node i.
F= factor used to simplified the derivation.
γ_{xz} = The shear strain .
g = shear factor.
[K] = stiffness matrix of arch member.
[D] = rigidity matrix of arch member.
E_i =modulus of elasticity of layer i.
G_i =shear modulus of layer i.

t_i = thickness of layer i.
GSH= E.I/(F.G.A)
I =second moment of area.
A= area of the section.
Ks = 12.F.g

Review of literature

A large amount of literature on the development of curved beam element exists, and continuous to grow. Some of them are based on exact strain energy or nature shape function. Others are based on assumed displacement fields. Some of those are presented herein.

Martin [1] derived an exact two dimensional stiffness matrix in which the axial and shear deformation are neglected. Morris [2] presented a derivation of three dimensional curved beam stiffness coefficients. The derivation was based on the node point reactions that results from unit displacement with all other displacements fixed. The reactive forces are the stiffness coefficients for one column of the stiffness. He repeated this procedure until all twelve nodal displacements had been given in terms of unit value. The results of those three authors give a good agreement with exact solution of arch beams.

Ashwell and Sabir [3,4] discussed the used and the limitations of shape function to the solution of arches by the method of finite element in their study they have drawn many conclusions for the usage and limitations of some shape functions. Dawe [5] analyzed shallow and deep arches using curved beam finite element. The individual elements were assumed to be shallow with respect to local base line, different types of strain displacement equations were utilized, his models showed very oscillatory behavior in the calculated force for thin arch. Dawe [6] extend his previous work to include circular arches and, the corresponding exact strain element equations were used, the same previous conclusions were stated in this work.

Yamada and Ezawa [7] derived an exact stiffness matrix for circular curved element from the natural shape function. These functions were assumed to give the exact deformations of circular arches and it was confirmed to be the inverse of the flexibility matrix the study gave a comparison between the inverse of the derived stiffness matrix with the exact one.. Litewaka and Rakowski [8] derived the exact stiffness matrix of curved beam element with constant curvature. The plane two nodes with six degree of freedom element were considered in which the effect of flexural, axial and shear deformations were taken into account. The analytical shape functions describing radial and tangential displacements as well cross section rotations were given in algebraic trigonometric form. They include the coupled influences of shear and membrane effects. Based on the shape functions, using the strain energy formula, the stiffness matrix for shear flexible and compressible arch element was formulated. Just [8] present exact stiffness matrix for circular curved beam subjected to in plane loading. The matrix was derived from the governing differential equations and from finite element procedure.

Aldaami [10] presented a development for the formulation of the space curved beam element. He used the exact strain energy expression. The contributions of axial, bending and bimoment to normal strain were included. Also, the energy due to shear strain which results from direct shear forces and torsional moments was considered , his study shows a good agreement with those obtain from **Msc NASTRAN** package program . Muhaisin[11] presented a new form of the stiffness matrix for reinforced concrete plane frame element including the effect of shear deformations is derived; the layered approach is adopted to take into account the variation of material properties throw the thickness, this approach gives a good result compared with those from other authors. Hadi [12] derived a curved beam stiffness matrix which takes into account the effect of shear, axial and bending deformations. For this purpose a curved beam element with hinges at one or both ends of the element has been formulated, also a good agreement obtained between his results and those from other authors.

In the present study the same shape functions used by Muhaisin[11] were modified to formulate a new stiffness matrix for arch beams using finite element method . Transverse shear deformation was included in the derivation.

Derivation of the Stiffness Matrix

Consider the element of arch beam having radius (R) and angle of curvature (β) as shown in Figure (1), the element with constant depth (prismatic) at any cross section shape. Three degrees of freedom are assumed at each node. Only the deformations in the plane of the element and the bending about the main centroidal axis are considered. All displacements, stresses and forces are positive if they are in the directions shown in Figure (1).

Basic Assumptions

The following assumptions were considered in the derivation of the stiffness matrix [11]:

1.The element is subdivided into the number of layers through the depth as shown in Figure (2).

2.Plane sections before bending remain plane after bending.

3.The stress in each layer is assumed to be uniform along the depth of the layer.

