Bayesian Fixed Sample Size Procedures for Selecting the Better of Two Exponential Populations With General Loss Function

أجراءات بيزينية بحجم عينة ثابت مع دالة خسارة مشتركة
لأختيار افضل مجتمع من بين مجتمعيين آسيين

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Abstract

In this paper an optimal (Bayesian) fixed sample size procedure for selecting the better of two Exponential populations is proposed and studied . Bayesian decision-theoretic approach with general loss function and Gamma priors are used to construct this procedure . A suboptimal procedure that is based on posterior estimate of the parameters and a method of obtaining an approximation to the optimal procedure using stirling's formula are also presented. Comparisons among these procedures are made using performance characteristics such as Bayes risk .

الخلاصة:

يتضمن هذا البحث طريقة ذات حجم عينة ثابت لاختيار أفضل مجتمع من بين مجتمعيين آسيين . وقد استخدم منهج القرار البيزيني مع دالة خسارة مشتركة لبناء هذه الطريقة . واحتوى البحث أيضا على طرق مثلى جزئيا احدها يعتمد على تقدير بعدي للمعالم والأخرى قاعدة اختيار تقريبية للمشكلة وباستعمال تقريب ستيرلنك للمفكوك الكبير . وتضمن البحث أيضا مقارنات بين هذه الإجراءات باستخدام خصائص انجاز مثل الخطورة البيزينية .

1.Introduction

Suppose that $\Pi_i, (i = 1,2)$ are two Exponential populations . The quality of the ith population is characterized by a positive real-valued parameter λ_i . The problem is to select the better of these Exponential populations on the basis of affixed number of observations N which is partitioned into n_1 and n_2 (not necessarily are equal) , the number of observations taken from populations Π_1 and Π_2 respectively . The ranked mean rates are denoted by $\lambda_{[1]} \le \lambda_{[2]}$, moreover we don't know which population is associated with $\lambda_{[2]}$. Our goal is to design fixed sample size selection procedures that enable us to select the population associated with $\lambda_{[2]}$, thus we have two-decision problem .

The following experimental conditions should be met

1.The observations produced by each population are independent each other .

2. λ_1 and λ_2 are constants during the experiment .

Many authors have considered the problem of selecting the largest population such as Gupta and Liang (1999) proposed a procedure to selecting good exponential populations compared with a control : a nonparametric empirical Bayes approach . Gupta and Liang (2001) proposed Bayes selection rules for the best exponential population with type-I censored data . Some contributions such as Nelson and Hung (2003) presented an indifference zone selection procedure which is sequential and has minimum number of switches . Nelson and Pichitlamken (2001) propose fully sequential indifference zone selection procedures . Some contribtion using Bayesian approach have been made by Mahi (1986) who presented Bayesian sequential procedures for Binomial and Multinomial selection problem . Chick

(1997) . Chen (1995) and Chen et al. (1996) have formulated the R&S problem as multi-stage optimization problem .

This paper is organized as follows .
In section 2 we present an optimal (Bayesian) fixed sample size scheme for selecting the better of two exponential populations using Gamma priors with general loss function . The two decision exponential selection formulation is given in subsection 2.1 . Subsection 2.2 contains the Bayesian selection procedure (opm). In subsection 2.3 we drive the posterior expected losses of making decision d_1 and d_2 for constant , linear and quadratic loss functions . In section 3 we present the suboptimal selection procedure (subopm1) as an approximation to opm procedure . In section 4 we describe the suboptimal selection procedure (subopm2) based on posterior Bayes estimate of the parameter . Comparisons of the schemes using posterior expected loss under constant loss function is given in section 5 .

