ON BINDING EXTENSION

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Abstract :

In this paper , we show that: there exist two integers u,v such that , for every relation R with cardinality greater than or equal to u , there exist v elements of the base, such that the restriction of R to its base with these v elements removed respects the embedding inequalities in the B_i's (B_i's be a finite relations), and has an extension of arbitrary large cardinality not respecting the non-embedding inqualities in the A_i's where A_1 ,... A_h be a finite set of finite relations with common arity .

الخلاصة :

في هذا البحث و جدت بأنه لعددين مثل u , v بحيث ان كل علاقة مثل R ذات عدد عناصر اكبر او يساوي u, يوجد عدد من عناصر R يساوي v بحيث ان اي تقصير للعلاقة R وذلك بحذف v من عناصرها و التي يحقق الشروط التالية : ان هذا التقصير يضم كل من العلاقات $B_1,...,B_k$ (B_i's علاقات محدودة العناصر , بدرجة واحدة مشتركة .) فأن التمديد (ذو عدد كبير من العناصر)لهذا التقصير لا يحقق الشرط: $R \not\geq A_1$ and ... and $\not\geq A_h$ حيث ان كل من ($A_1,...,A_h$ تكون مجموعة محدودة من علاقات محدودة العناصر بدرجة مشتركة.)

1. Introduction

Relation theory originates in the theory of order types , relation theory just extended to arbitrary relations the elementary notions of order type and embeddablity, in relation theory one considers equally the two truth values (+) and (-) taken on by a relation with base E for each element of E^2(or of E^n for the arity n) .We study here the problem , not yet solved , due to R.P.Dilworth [2] on the binding extension, the problem :(Does there exist two integers u , v such that , for every R with cardinality greater than or equal to u , there exist v elements of the base, such that the restriction of R to its base with these v elements removed respects the embedding inequalities in the B_i's (B_i's be a finite relations) and has an extension of arbitrary large cardinality respecting the non-embedding inqualities in the A_i's where A_1 ,... A_h be a finite set of finite relations with common arity .)

2 Definitions

2.1 let E be a set and n an integer , An-ary relation with base E is a function R which associates the value $R(x_1,...,x_n)$ =+ or – each and n an integer n-tuple $x_1,...,x_n$ in E , the integer will be called the arity of R . For n=2 R will be called the binary relation. **(**E.W.Miller)

2.2 A multirelation with base E is a finite sequence R of relations $R_1,\dots,R_h$, (h integer) each with base E .(R.P.Dilworth**)**

2.3 Restriction: let R be a relation with base E , and let F be a subset of E , we call the restriction of R to F , the relation taking the same value for each n-tuple with values in F.(S.Ginsburg)

2.4 Extension : Given a relation R with base E and superset E^+ of E , the extension of R to E^+ any relation with base E^+ whose restriction to E is R .

And is called finite (infinite) denumerable according to whether its base is finite , (infinite) , denumerable .(E. Faran)

2.5 Let R, S be two relations, we say that R is embeddable in S if and only if there exists a restriction of S isomorphic with R and write $S \geq R$. (E. Faran)

2.6 Let R, S be two relations and S does not admit an embedding of R, then there exists an extension T of S such that $T \ngeq R$ called a binding extension.(R.P.Dilworth)

2.7 A usual chain is a partial ordering whose elements are mutually comparable, for example Q is the chain of the rationals.(M.Aigner)

3. The main results:

Proposition3.1

Let $A_1,...,A_h$ be a finite set of finite relations with common arity, and R be a finite relation with $R \ngeq A_1$ and ... and $\ngeq A_h$ If there exist extensions of R with arbitrary large finite cardinalities, which are $\ngeq A_1$ and ... and $\ngeq A_h$, then there exists a denumerable extension of R which respects the same conditions. Proof: We can assume that R is defined on the integers l,...,p and that, for each integer i , there exists an extension R_i of R based on the integers 1 to p+i and respecting the conditions. For infinitely many integers i , the R_i have a same restriction S_1 to l,...,p+l . Among these, there are infinitely many integers i for which the R_i have a same restriction S_2 to l,...,p+2 . Iterating this, we thus define S_j for each integer j . It now suffices to take the common extension of the S_j , based on the set of all integers.

propositin3.2

Let $A_1,...,A_h$ and $B_{1},...,B_k$ be two finite sets of finite relations of common arity, and let R satisfy $R \ngeq A_1$ and ... and $\ngeq A_h$ as well as $R \geq B_1$ and ... and $\geq B_K$.
Then there exists an integer u such that every R with cardinality at least equal to u satisfying the preceding conditions has a restriction R' respecting the same conditions, and such that R' has a denumerable extension still respecting the conditions.