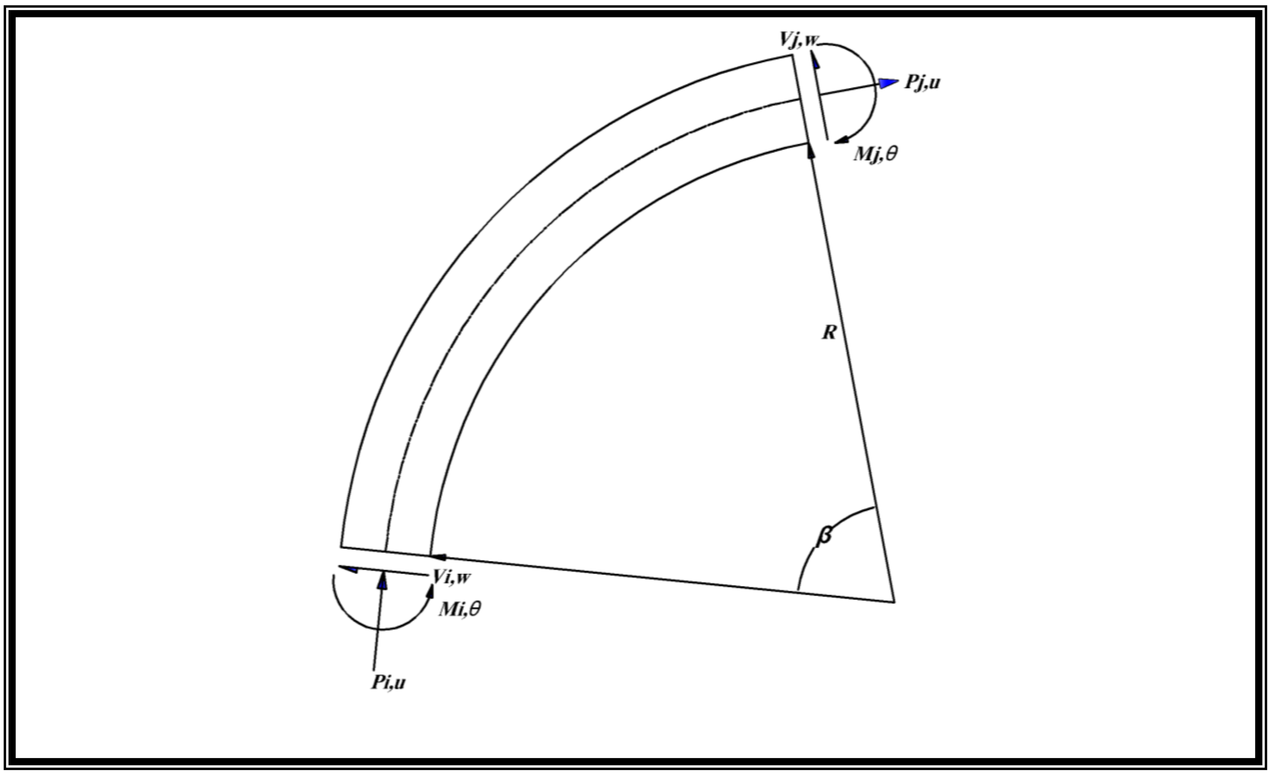


Figure (1) Degrees of Freedom and Sign Convention [10]

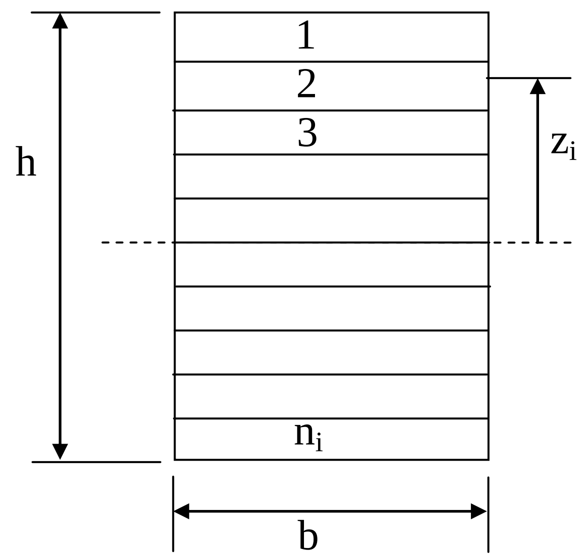


Figure (2) Section Discretization Into Layers.[11]

Assumed Displacement Field

A frame element with two nodes and three degrees of freedom per node is shown in Figure (1).The axial displacement u(x) is represented by the relationship[11]:

$$u(x) = N_1 u_1 + N_2 u_2$$
(1)

where:

$$N_1 = \left(1 - \frac{x}{L}\right)$$
(2)

$$N_2 = \frac{x}{L}$$
(3)