2.Bayesian (optimal)selection procedure(opm)

2.1 The Bayesian Decision-Theoretic Formulation

Let Π_1 and Π_2 be two Exponential populations with unknown parameters λ_1 and λ_2 respectively , and consider the following two-decision problem with decisions

d_1: $\lambda_1 > \lambda_2$

and …………..(2-1)

d_2: $\lambda_1 \le \lambda_2$

Corresponding to the two decision problem the parameter space $\Omega = \{(\lambda_1,\lambda_2): 0 \le \lambda_1 < \infty, 0 \le \lambda_2 < \infty\}$ is divided into disjoint sets : $\Omega_1 = \{(\lambda_1,\lambda_2): 0 \le \lambda_2 < \lambda_1 < \infty\}$ and $\Omega_2 = \{(\lambda_1,\lambda_2): 0 \le \lambda_1 \le \lambda_2 < \infty\}$.
To obtain an explicit Bayes rule (Bayes selection procedure) for this two decision problem we must specify loss function and prior distributions . Suppose the loss functions proposed are as follows :

$$L_1(\lambda_1,\lambda_2;d_1) = \begin{cases} 0 & \text{if } (\lambda_1,\lambda_2) \in \Omega_1 \\ k_1|\lambda_1 - \lambda_2|^r & \text{if } (\lambda_1,\lambda_2) \in \Omega_2 \end{cases} \qquad \text{…………….(2-2)}$$

$$L_2(\lambda_1,\lambda_2;d_2) = \begin{cases} k_2|\lambda_1 - \lambda_2|^r & \text{if } (\lambda_1,\lambda_2) \in \Omega_1 \\ 0 & \text{if } (\lambda_1,\lambda_2) \in \Omega_2 \end{cases} \qquad \text{…………….(2-3)}$$

Where r=0,1,2 gives the types of loss function , which are constants , linear and quadratic respectively . L_i (i=1,2) are the loss functions corresponding to decision d_i , and k_1 and k_2 are positive constants (the same for each pair of $\lambda's$) .
The Bayesian approach requires that we specify a prior density function $\Pi(\lambda_i), i = 1,2$, expressing our beliefs about λ_i before we obtain data . From a mathematical point of view , it would be very convenient if λ_i is assigned a prior distribution which is a member of the conjugate family , in this case is the family of Gamma distributions . Accordingly let λ_i ,(i=1,2) is assigned Gamma prior

distribution with parameters n_i', t_i' (Gamma(n_i', t_i')) . The normalized density function is given by :

$$\Pi(\lambda_i) = \frac{(t_i')^{n_i'}}{\Gamma(n_i')} e^{-t_i'\lambda_i} \lambda_i^{\,n_i'-1} \quad , n_i' > 0, t_i' > 0, \lambda_i > 0 \ldots\ldots\ldots\ldots.(2-4)$$

Where

n_i' : the number of independent events occurred in aunit of time .

t_i' : time .

λ_i : the mean rate of occurrence .

If $\underline{x}_i = (x_{i1}, x_{i2}, \ldots, x_{in})$, (i = 1,2) be a random sample from population Π_1 , then the likelihood function is given by

$$f(\underline{x}_i / \lambda_i) = \lambda_i^{\,n} e^{-\lambda_i \sum_{j=1}^{n} x_{ij}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2-5)$$

The posterior density function is derived from the prior function (2-4) and the assumed sampling model (2-5) by means the Bayes theorem as follows :

$$\Pi(\lambda_i / \underline{x}_i) = \frac{f(\underline{x}_i / \lambda_i)\Pi(\lambda_i)}{\int_{\lambda_i} f(\underline{x}_i / \lambda_i)\Pi(\lambda_i) d\lambda_i}$$

Where

$$\int_{\lambda_i} f(\underline{x}_i / \lambda_i)\Pi(\lambda_i) d\lambda_i = \frac{(t_i')^{n_i'}\Gamma(n + n_i')}{\Gamma(n_i')(t_i' + \sum x_{ij})^{n+n_i'}}$$

Then the posterior density function is given by

$$\Pi(\lambda_i / \underline{x}_i) = \frac{(t_i'')^{n_i''}}{\Gamma(n_i'')} \lambda_i^{\,n_i''-1} e^{-t_i''\lambda_i}$$, Where $t_i'' = t_i' + \sum_{j=1}^{n} x_{ij}$ and $n_i'' = n_i' + n$, $(i = 1,2)$

N is the number of observations taken from each population (N=2n is the total number of observations taken from each populations) .

The foregoing is all that need in order to obtain a Bayes rule (Bayes selection procedure) for the component two-decision problem .

