The General Solution Of the Partial Differential Equation which has The General form

$$P(x,y)\,Z_x + Q(x,y)\,Z_y + R(x,y,z)\,Z = G(x,y,z)\,Z^n \;\;, n \neq 1$$

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Abstract :

Our aim in this paper is to find the general solution of the first order linear partial differential quation which has the general form :- $P(x,y)\,Z_x + Q(x,y)\,Z_y + R(x,y,z)\,Z = G(x,y,z)\,Z^n \;,n \neq 1$ by using a new method

المستخلص :

هدفنا في هذا البحث هو أيجاد الحل العام إلى المعادلة التفاضلية الجزئية الخطية من الرتبة الأولى والتي صيغتها العامة $P(x,y)\,Z_x + Q(x,y)\,Z_y + R(x,y,z)\,Z = G(x,y,z)\,Z^n \;,n \neq 1$ باستخدام طريقة جديدة .

1- Introduction

The subject of partial differential equation is an important branch in mathematical science , since most physical phenomena whether field force fluid, electrical, mechanics, optics, heat effect can be described in general as a partial differential equations

In this work , we transform the partial differential equation

$$P(x,y)\,Z_x + Q(x,y)\,Z_y + R(x,y,z)\,Z = G(x,y,z)\,Z^n \;\;, n \neq 1 \;,$$

to the Lagrange [1] partial equation which has the form

$$P(x,y)\,t_x + Q(x,y)\,t_y = I(x,y,t)$$

by using the idea of Bernoulli [3] for solving some kinds of ordinary differential equations of the first order.

2- Lagrangian Partial Differential Equation [1]

The general form of this equation is given by

$$P(x,y,z)\,Z_x + Q(x,y,z)\,Z_y = R(x,y,z)$$

where P, Q, R are functions of x, y, z

Such equation is linear at least on partial derivatives, and it is not necessary linear on the dependent variable z .

For finding the general solution of this equation , we must to find two independent solutions, say $U = u(x,y,z) = a \quad , \quad V = v(x,y,z) = b$

from the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

So, we can set the general solution to the Lagrangian equation as follows :-

$$\Phi(U,V) = 0; \Phi = arbitrary\ function$$

3-Bernoulli equation [2]

The general form of Bernoulli equation is given by :-

$$y' + p(x)y = q(x)\,y^n \quad ; n \neq 1$$

where $p(x)$ and $q(x)$ are functions of x only , while y is the dependent variable , x is the independent variable.

For finding the solution of this equation see [2]

4- How to find the general solution of the partial differential equation which has the general form

$$P(x,y)\,Z_x + Q(x,y)\,Z_y + R(x,y,z)\,Z = G(x,y,z)\,Z^n \ , n \neq 1 ...(1)$$

Let

$$\left.\begin{aligned} t = Z^{1-n} \Rightarrow Z = (t)^{\frac{1}{1-n}} \\ t_x = (1-n)\,Z^{-n} Z_x \\ t_y = (1-n)\,Z^{-n}\,Z_y \end{aligned}\right\} ...(2)$$

By substituting equation (2) in equation (1) , follows,

$$P(x,y)t_x + Q(x,y)\,t_y + (1-n)R(x,y,t^{\frac{1}{1-n}})\,t = (1-n)G(x,y,t^{\frac{1}{1-n}})$$

$$\Rightarrow P(x,y)t_x + Q(x,y)\,t_y = (1-n)\left(G(x,y,t^{\frac{1}{1-n}}) - R(x,y,t^{\frac{1}{1-n}})t \right)$$

$$\Rightarrow P(x,y)t_x + Q(x,y)\,t_y = I(x,y,t) ...(3)$$

such that

$$(1-n)\left[G(x,y,t^{\frac{1}{1-n}}) - R(x,y,t^{\frac{1}{1-n}})\,t \right] = I(x,y,t)$$

Equation (3) is especial case of Lagrangian equation , where t is dependent variable , x and y are independent variables, so by using the idea of the Lagrangian method we can write the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dt}{I}$$

From the ordinary differential equation

$$\frac{dx}{P} = \frac{dy}{Q}$$

we can introduce the general solution

$$u(x,y) = a \quad ; \quad a \quad \text{be a constant}$$

Also from the ordinary differential equation

$$\frac{dx}{P} = \frac{dt}{I}$$

we can get the general solution

$$v(x,y,t)=b \quad , b \text{ be a constant}$$

So

$$\Phi(u(x,y),v(x,y,t))=0$$

represents the general solution of the partial differential equation (3),so the general solution of equation (1) is given by:-

$$\Phi(u(x,y),v(x,y,z))=0$$

5- Examples

Example 1:- to solve the partial differential equation

$$xZ_x - xZ_y + Z = xZ^3 \ldots(4)$$

we set

$$\left.\begin{aligned} t = Z^{-2} \Rightarrow t_x &= -2Z^{-3}Z_x \\ t_y &= -2Z^{-3}Z_y \end{aligned}\right\}\ldots(5)$$