The lateral displacement w(x) and the normal rotation $\theta_{(x)}$ are represented as[11]:

$$w(x) = \overline{N}_1 w_1 + \overline{N}_2 \theta_1 + \overline{N}_3 w_2 + \overline{N}_4 \theta_2$$
(4)

$$\theta(x) = \overline{\overline{N}}_1 w_1 + \overline{\overline{N}}_2 \theta_1 + \overline{\overline{N}}_3 w_2 + \overline{\overline{N}}_4 \theta_2$$
(5)

where:

$$\left.\begin{aligned}
\overline{N}_1 &= 1 - F\left[12gx + 3Lx^2 - 2x^3\right]\\
\overline{N}_2 &= F\left[\left(L^2+6g\right)Lx - \left(2L^2+6g\right)x^2 + Lx^3\right]\\
\overline{N}_3 &= F\left[12gx + 3Lx^2 - 2x^3\right]\\
\overline{N}_4 &= F\left[-6gLx + \left(6g - L^2\right)x^2 + Lx^3\right]\\
\overline{\overline{N}}_1 &= F\left[6x^2 - 6Lx\right]\\
\overline{\overline{N}}_2 &= F\left[L^3 + 12gL - \left(4L^2+12g\right)x + 3Lx^2\right]\\
\overline{\overline{N}}_3 &= F\left[6Lx - 6x^2\right]\\
\overline{\overline{N}}_4 &= F\left[3Lx^2 - \left(2L^2 - 12g\right)x\right]\\
F &= \frac{1}{L\left(L^2+12g\right)}
\end{aligned}\right\} \qquad (6)$$

Strain – Displacement Relationships

The axial displacement at any point (x,z) may be expressed directly in term of θ(x) which is the rotation of the normal to the centroidal axis due to flexural effect, and u(x) which is the axial displacement due to axial effect and curvature

$$\bar{u}(x,z) = u(x) + \frac{w}{R} - z\left(\theta_{(x)} + \frac{1}{R}u(x)\right)$$

(7)

where ***z*** is the distance from the centroidal axis of the cross section to the point (x,z). The normal rotation θ(x) is equal to the slope of the centroidal axis minus a rotation, which is due to the transverse shear deformation

$$\theta_{(x)} = \frac{d\bar{w}(x,z)}{dx} - \beta$$

(8)

where:

$\bar{w}$ is the lateral displacement at any point (x,z) and equals to the lateral displacement at the centroidal axis.

According to Equation (7), the axial strain in at any layer is:

$$\varepsilon_x = \frac{dN1}{dx}u_1 + \frac{\overline{N}1}{R}w1 + \frac{\overline{N}2}{R}\theta 1 - Z\left[\frac{d\,\overline{\overline{N}}1}{dx}w_1 + \frac{d\overline{\overline{N}}1}{dx}\theta_1 + \frac{1}{R}\frac{d\,N1}{dx}u1\right]$$

$$+\frac{dN2}{dx}u_2 + \frac{\overline{N}3}{R}w2 + \frac{\overline{N}4}{R}\theta 2 - z\left[\frac{d\overline{\overline{N}}3}{dx}.w_2 + \frac{d\,\overline{\overline{N}}4}{dx}\theta_2 + \frac{1}{R}\frac{d\,N2}{dx}u2\right] \qquad (9)$$

In a matrix form;

$$\varepsilon_x = [K1 \quad K2 \quad K3 \quad K4 \quad K5 \quad K6]\,[\psi]$$

(10)

where:

K1=1/L(Z/R-1)
K2=1/R(1-F(12gx+3Lx^2-2x^3))- ZF(12x-6L)
K3=1/R(F((L^2+6g)Lx-(2L^2+6g)x^2+Lx^3))- ZF(6Lx-(4L^2+12G))
K_4 =-K1
K_5 = 1/R (F(12gx+3Lx^2-2x^3))- ZF(6L -12x)
K_6 = 1/R (F(-6gLx+(6g-L^2)x^2+Lx^3)- ZF[6Lx-(2L^2 –12g)] (11)