2.2 The procedure (opm)

For the two-decision problem considered above , the Bayesian selection procedure is given as follows :

Make decision d_1 : $\lambda_1 > \lambda_2$ that is selecting Π_1 as the better population if $R_1(\lambda_1, \lambda_2; d_1) < R_2(\lambda_1, \lambda_2; d_2)$

and

Make decision d_2 : $\lambda_1 \le \lambda_2$ that is selecting Π_2 as the better population if $R_1(\lambda_1, \lambda_2; d_1) \ge R_2(\lambda_1, \lambda_2; d_2)$

Where $R_i(\lambda_1, \lambda_2; d_i)$, $(i = 1,2)$ is the posterior expected loss for the decision d_i and calculated as follows :

$R_i(\lambda_1, \lambda_2; d_i) = E_{\Pi(\lambda_1, \lambda_2 / n_1'', t_1'', n_2'' t_2'')}[L_i(\lambda_1, \lambda_2; d_i)], i = 1,2$ where $\Pi(\lambda_1, \lambda_2 / n_1'', t_1'', n_2'', t_2'')$ on the expectation sign is the joint posterior of λ_1 and λ_2 with respect to which the expectation is being performed .

The optimal posterior expected loss using opm is $R(\lambda_1,\lambda_2)=\min(R_1,R_2)$

2.3 The Posterior Expected Loss for General Loss Function

In this subsection we derive the posterior expected losses of making decision d_1 *and* d_2 for general loss function , then the posterior expected losses are defined as follows :

$$R_1(\lambda_1,\lambda_2;d_1)=k_1\left[\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\frac{\Gamma(n_2''-i+r)\Gamma(n_1''+i)}{(t_2'')^{r-i}\Gamma(n_2'')\Gamma(n_1'')(t_1'')^{i}}-\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\frac{\Gamma(n_2''-i+r)(t_1'')^{n_1''}}{(t_2'')^{r-i}\Gamma(n_2'')\Gamma(n_1'')}\sum_{j=0}^{n_2''-i+r-1}\frac{(t_2'')^{j}\Gamma(n_1''+i+j)}{j!(t_1''+t_2'')^{n_1''+i+j}}\right]$$

$$R_2(\lambda_1,\lambda_2;d_2)=k_2\left[\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\frac{\Gamma(n_1''-i+r)\Gamma(n_2''+i)}{(t_1'')^{r-i}\Gamma(n_2'')\Gamma(n_1'')(t_2'')^{i}}-\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\frac{\Gamma(n_1''-i+r)(t_2'')^{n_2''}}{(t_1'')^{r-i}\Gamma(n_2'')\Gamma(n_1'')}\sum_{j=0}^{n_2''-i+r-1}\frac{(t_1'')^{j}\Gamma(n_2''+i+j)}{j!(t_1''+t_2'')^{n_2''+i+j}}\right]$$

This section contains some numerical result about this procedure , various N and various priors. All the programs in this paper are applied by Microsoft Matlab ver 6.5 . from this numerical result we note that :

1-As N increase , the Bayes Risk decreases .

2- If we increase priors parameters the posterior expected loss decreases .

