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The Module of A-derivations

Hanan Ali Hussein
AL-Shaibany

Abstract:

In this work we have dealt with module $D(A)$ and produced the arrangement as one of its example.We have also shown some of the basic properties related to it and one of them is the module of the product of two arrangements.

الخلاصة :

في هذا العمل تناولنا الموديول D(A) وقدمنا الترتيبة $Q(A) = x_1 \dots x_\ell$ كأحد الأمثلة عليه ،كما استعرضنا بعض الخصائص الأساسية الخاصة به ومنها موديول حاصل ضرب ترتيبتين .

1-Introduction :

Let V be an ℓ-dimensional vector space ,an arrangement is a finite collection of affine subspaces (H) of codimension one .The product $Q(A) = \prod_{H\in A} \alpha_H$ is called a defining polynomial of A.Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V .If $x_1,\dots,x_\ell$ is a basis for V^* ,then $S \cong K[x_1,\dots,x_\ell]$ .Let $Der_K(S)$ be the set of K-linear maps $\theta : S \to S$ such that $\theta(fg) = f\theta(g) + g\theta(f)$ f,g $\in$ S .An element of $Der_K(S)$ is called a derivation of S over K .For $f \in S$ *and* $\theta_1,\theta_2 \in Der_K(S)$,we defined $f\theta_1 \in Der_K(S)$ and $\theta_1 + \theta_2 \in Der_K(S)$ by $f\theta_1(g) = f(\theta_1(g))$ and $(\theta_1 + \theta_2)(g) = \theta_1(g) + \theta_2(g)$ for any $g \in S$ .Any K-linear map from V^* to S can be extended uniquely to a derivation of S over K. In particular,for any $v \in V$ there exists a unique $D_v \in Der_K(S)$ such that $D_v(\alpha) = \alpha(v)$ for any $\alpha \in V^*$ .Let $e_1,\dots,e_\ell \in V$ be the dual basis of $x_1,\dots,x_\ell$ .Define $D_i = D_{ei}$, $1 \le i \le \ell$ .Then D_i is the usual derivation $\partial/\partial x_i : D_i(f) = \partial/\partial x_i$; $f \in S$. It is easy to see that $D_1,\dots,D_\ell$ is a basis for $Der_K(S)$ over S.Thus any derivation θ of S over K is expressed uniquely as $\theta = f_1D_1 + \dots + f_\ell D_\ell$; $f_1,\dots,f_\ell \in S$.It follows that $Der_K(S)$ is a free
S-module of rank ℓ .Let S_p denote the K-vector subspace of S consisting of 0 and the homogenous polynomials of degree P for P$\ge$ 0. For P<0 defin S_p=0 Then $S = \bigoplus_{P\in Z} S_p$ is a graded K-algebra .It follows that deg x=1 for x $\in$ V^* and x$\neq 0$.

2- The module D(A) :-

2.1 Definition

Ahyperplane in V is an affine subspace of dimensional $(\ell - 1)$ or it is a maximal subspace of codimension one .

2.2Examples

1) {0} is a hyperplane in Euclidean space R .
2) If $V = R^2$,then any line in V is a hyperplane .
3) If $V = R^3$, then any plane in V is a hyperplane .

2.3 Definition

An arrangement $A = (A,V)$ is a finite collection of hyperplanes in V .

2.4 Definition

A defining polynomial of an arrangement A is the product $Q(A)=\prod_{H\in A}\alpha_H$ of polynomials of degree one .

2.5 Remark

We agree that $Q(\phi_\ell)=1$ is the defining polynomial of the empty arrangement .

2.6 Definitions

An arrangement A is called

i) centerless ,if $\bigcap_{H\in A} H=\phi$.

ii) centered with center T ,if $T=\bigcap_{H\in A} H\neq\phi$.

iii) central ,if each hyperplane contains the origin .

2.7 Examples

1) Define A by $Q(A)=xy(x+y)$,then A consists of three lines through the origin

2) Define A by $Q(A)=xy(x+y-1)$,then A consists of three affine lines

3) The arrangement which is defined by $Q(A)=x_1x_2...x_\ell$ called the Boolean arrangement .

4) For $1\leq i\leq j\leq\ell$ and $H_{ij}=\ker(x_i-x_j)$.The arrangement which defined by $Q(A)=\prod_{1\leq i\leq j\leq\ell}(x_i-x_j)$ is called a Braid arrangement .

2.8 Definition

Let A be an arrangement and let $L=L(A)$ be the set of all non-empty intersections of elements of A.Define a partial order on L by $X\leq Y\Leftrightarrow Y\subseteq X$.

2.9 Remarks

1) The Poset (partially order set) $L(A)$ includes V as the intersection of the empty collection of hyperplanes.

