THE MAXIMUM NUMBER IN WHICH EVERY STRONG TOURNAMENT CONTAINS A TRANSITIVE SUBTOURNAMENTS

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ABSTRACT :
In this paper , we find the maximum number in which every strong tournament contains a Transitive subtournaments

الخلاصة :
في هذا البحث توصلت الى إيجاد الحد الأعلى لعدد من العلاقات الدوريـة الجزئيـة المتعديـة المحتـواة فـي العلاقـة الدوريـة الغير مجزأة.

1. Introduction :

Tournaments provide a model of the statistical technique , called the method of paired comparisons . This method is applied when there are a number of objects to be judged on the basis of some criterion and it is impracticable to consider them all simultaneously. The objects are compared two at a time and one member of each pair is chosen .This method and related topics are discussed in K.B Reid [4] Tournament have also been studied in connecting with sociometric relations in small groups. A survey of some of these investigations is given by R.Fraisse [2] . Our main object here to derive the maximum number where every strong tournament contains a transitive subtournaments.

2. Definitions:-

2.1 A tournament T_n consists of n nodes $p_1, p_2, \ldots\ldots, p_n$ such that each pair of distinct nodes p_i and p_j is joined by one and only one of the oriented arcs $\overrightarrow{p_i p_j}$ or $\overrightarrow{p_j p_i}$. The relation of dominance thus defined is a complete , irreflexive, antisymmetric , binary relation , every restriction of a tournament is subtournament . [2]

2.2 The score of p_i is the number s_i of nodes that p_i dominates the score vector of T_n in the ordered n - tuple ($s_1, s_2, \ldots\ldots, s_n$) .We usually assume that the nodes are labeled in such a way that $s_1 \leq s_2 \leq \ldots.. \leq s_n$ [2]

2.3 Strong tournament: For any subset X of the nodes of a tournament T_n, let

$$\Gamma(x) = \{q : p \to q \quad \text{for some } p \in X \}$$

A tournament T_n is strong if and only if for every node p of T_n the set:

$$\{p\} \cup \Gamma(p) \cup \Gamma^2(p) \cup \ldots\ldots\ldots \cup \Gamma^{n-1}(p)$$ contains every node of T_n. [2]

2.4 A tournaments is transitive if , whenever p→q and q→r , then p→r . [2]

2.5 A tournaments T_n is reducible if it is possible to partition its nodes into two nonempty sets B and A in such a way that all the nodes in B dominate all the nodes in A; the tournament is irreducible if this is not possible . [2]

3. NOTATIONS:

1) Let S(n,k) denote the minimum number of strong subtournament T_k that a strong tournament T_n can have . [2]

2) Let v = v(n) is the Largest integer which every strong tournament contains a transitive Subtournament. [2]

4. Theorems:

The following theorem gives some properties of a transitive tournaments whose scores ($s_1, s_2 \ldots, s_n$) are in nondecreasing order .

Theorem 4.1 [5] The following statements are equivalent .
(1)T_n is transitive.
(2)Node p_j dominates node p_i if and only if j>I
(3)T_n has score ($0,1,\ldots,n-1$)
(4)The score vector of T_n satisfies the equation:

$$\sum_{i=1}^{n} s_i^2 = \frac{n(n-1)(2n-1)}{6}$$

(5)T_n contains no cycles .
(6)T_n contains exactly $\binom{n}{k+1}$ paths of length K, if $1 \le k \le n-1$
(7)T_n contains exactly $\binom{n}{k}$ transitive subtournaments T_k, if $1 \le k \le n$
(8)Each principal submatrix of the dominance matrix $M(T_n)$ contains a row and column of zeros.

Every tournament T_n ($n \ge 4$) contains at least one transitive subtournament T_3,but not every tournament T_n is itself transitive .

THEOREM 4.2 [1] if $3 \le k \le n$, then

$$S(n,k) = n-k+1 .$$

THEOREM 4.3 [3] Each node of an irreducible tournament T_n in contained in some k- cycle, for k=3,4,….,n .

The main results

THEOREM 4.4 if $[\log_2 n]$ denotes the greatest integer not exceeding $\log_2 n$ then

$$[\ \log_2 n\ ] + 1 \le v(n) \le [\ 2\log_2 n\ ] + 1$$

Proof:- Consider a tournament T_n in which the node P_n has the largest score S_n. It must be that $S_n \ge [\frac{1}{2} n]$,so there certainly exists a subtournament $T[\frac{1}{2} n]$ in T_n each node of which is dominated by P_n. We may suppose that $[\log_2 [\frac{1}{2} n]] + 1$ nodes . These nodes together with P_n determine a transitive subtournament of T_n with at least

$$[\ \log_2 [\frac{1}{2} n]\ ] + 2 = [\ \log_2 n\ ] + 1$$

nodes. The lower bound now follows by induction there are $2^{\binom{n}{2}-\binom{v}{n}}$ tournaments T_n, containing a given transitive subtournament T_v, and there are $\binom{n}{v}v!$ such subtournaments T_v possible. Therefore,

$$\binom{n}{v}v!\ 2^{\binom{n}{2}-\binom{v}{n}} \ge 2^{\binom{n}{2}}$$

Since every tournament T_n contains at least one, transitive subtournament T_v, this inequality implies that $n^v \ge 2^{\binom{v}{2}}$. Consequently , $v \le [\ 2\log_2 n\ ] + 1$, and the theorem is proved.