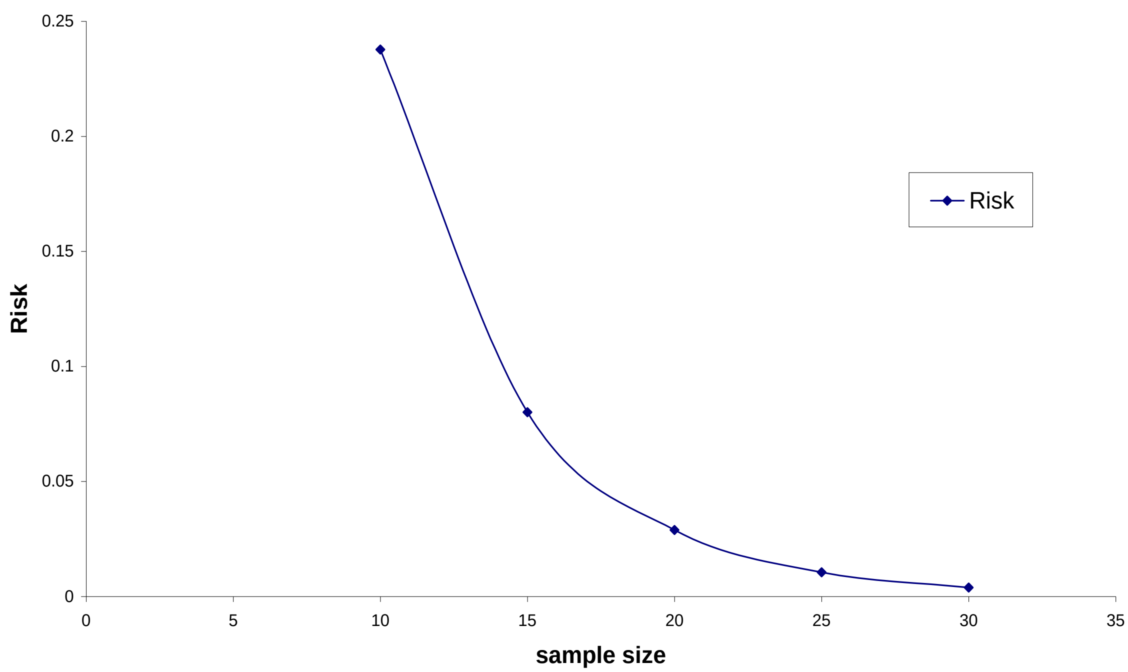

Figure (1) : The influence of the sample size on the posterior expected loss in opm procedure for constant loss function

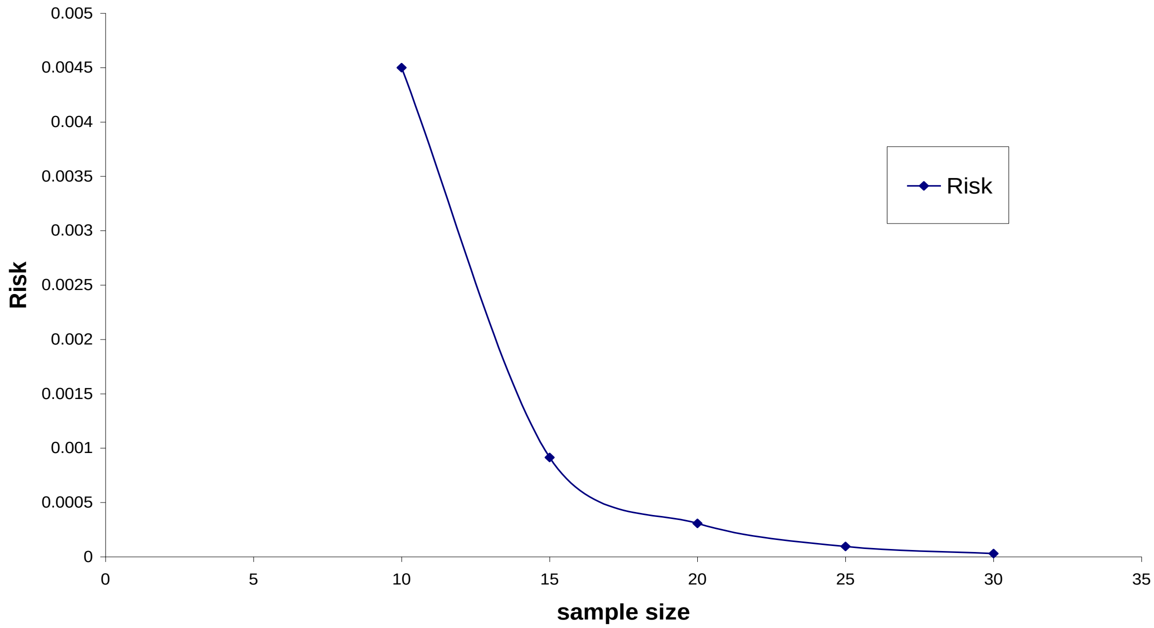

Figure (2) : The influence of the sample size on the posterior expected loss in opm procedure for linear loss function