2) If $X\in L(A)$,then $X\subseteq V$ as a subspace .

2.10 Definition

A non-zero element $\theta\in Der_K(S)$ is homogenous of polynomial degree P if $\theta=\sum_{K=1}^{\ell} f_K D_K$ and $f_K\in S_P$ for $1\leq k\leq\ell$. In this case we write $p\deg ree\theta=p$ .Note that $p\deg D_i=0$ .Let $Der_K(S)_P$ denote the vector space consisting of all homogenous elements of $p\deg reep$ for $\text{p}\geq 0$.Let $Der_K(S)_P$=0 if P<0,with this P degree function . $Der_K(S)$ is a graded S-module $Der_K(S)=\bigoplus_{P\in Z} Der_K(S)_P$ .If we view derivations as a subset of the set of K-linear endomorphism of S , then there is another natural grading of $Der_K(S)$.

2.11 Definition

For any $f\in S$,define $D(f)=\{\theta\in Der_K(S)\ /\ \theta(f)\in fS\}$.Note that D(f) is an S-submodule of $Der_K(S)$.

2.12 Definiton

Let A be an arrangement in V with defining polynomial $Q(A)=\prod_{H\in A}\alpha_H$ where $H=Ker(\alpha_H)$ .Define the module of A-derivations by $D(A)=D(Q(A))$.

Clearly, $D(A)$ does not depend on the choice of $Q(A)$. In particular $D(\phi_\ell)=Der_K(S)$ because $Q(\phi_\ell)=1$.An element of $D(A)$ is called a derivation tangent to A.This terminology is justified by the topological significance of the module $D(A)$ in case K=C.

2.13 Example

Let A be the Boolean arrangement .Then $\sum_{i=1}^{\ell} f_i D_i \in D(A).$

$$\Leftrightarrow \sum_{i=1}^{\ell} f_i(\partial(x_1...x_\ell)\big/\partial x_i)\in x_1...x_\ell S \Leftrightarrow (x_1...x_\ell)\sum_{i=1}^{\ell}(f_i/x_i)\in x_1...x_\ell S \Leftrightarrow f_i\in x_i S \quad (1\le \text{i} \le \ell)$$

This implies that $D(A)$ is a free S-module with basis $\{x_1D_1,...,x_\ell D_\ell\}$.

3.Basic Properties

In this section we assume that all arrangements are central and use "arrangement" in place of central arrangement.

3.1 Definition

The Euler derivation $\theta_E\in Der_K(S)$ is defined by $\theta_E=\sum_{i=1}^{\ell}x_iD_i$ for any homogenous $f\in S$, $\theta_E(f)=(\deg f)f$ .Thus θ_E is independent of choice of $\{x_1,...,x_\ell\}$

Taking $f=Q=Q(A)$, we get $\theta_E(Q)=|A|Q\in \text{QS}$.Thus $\theta_E\in D(A)$ for any arrangement A.

3.2 Proposition

$D(A)=\bigcap_{H\in A}D(\alpha_H)=\{\theta\in Der_K(S)\ /\ \theta(\alpha_H)\in\alpha_H S \quad \text{for all H}\in \text{A}\}.$

Proof

It is sufficient to prove

$D(f_1f_2)=D(f_1)\bigcap D(f_2)$ for any $\text{f}_1, f_2\in S$ such that f_1 and f_2 are coprime.

If $\theta\in Der_K(S)$,then

$\theta\in D(f_1f_2)$

$\Leftrightarrow \theta(f_1f_2)\in f_1f_2S$

$\Leftrightarrow f_1\theta(f_2)+f_2\theta(f_1)\in f_1f_2S$

$\Leftrightarrow \theta(f_i)\in f_iS \quad (i=1,2)$

$\Leftrightarrow \theta\in D(f_1)\bigcap D(f_2)$.

3.3 Corollary

Let A_1 and A_2 be two arrangements in V such that $A_1\subseteq A_2$ .Then $D(A_2)\subseteq D(A_1)$.

Proof

$D(A_2)=\bigcap_{H\in A_2}D(\alpha_H)=\{\theta\in Der_K(S)\ /\ \theta(\alpha_H)\in\alpha_H S \quad \text{for all H}\in \text{A}_2\}$

Let $\theta \in D(A_2)$. Then $\theta \in Der_K(S)$ / $\theta(\alpha_H) \in \alpha_H S$ for all $H \in A_2\}$
Since $A_1 \subseteq A_2$ .Then $\theta \in Der_K(S)$ / $\theta(\alpha_H) \in \alpha_H S$ for all $H \in A_1\}$
Then $\theta \in D(A_1)$. Hence $D(A_2) \subseteq D(A_1)$.