By substituting equation (5) in equation (4) we produce,

$$xt_x - xt_y = 2t - 2x \ldots(6)$$

Equation (6) represents Lagrange equation.To solve it, we observe that the auxiliary equations are given by

$$\frac{dx}{x}=\frac{dy}{-x}=\frac{dt}{2t-2x}$$

By taking first and second ratio

$$\frac{dx}{x}=\frac{dy}{-x}$$

follows , $x+y=a$;a be a constant

And by taking first and third ratio , follows

$$\frac{dx}{x}=\frac{dt}{t-2x}\Rightarrow\frac{2}{x}t-2=\frac{dt}{dx}\Rightarrow\frac{dt}{dx}-\frac{2}{x}t=-2$$

The last equation is linear ordinary differential equation and it’s integrating factor is given by

$$I.f = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = \frac{1}{x^2}$$

So,

$$x^{-2}\,dt - 2x^{-3}tdx = -2x^{-2}dx$$

$$\Rightarrow x^{-2}t = 2x^{-1}+b \Rightarrow x^{-2}t - 2x^{-1} = b \Rightarrow x^{-2}z^{-2} - 2x^{-1} = b$$

Moreover ,

$$\Phi(x+y, x^{-2}z^{-2} - 2x^{-1}) = 0$$

represents the general solution of the above partial differential equation

Example 2:-To solve the partial differential equation

$$xZ_x - 2yZ_y - (x^2y + Z^2)Z = -Z^2 \ldots(7)$$

We set

$$\left.\begin{aligned} t = Z^{-1} \Rightarrow t_x &= -Z^{-2}Z_x \\ t_y &= -Z^{-2}Z_y \end{aligned}\right\}\ldots(8)$$

By substituting equation (8) in equation (7) we produce,

$$xt_x - 2yt_y = 1 - x^2 yt - t^{-1} \quad ...(9)$$

Equation (9) represents Lagrange equation .To solve it, we consider the auxiliary equations which are given by

$$\frac{dx}{x} = \frac{dy}{-2y} = \frac{dt}{1 - x^2 yt - t^{-1}}$$

By taking first and second ratio

$$\frac{dx}{x} = \frac{dy}{-2y}$$

we get

$x^2 y = a$; a be a constant

And by taking first and third ratio , follows,

$$\frac{dx}{x} = \frac{dt}{1 - at - t^{-1}} = \frac{t}{t - at^2 - 1}dt = \frac{t}{-a(t - \frac{1}{2a})^2 - 1 + \frac{1}{4a}}dt$$

Now,

$\int \frac{dx}{x} = \ln x + a_1$; a_1 be a constant

To get the integral $\int \frac{t}{-a(t - \frac{1}{2a})^2 - 1 + \frac{1}{4a}}dt$

Let $t - \frac{1}{2a} = s$

$$\Rightarrow \int \frac{t}{-a(t - \frac{1}{2a})^2 - 1 + \frac{1}{4a}}dt = -\int \frac{s + \frac{1}{2a}}{as^2 + c^2}ds \; ; c^2 = 1 - \frac{1}{4a}$$

$$= -\frac{1}{2a}\ln(as^2 + c^2) - \frac{1}{2ac\sqrt{a}}\tan^{-1}\frac{\sqrt{a}s}{c} + b_1$$

$$= -\frac{1}{2a}\ln(a(t - \frac{1}{2a})^2 + c^2) - \frac{1}{2a\sqrt{(1 - \frac{1}{4a})}\sqrt{a}}\tan^{-1}\frac{\sqrt{a}}{\sqrt{1 - \frac{1}{4a}}}(t - \frac{1}{2a}) + b_1$$

$$= -\frac{1}{2a}\ln(a(z^{-1} - \frac{1}{2a})^2 + 1 - \frac{1}{4a}) - \frac{1}{2a^{3/2}\sqrt{(1 - \frac{1}{4a})}}\tan^{-1}\frac{\sqrt{a}}{\sqrt{1 - \frac{1}{4a}}}(z^{-1} - \frac{1}{2a}) + b_1 \quad ; b_1 \text{ be a constant}$$

So,

$$\Phi\left(x^2 y\ , \ln x + \frac{1}{2x^2 y}\ln\left(x^2 y\left(z^{-1} - \frac{1}{2x^2 y}\right)^2 + 1 - \frac{1}{4x^2 y}\right) - \frac{1}{2(x^2 y)^{3/2}\sqrt{\left(1 - \frac{1}{4x^2 y}\right)}}\tan^{-1}\frac{\sqrt{x^2 y}}{\sqrt{1 - \frac{1}{4x^2 y}}}\left(z^{-1} - \frac{1}{2x^2 y}\right)\right)$$

represents the general solution of the above partial differential equation

References :

[1] A.D.Polyanin, V.F.Zaitsev, and A.Moussiaux, Handbook of "First Order Partial differential Equations ", Taylor&Francis,London, 2002.

[2]Braun,M.,"Differential Equation and their Application",4 th ed, NewYork :Spring -Verlag, 1993.

[3] Wu.cheng http://www.efunda.com/math/ode/Ordinary Differential Equation .cfm , 2005