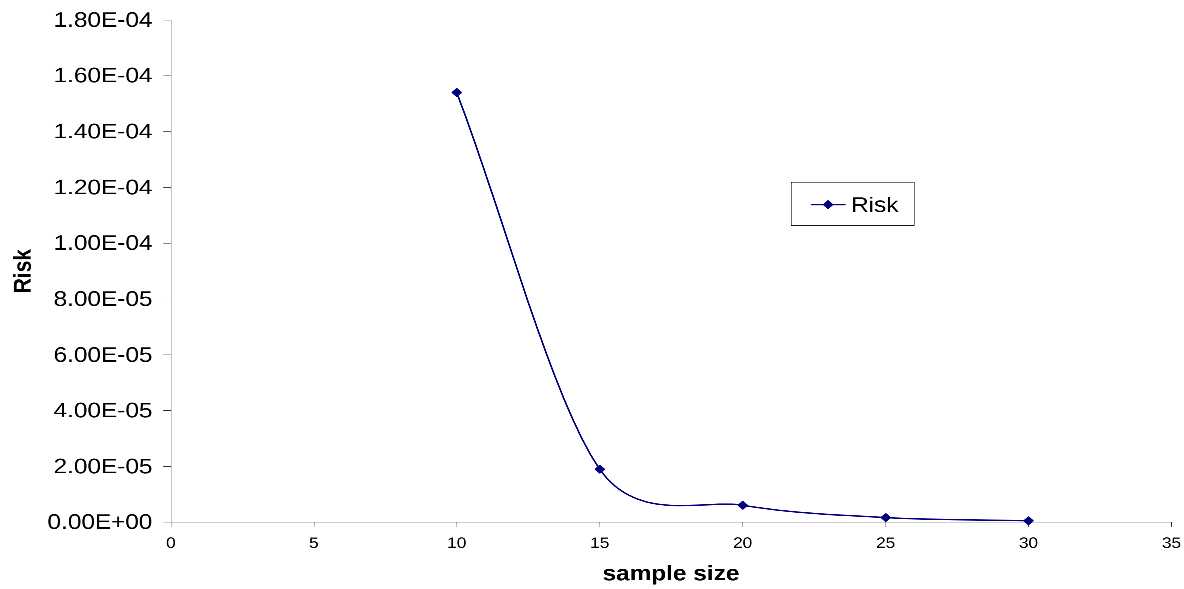

Figure (3) : The influence of the sample size on the posterior expected loss in opm procedure for quadratic loss function