The shear strain γ_{xz} is then:

$$\gamma_{xz} = \frac{dw_{(x)}}{dx} - \theta_{(x)} - \frac{u}{R}$$

12)

$$\gamma_{xz} = \begin{bmatrix} w_1.\frac{d\,\overline{N}_1}{dx} - w_1.\overline{\overline{N}}_1 + \theta_1.\frac{d\,\overline{N}_2}{dx} - \theta_1.\overline{\overline{N}}_2 + w_2.\frac{d\,\overline{N}_3}{dx} - w_2.\overline{\overline{N}}_3 \\ +\theta_2.\frac{d\overline{N}_4}{dx} - \theta_2.\overline{\overline{N}}_4 - \frac{N1}{R}u1 - \frac{N2}{R}u2 \end{bmatrix}$$

(13)
or in other form:

$$\gamma_{xz} = \left[\left(\frac{x}{L}-1\right)\frac{1}{R} \quad -ks \quad -\frac{1}{2}ksL \quad \frac{-x}{RL} \quad ks \quad -\frac{1}{2}ksL\right][\psi]$$

(14)
Ks = 12Fg
(15)
Hence, the strain displacement matrix can be written as:

$$[\varepsilon] = [B][\varphi]$$

(16)
where:

$$[B] = \begin{vmatrix} K1 & K2 & K3 & K4 & K5 & K6 \\ \left(\frac{x}{L}-1\right)\frac{1}{R} & -ks & -\frac{1}{2}ksL & \frac{-x}{RL} & ks & -\frac{1}{2}ksL \end{vmatrix}$$

(17)

Stiffness Matrix Evaluation

The element stiffness matrix can be written as:

$$\mathbf{k} = \int_{\mathbf{v}} [\mathbf{B}]^{\mathrm{T}} [\mathbf{D}][\mathbf{B}]\mathbf{dv}$$

(18)
and for one dimensional stress problems

$$\mathbf{k} = \int_{\mathbf{o}}^{\mathbf{L}} [\mathbf{B}]^{\mathrm{T}} [\mathbf{D}][\mathbf{B}]\mathbf{b.h.dx}$$

(19)
According to the first assumption, each element will be subdivided into a chosen number of layers in z direction as shown in Figure (2). The stiffness matrix for element *k* can be obtained as

$$K=\int_{o}^{L}\sum_{i=1}^{N_c}[B_i]^T[D_{ci}][B_i]b_i.t_i.dx$$

(20)

where:

$$[D_i]=\begin{bmatrix} E_i & 0 \\ & \\ 0 & G_i \end{bmatrix}$$

(21)

Then the total stiffness matrix will be as follows:

$$K=\int_{0}^{L}\sum_{i=1}^{N_C}b_i\,t_i\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{51} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix}dx$$

(21)

After completing the summation for an element the element of stiffness matrix will be as:

$$K=\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{51} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix}$$

(22)

The element are shown in appendix A

Applications

In order to verify the efficiency and the accuracy of the stiffness matrix of arch members, two different examples with different supports and load conditions were analyzed. These examples are:

Curved Cantilever

The end displacements of curved cantilever shown in Figure (3) were obtained by Just [9] using an exact circularly curved beam element derived by him. In this element, the strain energy contributed by axial and flexural actions was considered. Three subtended angles of the curved cantilever were investigated. The beam was analyzed by using the present curved element and also by approximating it into a various numbers of equal straight segments. The results of analyses are listed in table (1). Through the comparison of these results some notes can be stated as follows:

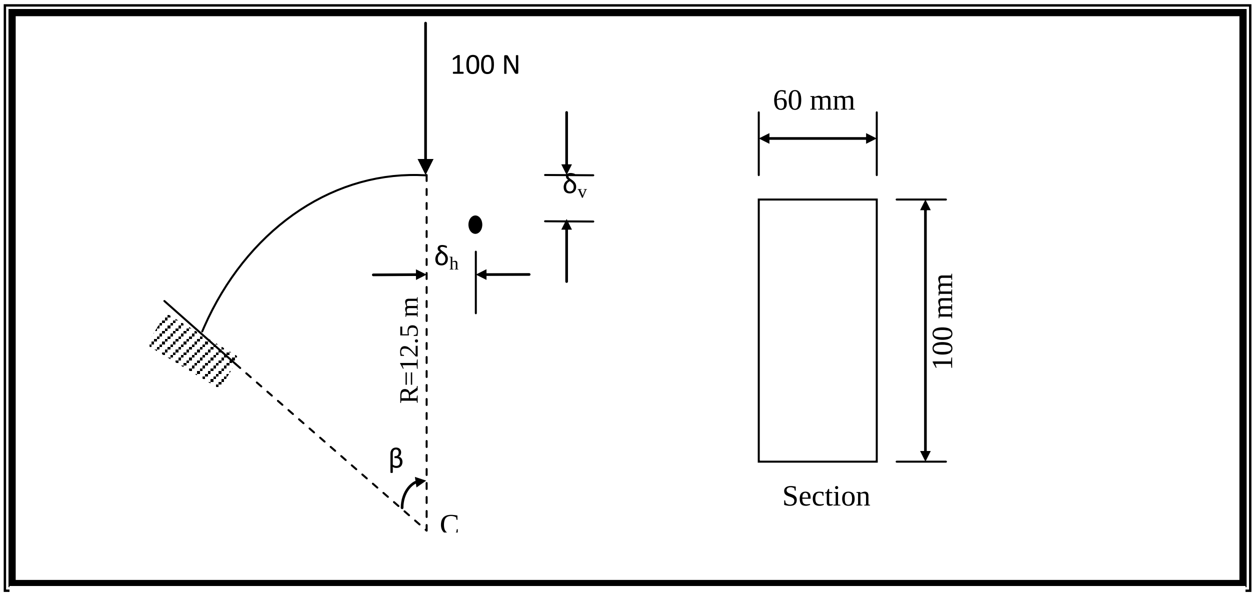


Figure (3) Data of Curved Cantilever Beam

Table (1) Results of Curved Cantilever Beam

Defl. (mm)	β	Number of Straight Segments			Exact Rf.[8]	present study
		1	2	4		
δv δh	90°	94.28 94.28	138.13 97.67	152.11 99.37	157.08 100.00	150.32 96.08
δv δh	60°	50.00 28.87	58.08 25.88	60.55 25.22	61.42 25.00	57.77 23.46
δv δh	30°	8.63 2.31	8.93 1.92	9.03 1.83	9.06 1.79	8.01 1.63

From table (1) a good agreement could be noticed between the results obtained from the suggested element and that obtained form of the proposed model of analyses, also no significant differences appear through the inclusion of the effect of shear deformations to the curved beam element derived in the present study. This can be attributed to the use of very small cross-sectional depth with respect to radius of curvature.

Thick Arch Beam

A thick arch with the properties listed in Figure (2) was analyzed using exact curved beam element stiffness matrix which was derived by Litewka and Rakowski[8]. The arch was divided into four elements. Different boundary and loading conditions were investigated. Effects of shear and membrane deformations were studies. Also, this arch is analyzed using the curved beam element derived herein by dividing it into four elements.

Radius of Curvature (R) =4 m : Subtended angle of the arch (β) =2π/3

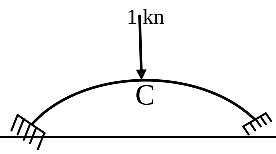
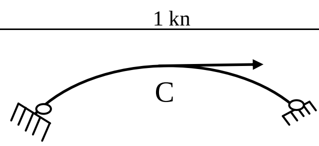
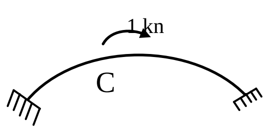
Length of the arch (L) =8π/3 m