3.4 Proposition [2]

Let $D(A)_p = D(A) \bigcap Der_K(S)_P$ .Then $D(A) = \bigoplus_{P \in Z} D(A)_P$ .Thus D(A) is a graded S-submodule of $Der_K(S)$.

3.5 Definition

If $\theta \in Der_K(S). Then \theta = \sum \theta(x_i) D_i$ .Given derivations $\theta_1,...,\theta_\ell \in D(A)$,define the coefficient matrix $M(\theta_1,...,\theta_\ell)$ by $M_{ij} = \theta_j(x_i)$.Thus

$$M(\theta_1,...,\theta_\ell) = \begin{bmatrix} \theta_1(x_1)...\theta_\ell(x_1) \\ . \\ . \\ . \\ \theta_1(x_\ell)...\theta_\ell(x_\ell) \end{bmatrix}$$

And $\theta_j = \sum M_{ij} D_i$.

3.6 Proposition

If $\theta_1,...,\theta_\ell \in D(A)$,then $\det M(\theta_1,...,\theta_\ell) \in QS$.

Proof

This is clear for $A = \phi_\ell$ .Since $Q(A) = 1$

Let $H \in A$ and let $H = \ker(\alpha_H)$,where $\alpha_H = \sum_{i=1}^{\ell} c_i x_i \in V^*$.May assume that $c_i = 1$ for *some* i .Then

$$\det M(\theta_1,...,\theta_\ell) = \det \begin{bmatrix} \theta_1(x_1)...\theta_\ell(x_1) \\ . \\ . \\ . \\ \theta_1(\alpha_H)...\theta_\ell(\alpha_H) \\ . \\ . \\ . \\ \theta_1(x_\ell)...\theta_\ell(x_\ell) \end{bmatrix} \in \alpha_H S$$

Since H is arbitrary, $\det M(\theta_1,...,\theta_\ell)$ is divisible by all α_H ,and hence by Q.
as required .

Let (A_1,V_1) and (A_2,V_2) be two arrangement .let $S_i = S(V_i^*)$ for $i = 1,2$ and let $V = V_1 \oplus V_2$ .Then S_1 and S_2 may be regarded as K-subalgebras of $S = S(V^*)$ .An element $\theta \in Der(S_1)$ is uniquely extended to an element γ of Der(S) such that $\gamma / S_2 = 0$.By this extension , Der(S_i) may be regarded as a subset of Der(S) for i=1,2. The following is easy to see.

3.7 Proposition [3]

$Der(S) = SDer(S_1) \oplus SDer(S_2)$.

3.8 Proposition

For two arrangements (A_1, V_1) and (A_2, V_2) ,we have $D(A_1 \times A_2) = SD(A_1) \oplus SD(A_2)$.

Proof

Write $A = A_1 \times A_2$ .For i=1,2 , let $S_i = S(V_i^*)$ and let $Q_i \in S_i$ be defining polynomials for A_i .Then $Q_1 Q_2$ is a defining polynomial for A .Since an element of Der(S_1) annihilates every element in S_2 ,we get $SD(A_1) \subseteq D(A)$. Thus we obtain $SD(A_1) \oplus SD(A_2) \subseteq D(A)$.

By Proposition (3.7) ,any element $\theta \in D(A)$ can be written as $\theta = \theta_1 + \theta_2$ for some $\theta_i \in SDer(S_i)$ with i=1,2 .By symmetry we only have to prove $\theta_1 \in SD(A_1)$.For that purpose we may assume that $\theta = \theta_1$.

Since $\theta(Q_1 Q_2) = Q_2 \theta(Q_1) + Q_1 \theta(Q_2) = Q_2 \theta(Q_1) \in Q_1 Q_2 S$,we have $\theta(Q_1) \in Q_1 S$.

Let $G = \{g_1, g_2, ...\}$ be a K-basis for S_2 .For example,take G to be the set of all monomials.Note that G is linearly independent over S_1 also there is a unique expression

$\theta = \sum_{i \geq 1} g_i \eta_i$ with $\eta_i Der(S_1)$.There is also a unique expression

$\theta(Q_1) = Q_1 \sum_{i \geq 1} g_i h_i$ with $h_i \in S_1$.Thus we have $\sum_{i \geq 1} g_i (h_i Q_1) = \theta(Q_1) = \sum_{i \geq 1} g_i \eta_i (Q_1)$

By the uniqueness of the expression,we have for $i \geq 1$

$\eta_i (Q_1) = h_i Q_1 \in Q_1 S_1$,so $\eta_i \in D(A_1)$. Thus $\theta = \sum_i g_i \eta_i \in SD(A_1)$.

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