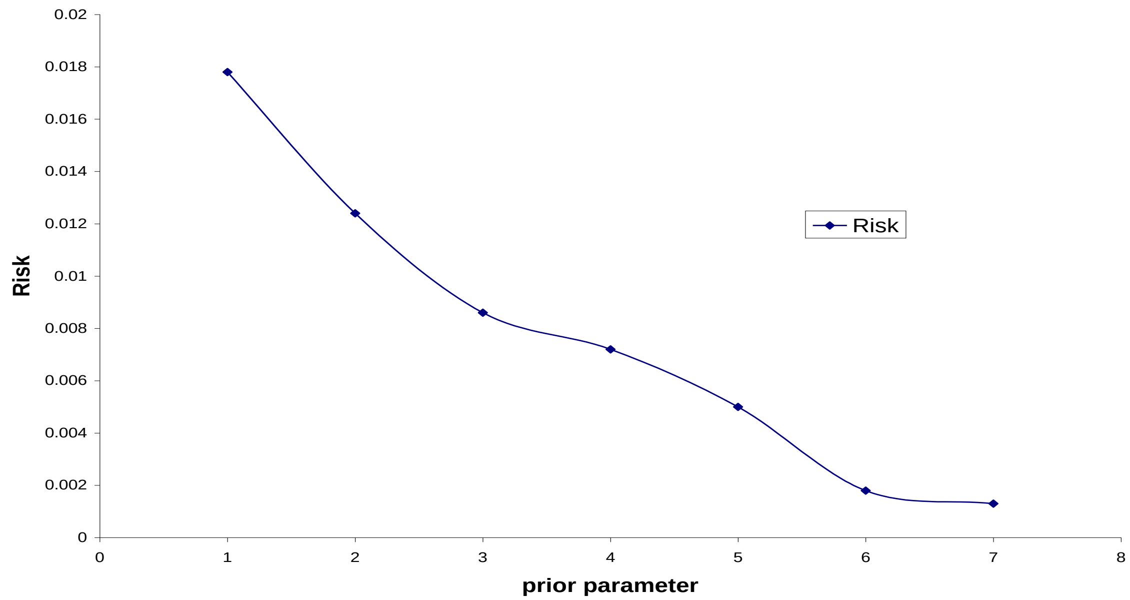

Figure (4) : The influence of the prior information on the posterior expected loss in opm procedure for constant loss function

$k_1 = 4, k_2 = 6, \underline{\lambda} = (12,5)$					
$(n_1^{'}, t_1^{'}), (n_2^{'}, t_2^{'})$	n	Risk	constant	linear	quadratic
(5,3),(2,3)	10	R(d$_1$)	0.2377	0.0045	1.5399e-004
		R(d$_2$)	5.6434	0.4823	0.0552
	20	R(d$_1$)	0.0289	3.0762e-004	6.0241e-006
		R(d$_2$)	5.9566	0.5348	0.0572
	30	R(d$_1$)	0.0039	2.9987e-005	4.3818e-007
		R(d$_2$)	5.9942	0.5850	0.0643
(7,3),(4,3)	10	R(d$_1$)	0.1575	0.0030	1.0470e-004
		R(d$_2$)	5.7638	0.5775	0.0752
	20	R(d$_1$)	0.0194	2.0741e-004	4.0961e-006
		R(d$_2$)	5.9709	0.5886	0.0681
	30	R(d$_1$)	0.0026	2.0334e-005	2.9861e-007
		R(d$_2$)	5.9961	0.6242	0.0727
(9,3),(6,3)	10	R(d$_1$)	0.1050	0.0020	7.1371e-005
		R(d$_2$)	5.8425	0.6734	0.0983
	20	R(d$_1$)	0.0131	1.4016e-004	2.7888e-006
		R(d$_2$)	5.9804	0.6424	0.0801
	30	R(d$_1$)	0.0018	1.3805e-005	2.0365e-007
		R(d$_2$)	5.9974	0.6634	0.0815

Table (I)

3- The Suboptimal Selection Procedure(subopm1)

Factorials are not very convenient for mathematical manipulation , and it is often useful to replace r! by an approximation . The most common approximation is striling's formula , namely

$$r!\approx (2\pi)^{\frac{1}{2}} r^{r+\frac{1}{2}} e^{-r} \quad ....(3\text{-}1)$$

Using stirling's formula for approximation factorials , we can obtain the approximations posterior expected loss $R_1^{*}(\lambda_1,\lambda_2;d_1)$ and $R_2^{*}(\lambda_1,\lambda_2;d_2)$ for general loss function .

The procedure subopm1 is given as follows :

make decision d_1 (select Π_1 as the better population) if $R_1^{*}(\lambda_1,\lambda_2;d_1) < R_2^{*}(\lambda_1,\lambda_2;d_2)$

and

make decision d_2 (select Π_2 as the better population) if $R^{*}{}_1(\lambda_1,\lambda_2;d_1) \geq R_2^{*}(\lambda_1,\lambda_2;d_2)$

where

$$R^{*}{}_1(\lambda_1,\lambda_2;d_1)=k_1\left[\frac{(-1)^r e^{-r}(n_2''+r-1)^{n_2''+r-\frac{1}{2}}\left((t_1''+t_2'')^{n_1''}-(t_1'')^{n_1''}\right)}{(n_2''-1)^{n_2''-\frac{1}{2}}(t_2'')^r(t_1''+t_2'')^{n_1''}}+\sum_{i=1}^{r}\frac{r^{r+\frac{1}{2}}(-1)^{r-i}e^{-r}(n_2''-i+r-1)^{(n_2''-i+r-\frac{1}{2})}(r-i)^{-r+i-\frac{1}{2}}}{i^{i+\frac{1}{2}}(t_2'')^{r-i}(t_1'')^{i}}\cdots\right.$$