Proof: Let v be the sum of the cardinalities of the relations B_1 through B_k . For each R satisfying the conditions, there exists a restriction R' of R with cardinality at most equal to v , which satisfies the conditions. Consider all these R' , which are only finitely many, up to isomorphism. For each, either there exists a denumerable extension satisfying the conditions. Or there exists an integer u(R') which is strictly greater than the cardinalities of all extensions of R' respecting the conditions. Then it suffices to set u to be the maximum of these u(R') .

Theorem 3.3

There is no an integer u such that, if R has cardinality greater than or equal to u and satisfies the conditions ($R \geq B_1$ and ... and $\geq B_K$ as well as $R \ngeq A_1$ and ... and $\ngeq A_h$) then there exists a denumerable extension of R satisfying them.

Proof: Take the base of integers from 0 to n-1 . Let I_n be the usual chain of these integers; let C_n be the consecutivity relation (y = x+1) ; let 0_n be the unary relation called the singleton of zero, i.e. the relation taking (+) for 0 and (-) elsewhere; and let U_n be the relation singleton of n-1 . Finally let R_n be the quadrirelation (I_n ,C_n,0_n ,U_n) . From n = 7 on, all the R_n have the same restrictions of cardinalities 1, 2 and 3 , up to isomorphism. Let $A_1,...,A_h$ be those quadrirelations of the same arity and cardinalities 1, 2, 3 which are not embeddable in R_7, , and hence in R_n ($n \geq 7$) . We see that every extension of an R_n to a new element added to its base admits an embedding of one of the $A_1 ..., A_h$ ⊟

Theorem 3.4

Given the finite relations A_1 ,..., A_h and B_1,..., B_k then there is not two integers u, v such that, for every R with cardinality greater than or equal to u , there exist v elements of the base, such that the restriction of R to its base with these v elements removed respects the embedding inequalities in the B_i's and has an extension of arbitrary large cardinality respecting the non-embedding inequalities in the A_i's.

Proof: For the base, take the set of points, or ordered pairs of integers called the abscissa and ordinate, and which vary from 0 to n-1 . Let R_n be the multirelation on this base, which is composed of the following 4 unary relations and 6 binary relations. The unary relation 0_n takes the value (+) for the points with abscissa 0 . The relation U_n takes (+) for the points with abscissa n-1 . Similarly $0'_n$ and U'_n are defined by interchanging the abscissas and ordinates. The stratified partial ordering I_n takes the value (+) for each ordered pair of points (i,x), (j,y) whose abscissas satisfy $i < j < n$, with arbitrary ordinates x, y ; moreover I_n is reflexive. The equivalence relation E_n takes (+) for any two points with a same abscissa and arbitrary ordinates. The equivalence classes of this relation are thus the classes of elements which are pairwise incomparable modulo I_n . The binary relation C_n , which by abuse of notation we shall call a consecutivity, takes the value (+) for each ordered pair of points (i,x), (i+l,y) whose abscissas are consecutive. Finally, the stratified partial ordering I'_n , the equivalence relation E'_n and the consecutivity C'_n are obtained from the preceding by interchanging abscissas and ordinates.

From n = 7 on, every R_n has the same restrictions $B_1,...,B_k$ with cardinalities 1, 2, 3 (up to isomorphism). Let $A_1,...,A_h$ be the other multirelations of the same arity and cardinalities 1, 2, 3 . We see that every proper extension of R_n ($n \geq 7$) admits an embedding of at least one of the A_i's . Indeed, add a new element t to the base of R_n.Consider the case where either 0_n or U_n or $0'_n$ or U'_n takes the value (+) for t Now consider the case where all the preceding unary relations take the value (-) for t. Then either there exists an equivalence class of E to which t belongs: Or t occurs between two consecutive equivalence classes of E_n . In this case, use the consecutivity C_n to see that the extension of R_n thus obtained admits an embedding of one of the A_i's . Now suppose the existence of u and v satisfying our hypothesis; take n >u and >v . Let S_n be a restriction of R_n in which the B_i's are embeddable, and which is obtained by removing v points. Then in each equivalence class of E_n , there remains at least one element of $|S_n|$; similarly for E'_n . Add a new element t to the base of S_n , and attempt to require that the extension of S_n to its base with t added admit only embeddings of the B_i's and not of the A_i's . This leads us to situate t in the chain of the equivalence classes of E_n . By using C_n , one sees that t necessarily belongs to one of the equivalence classes: t cannot be situated between two consecutive classes. Thus we obtain an element in the base of S_n , which is equivalent with t (mod E_n) , and another element equivalent with t (mod E'_n) . From this, we deduce that t is the unique element common to both equivalence classes. Thus we have again a restriction of R_n obtained by removing v-1 points: this is our extension of S_n . Iterating this, we obtain R_n itself, and at the following step we obtain a proper extension of R_n , in which necessarily one of the A_i's is embeddable.

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