Young's modulus (E) = 30 GPA : Poisson's Ratio (ν) = 0.17

Rectangular cross section with depth (h) =0.6 m, and width (b) =0.4 m

Figure (4) Data of Thick Arch Beam

Table (2) Results of Thick Arch Beam

Arch	Displ. ($m\times10^{-6}$)	With shear effect		Without shear effect	
		Ref[8]	Present study	Ref[8]	Present study
	$\bar{u}_c = u_c / l$ $\bar{v}_c = \bar{v}_c / l$ $\bar{\theta}_c = \theta_c / l$	0.0000 0.2488 0.0000	0.0000 0.2411 0.0000	0.0000 0.2205 0.0000	0.0000 0.2191 0.0000
	$\bar{u}_c = u_c / l$ $\bar{v}_c = \bar{v}_c / l$ $\bar{\theta}_c = \theta_c / l$	0.1252 0.0000 0.3796	0.1242 0.0000 0.3790	0.1136 0.0000 0.3638	0.1125 0.0000 0.3629
	$\bar{u}_c = u_c / l$ $\bar{v}_c = \bar{v}_c / l$ $\bar{\theta}_c = \theta_c / l$	-0.0949 0.0000 -1.0824	-0.0942 0.0000 -1.0818	-0.0921 0.0000 -1.0375	-0.0917 0.0000 -1.0366
	$\bar{u}_c = u_c / l$ $\bar{v}_c = \bar{v}_c / l$ $\bar{\theta}_c = \theta_c / l$	0.0000 0.3047 0.0000	0.0000 0.2801 0.0000	0.0000 0.2546 0.0000	0.0000 0.2535 0.0000
		0.2884 0.0000 0.8064	0.2881 0.0000 0.8058	0.2770 0.0000 0.8122	0.2770 0.0000 0.8118
	$\bar{u}_c = u_c / l$ $\bar{v}_c = \bar{v}_c / l$ $\bar{\theta}_c = \theta_c / l$	-0.2016 0.0000 -1.3613	-0.2015 0.0000 -1.3612	-0.2030 0.0000 -1.3383	-0.2029 0.0000 -1.3382
	$\bar{u}_c = u_c / l$ $\bar{v}_c = \bar{v}_c / l$ $\bar{\theta}_c = \theta_c / l$	0.0000 0.1180 0.0000	0.0000 0.1328 0.0000	0.0000 0.1190 0.0000	0.0000 0.1249 0.0000

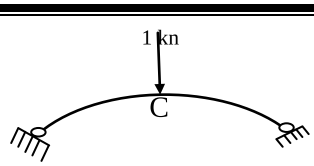


Table (2) presents the results of the calculation of the normalized displacement with respect to the length of beam $\left(\bar{u}_c = u_c / l, \bar{v}_c = \bar{v}_c / l, \bar{\theta}_c = \theta_c / l\right)$ of the arch midpoint ***C***.

This table shows that the influences of shear deformations are varied as the load and support conditions are changed. The maximum effect appears in the clamped- clamped arch with vertical loading, the maximum difference is about (10%).While, for all cases, neglecting shear deformations does not affect the rotational displacement significantly. Maximum error is (4 %) for the case clamped-clamped arch loaded by the moment .as a result a good agreement between two methods is found which indicates the validity of the proposed method of analysis.

Conclusions

A new stiffness matrix of arch beam with layered section was formulated using the finite element method. The shear effect was taken into account. Two different examples with different supports and load conditions were analyzed using the present element and compared with the exact solution carried out by Just [9] and Litewka and Rakowski [8]. A good agreement was found between the present study and the exact solutions. The maximum difference in deflection was found to be (9%). Also the influence of shear deformations was found to be significant in the thick arch example with a range of (10%).

References

1.Martin, H.C., “Introduction to Matrix Method of Structural Analysis”, McGraw-Hill Company, NewYork, 1966.

2.Morris, D.L., Curved Beam Stiffness Coefficients, Journal of the structural Division, Vol.94, no.st5, 1968, pp.1165-1174.

3.Ashwell, D.G., and Sabir, A.B., “Limitation of Certain Curved Finite Element When applied to Arches”, International Journal of Mechanical Sciences, Vol.13, 1971, pp.133-139.

4.Ashwell, D.G., Sabir, A.B. and Roberts, T.M., “Further Studies in The Application of Curved Finite Element to the Circular Arches”, International Journal of Mechanical Sciences, Vol.13-1971-pp.507-517.

5.Dawe, D.J., “Curved Finite Element for the Analysis of Shallow and Deep Arches”, Journal of Computers and Structures.Vol.4, 1974, pp.559-580.

6.Dawe, D.J., “Numerical Studies Using Circular Arch Finite Elements”, Journal of Computers And Structures.Vol.4, 1974, pp.729-740.

7.Yamada, Y. and Ezawa, Y., On Curved Finite Element For The Analysis of Circular Arches, International Journal for Numerical Methods in Engineering,Vol.11-1977-pp.1635-1651.

8.Litewka, p. And Rakowski, j.,” The Exact Thick Arch Finite Element”, Journal of Computer And Structures, Vol.68, 1998, pp.369-379.

9.Just, D.J., “Circularly Curved Beams Under Plane Loads”, Journal of The Structural Division, Vol.1108, no.st8, August, 1982, pp.1858-1873.

10. Aldaami, H.H., “Linear and Nonlinear Static and Free Vibration Analysis of Thin-Walled Cellular and Ribbed Spherical Domes by Spherical Grillage Analogy”,Ph.D. Thesis, Department of Civil Engineering, University of Baghdad-Iraq, 2000.