$$\frac{(n_1''+i-1)^{n_1''+i-\frac{1}{2}}\left((t_1''+t_2'')^{n_1''+i}-(t_1'')^{n_1''+i}\right)}{(2\pi)^{\frac{1}{2}}(n_1''-1)^{n_1''-\frac{1}{2}}(n_2''-1)^{n_2''-\frac{1}{2}}(t_1''+t_2'')^{n_1''+i}}-\frac{(-1)^r(t_1'')^{n_1''}e^{-r}(n_2''+r-1)^{n_2''+r-\frac{1}{2}}}{(n_1''-1)^{n_1''-\frac{1}{2}}(2\pi)^{\frac{1}{2}}(n_2''-1)^{n_2'''-\frac{1}{2}}(t_2'')^r}\sum_{j=1}^{n_2''+r-1}\frac{(t_2'')^j(n_1''+j-1)^{n_1''+\cdots}}{j^{j+\frac{1}{2}}(t_1''+t_2'')^{n_1''+j}}$$

$$\left.-\sum_{i=1}^{r}\frac{r^{r+\frac{1}{2}}(-1)^{r-i}(r-i)^{-r+i-\frac{1}{2}}(t_1'')^{n_1''}e^{-r}(n_2''-i+r-1)^{n_2''-i+r-\frac{1}{2}}}{i^{i+\frac{1}{2}}(2\pi)(n_1''-1)^{n_1''-\frac{1}{2}}(n_2''-1)^{n_2''-\frac{1}{2}}(t_2'')r-i}\sum_{j=1}^{n_2''-i+r-1}\frac{(t_2'')^j(n_1''+i+j-1)^{n_1''+i+j-\frac{1}{2}}}{j^{j+\frac{1}{2}}(t_1''+t_2'')^{n_1''+i+j}}\right]$$

$$R^{*}{}_2(\lambda_1,\lambda_2;d_2)=k_2\left[\frac{(-1)^r e^{-r}(n_1''+r-1)^{n_1''+r-\frac{1}{2}}\left((t_1''+t_2'')^{n_2''}-(t_2'')^{n_2''}\right)}{(n_1''-1)^{n_1''-\frac{1}{2}}(t_1'')^r(t_1''+t_2'')^{n_2''}}+\sum_{i=1}^{r}\frac{r^{r+\frac{1}{2}}(-1)^{r-i}e^{-r}(r-i)^{-r+i-\frac{1}{2}}(n_1''-i+r-1)^{(n_1''-i+r-\cdots}}{i^{i+\frac{1}{2}}(t_1'')^{r-i}(t_2'')^{i}}\cdots\right.$$

$$\frac{(n_2''+i-1)^{n_2''+i-\frac{1}{2}}\left((t_1''+t_2'')^{n_2''+i}-(t_2'')^{n_2''+i}\right)}{(2\pi)^{\frac{1}{2}}(n_1''-1)^{n_1''-\frac{1}{2}}(n_2''-1)^{n_2''-\frac{1}{2}}(t_1''+t_2'')^{n_2''+i}}-\frac{(-1)^r(t_2'')^{n_2''}e^{-r}(n_1''+r-1)^{n_1''+r-\frac{1}{2}}}{(n_1''-1)^{n_1''-\frac{1}{2}}(2\pi)^{\frac{1}{2}}(n_2''-1)^{n_2'''-\frac{1}{2}}(t_1'')^r}\sum_{j=1}^{n_1''+r-1}\frac{(t_1'')^j(n_2''+j-1)^{n_2''+\cdots}}{j^{j+\frac{1}{2}}(t_1''+t_2'')^{n_2''+j}}$$

$$\left.-\sum_{i=1}^{r}\frac{r^{r+\frac{1}{2}}(-1)^{r-i}(r-i)^{-r+i-\frac{1}{2}}(t_2'')^{n_2''}e^{-r}(n_1''-i+r-1)^{n_1''-i+r-\frac{1}{2}}}{i^{i+\frac{1}{2}}(2\pi)(n_1''-1)^{n_1''-\frac{1}{2}}(n_2''-1)^{n_2''-\frac{1}{2}}(t_2'')r-i}\sum_{j=1}^{n_1''-i+r-1}\frac{(t_1'')^j(n_2''+i+j-1)^{n_2''+i+j-\frac{1}{2}}}{j^{j+\frac{1}{2}}(t_1''+t_2'')^{n_2''+i+j}}\right]$$