The exact value of v(n) is known only for some small values of n. For example , $v(7) \ge 3$. By theorem 4.4 the tournament T_7, in which $P_i \to P_j$ if and only if $j \to i$ is a quadratic residue module 7 contains no transitive subtournament T_4. It follow that $v(7) = 3$. We examine other similarly constructed tournaments and we deduced the information about v(n) given in the following table .

Table *v(n)*, the largest integer v such that every tournament T_n contains a transitive subtournament T_v.

$$v(2) = v(3) = 2$$
$$v(4) = \ldots = v(7) = 3$$
$$v(8) = \ldots = v(11) = 4$$
$$4 \le v(12) \le \ldots \le v(15) \le 5$$
$$v(16) = \ldots = v(23) = 5$$
$$5 \le v(24) \le \ldots \le v(31) \le 7$$
$$6 \le v(32) \le \ldots \le v(43) \le 7$$

Let u(n,k) denote the maximum number of transitive subtournaments T_k that a strong tournament T_n can have. (The problem is trivial if T_n is not strong)

Theorem 4.5 if $3 \le k \le n$ then $u(n,k) = \binom{n}{k} - \binom{n-2}{k-2}$

Proof:- When k = 3 the theorem follows from theorem 4.2 , since every subtournament T_3 is either strong or transitive. We now show that $u(n,k) \le \binom{n}{k} - \binom{n-2}{k-2}$, for any larger fixed value of k . This inquality certainly holds when n = k , if $n > k \ge 4$, then any strong tournament T_n , contains a strong subtournament T_{n-1} by theorem 4.3. Let p be the node not in T_{n-1} , there are at most $u(n-1, k-1)$ transitive subtournaments T_k of T_n that contain the node p and at most $u(n-1, k)$ that do not , We may suppose

$$u(n-1, k-1) \le \binom{n-1}{k-1} - \binom{n-3}{k-3},$$

and

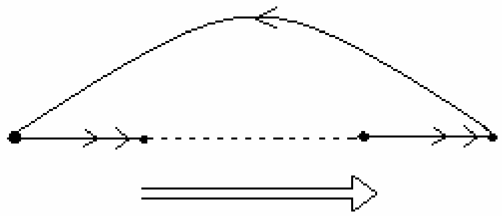
$$u(n-1, k) \le \binom{n-1}{k} - \binom{n-3}{k-2}.$$

Therefore

$$\begin{aligned} u(n,k) &\le u(n-1,k-1) + u(n-1,k) \\ &\le \binom{n-1}{k-1} + \binom{n-1}{k} - \binom{n-3}{k-3} - \binom{n-3}{k-2} \\ &= \binom{n}{k} - \binom{n-2}{k-2} \end{aligned}$$

The inquality now follows by induction :

To show that $u(n,k) \ge \binom{n}{k} - \binom{n-2}{k-2}$ consider the strong tournament T_n in which $p_1 \to p_n$ but otherwise $p_j \to p_i$ if $j > i$ (this tournament is illustrated in the following figure) this tournament has exactly $\binom{n}{k} - \binom{n-2}{k-2}$ transitive subtournament T_k if $3 \le k \le n$ because every subtournament T_k is transitive except those containing both p_1 and p_n , this completes the proof of the theorem .



Corollary 4.6 the maximum number of transitive subtournaments a strong tournament T_n ($n \geq 3$) can contain, including the trivial tournaments T_1 and T_2 , is $3 . 2^{n-2}$.
Let r (n , k) denote the minimum number of transitive subtournaments T_k a tournament T_n can have . it follows from theorem 4.4 that r (n,k) = 0
If k > [2 $\log_2$] + 1 and that r(n,k) > 0 if k ≤ [$\log_2$ n] +1 .

Theorem 4.7 let

$$\tau(n,k)=\begin{cases} n.\frac{(n-1)}{2}.\frac{(n-3)}{4}\cdots\cdots\frac{(n-2^{k-1}+1)}{2^{k-1}} & \text{if } n > 2^{k-1}-1, \\ 0 & \text{if } n \leq 2^{k-1}-1 \end{cases}$$

k denotes the number of nodes in a subtournament T_k

Then

$$r(n, k) \geq \tau(n, k)$$

Proof:- when k = 1 , the result is certainly true if we count the trivial tournament T_1 as transitive , if $k \geq 2$, then clearly

$$r(n, k) \geq \sum_{i=1}^{n} r(s_i, k-1),$$

Where (s_1 , s_2 , ….. , s_n) denote the score vector of the tournament T_n , We may suppose that r(s_i , k -1) ≥ τ (s_i , k-1); since τ (n,k) is convex function of n for fixed values of k , we may apply Jensen's inequality and conclude that:

$$r(n,k) \geq \sum_{i=1}^{n} \tau(s_i,k-1) \geq nr(1/2(n-1),k-1)=\tau(n,k).$$

The theorem now follows by induction on k .
Notice that the lower bound in theorem 4.4 follows from theorem 4.7

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