11. Muhaisin M. H. ,”Influence of Moving Load Upon Deformations of Reinforced Concrete Frames”, M.Sc. Thesis, Department of Civil Engineering, University of Babylon-Iraq, 2003.

12. Hadi, W.K., Elastic Plastic Analysis of Reinforced Concrete Shallow Arched Frames Using Curved Beam Element, M.Sc. Thesis, Department of Civil Engineering, University of Babylon-Iraq, 2002.

Appendix A

Stiffness Matrix Elements	
K11	$E/L(I/R^2+A)+GAL/(3R^2)$
K12	$-EA/R(1-F(6G_{SH}L+L/2))$
K13	$EI\ F/R(L^2-12G_{SH})-EA\ F\ L^2/R(L^2/12+G_{SH})+GAK_SL^2/(4R)$
K14	$-E/L(I/R^2-A)+GA\ K_S\ L^2/(4R)$
K15	$-EA\ F\ L/R(6\ G_{SH}+L^2/2)+AG\ K_SL/(2R)$
K16	$-EI\ F/R(L^2-12\ G_{SH})+EA\ F/R(4\ G_{SH}\ L^2-L^4/12)+GA\ K_SL^2/(4R)$
K22	$EA\ F^3\ L^3/R^2(48\ G_{SH}^2+42/5\ G_{SH}\ L^2+13/35L^4)+EAL/R^2(1-F(12\ G_{SH}\ L+L^3))+12EI\ F^2\ L^3+GA\ K_S^2\ L$
K23	$EA/R2(F\ L^3(L^2/12+G_{SH})+F^2(-6\ G_{SH}^2\ L^4+0.1\ G_{SH}\ L^6+367/420L^8))+EI\ F^2(6L^4)+GAL^2\ K_S^2/2$
K24	$-EA/R(1-F(6\ G_{SH}\ L+L^3/2))+GAL^2\ K_S^2/(RL)$
K25	$EA/R^2(F(6\ G_{SH}\ L^2+L^4/12)-F^2\ L^3(48\ G_{SH}^2+42/5\ G_{SH}\ L^2+13/35L^4))-EI\ F^2\ 12L^3-GA\ K_S^2\ L$
K26	$EA\ F/R^2(F(6\ G_{SH}^2\ L^4+11/10\ G_{SH}\ L^6+11/210L^8)-L^3(G_{SH}+L^2/12))+EI\ F^2\ 6L^4+GA\ K_S^2\ L^2/2$
K33	$-EAF^2/R^2(4L^9/15+7/5\ G_{SH}\ L^7+18/5\ G_{SH}^2\ L^5)+EIF^2\ L(2L^2-12\ G_{SH})^2+GK_S^2\ L^3\ A/4$
K34	$EAFL^2/R(L^2/12.0+G_{SH})-EIF^2/R(L^2+12\ G_{SH})+GAK_SL^2/(4R)$
K35	$EA\ F^2\ L^4/R^2(6\ G_{SH}^2+9/10\ G_{SH}\ L^2+13/420L^4)-EI\ F^2\ 6L^4+GA\ K_SL^2/2$
K36	$EA\ F^2/R^2(24/5\ G_{SH}^2\ L^5-11/30\ G_{SH}\ L^7-113/140L^9)+EI\ F^2\ L(2L^4-24\ G_{SH}\ L^2-144\ G_{SH}^2)+GA\ K_S^2\ L^3/4$
K44	$E/L(A+I/R^2)+GAL/(3R^2)$
K45	$EA\ F\ L/R(6\ G_{SH}+L^2/2)$
K46	$-EA\ F\ L^2/R(G_{SH}+L^2/12)+EI\ F/R(L^2+12\ G_{SH})+GA\ K_SL^2/(4R)$
K55	$EA\ F^2\ L^3/R^2(48\ G_{SH}^2+42/5\ G_{SH}\ L^2+13/35L^4)+EI\ F^2\ 12L^3+GA\ K_S^2\ L$
K56	$EA\ F^2\ L^4/R^2(6\ G_{SH}^2+11/10\ G_{SH}\ L^2+11/210L^4)-EI\ F^2\ 6L^4-GA\ K_S^2\ L^2/2$
K66	$EAF^2\ L^5/R^2(6\ G_{SH}^2/5+G_{SH}\ L^2/5+L^4/105)+EIF^2\ L(4L^4+24\ G_{SH}\ L^2+144\ G_{SH}^2)+GAK_S^2\ L^3/4$