The optimal posterior expected loss under subopm1 is $R^{*}=\min(R^{*}{}_1,R^{*}{}_2)$

4- The Suboptimal Procedure(subopm2)

A Bayesian suboptimal scheme is proposed with decision criteria based on the posterior probabilities of λ_1 and λ_2, the posterior Bayes estimator of $\lambda_i,(i=1,2)$ with respect to the gamma posterior distribution is given by $E(\lambda_i/\underline{x}_i)=\dfrac{n_i''}{t_i''}$. This is prompted by the need for a quick , easy procedure , to select the better of two poisson populations , which allow for the incorporation of information about the parameters with sampling information , ignoring the decision-theoretic structure and indifference-zone formulation . Suppose n observations are taken from each population and are assumed to be independent , this procedure is given as follows:

Select $d_1:\lambda_1>\lambda_2$ if $\hat{\lambda}_1>\hat{\lambda}_2$

and

select $d_2:\lambda_1\le\lambda_2$ if $\hat{\lambda}_1\le\hat{\lambda}_2$

for the sake of risk comparison , we use the following procedure

let $R'=R_1(\lambda_1,\lambda_2;d_1)$ if $\hat{\lambda}_1>\hat{\lambda}_2$,

$R'=R_2(\lambda_1,\lambda_2;d_2)$ if $\hat{\lambda}_1\le\hat{\lambda}_2$

so R' will be the optimal risk using subopm2

5- Comparisons and Discussion Under Posterior Expected Losses

This section contains some numerical results about the efficiency of these schemes relative to opm for the constant loss function , various N and various priors all the programs in this paper are applied by Microsoft Matlab ver 6.5 .

From table (II) we note that , the posterior expected loss for the suboptimal 1 procedure is less than the posterior expected loss for the optimal and suboptimal 2 procedures . Also it is clear from the table that as N increases , the Bayes risk decreases in all schemes, and if we increases the prior parameters the Bayes risk decreases in all schemes .

5.5- The Algorithms for the opm and subopm1 procedures

1-Specify prior parameters , $n_i'',t_i'',i=1,2$, sample size n , parameters for populations $\lambda_i>0$ and constant losses k_i .

2-Generate a random sample of size n from populations $\Pi_i,(x_{i1},x_{i2},...,x_{i_{in}})$ and find $s=\sum_{j=1}^{n}x_{ij}$ by using the function poissrnd in the Microsoft Matlab ver 6.5 .

3-Calculate the posterior parameter for populations $\Pi_i,i=1,2$, $n_i''=n_i'+n$, $t_i''=t_i'+s$

4-Find the posterior expected loss for decision d_1 and d_2 ($R_1(\lambda_1,\lambda_2;d_1)$, $R_2(\lambda_1,\lambda_2;d_2)$ for opm procedure and $R^*{}_1(\lambda_1,\lambda_2;d_1)$, $R^*{}_2(\lambda_1,\lambda_2;d_2)$ for subopm1 procedure) .

$k_1 = k_2 = 5, \underline{\lambda} = (9,6)$				
Prior prob. $(n_1', t_1'),(n_2', t_2')$	n	opm	Subopm1	Subopm2
(3,5),(3,5)	10	0.8283	0.7601	0.8283
	20	0.4613	0.4243	0.4613
	30	0.2718	0.2459	0.2718
(6,5),(6,5)	10	0.7007	0.6468	0.7007
	20	0.3951	0.3624	0.3951
	30	0.2345	0.2106	0.2345
(7,5),(7,5)	10	0.6638	0.6134	0.6638
	20	0.3755	0.3440	0.3755
	30	0.2233	0.2001	0.2233
(9,5),(9,5)	10	0.5970	0.5523	0.5970
	20	0.3395	0.3102	0.3395
	30	0.2026	0.1805	0.2026

Table (I I)
Comparisons of the schemes using Bayes risk under constant loss function for various prior and various N

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