Collecting between the Collage Theorem and the optimization method to solve inverse problem of fractal image

زينب حسن عبود   م.م. مشتاق كريم

جامعة كربلاء/ كلية الإدارة والاقتصاد

Abstract

The inverse problem of fractal shapes can be considered as the main difficulty in fractal geometry, because the problem of determining and identifying the parameters of the affine mappings that constitute the Iterated Function System (IFS). Barinsly state the most famous theorem in this subject, which is called **The Collage Theorem.** Using the optimization method to find inverse problem of fractal set also discussed this problem.

The purpose of this paper is to find Collecting between the Collage Theorem and the optimization method, so as to use the results of the Collage theorem as initial values to the optimization method, which is given speed and accuracy to find the parameter of IFS.

الملخص :

المسألة العكسية للأشكال الكسورية يمكن أن توصف بالصعوبة الرئيسية في الهندسة الكسورية، بسبب وجود مشكلةً فـي حسـاب وتَمييز معاملات مخططات التقريب الذي يشكلِ نظام الوظيفة المتكرر (IFS). يَذْكرُ Barinsly النظريـة الأكْثَر شَـهْرَة فـي هذا الموضـوع، والتـي تُدْعَى نظريـةَ الفن التصـويري (Collage Theorem). أن استعمال طريقةِ تحقيقَ الأمثلية لحل المسـألة العكسيةِ للمجاميع الكسورية ناقشتْ هذه المشكلةِ أيضاً.

إنّ غرضَ هذا البحثِ أَنْ نجدَ الجمـع بـين نظريـةِ الفن التصـويري وطريقةَ تحقيقِ الأمثليـة، لكـي يَستعملَ نَتـائِجَ نظريـةِ الفن التصويري كَقِيَم أولية إلى طريقةِ تحقيقِ الأمثلية، مما يعطي سرعة ودقة ولإيجاد عوامل نظامِ الوظيفةِ المكَرر.

1- Introduction

It is a new branch of mathematics provide a general framework for irregular objects, recently gained a remarkable growth popularity and a broad variety of applications within the mathematical community and among physics, computer graphics, and other sciences.

Fractals possess many different aspects through which they can be characterized. Among these are:
- Self-similarity.
- Non-integer dimensionality.
- Being attractors of some peculiar dynamical system.

In space of fractal geometry, Franklin wrote in [5] "The so-called "Inverse Problem" of finding IFS parameters for a given image is nontrivial", and Ovanessian in [9] wrote "The tackling of this inverse problem, as it stands, is difficult, if it is not impossible". While Barnsley presents a method for obtaining approximate solution of the inverse problem, which is the **Collage Theorem**. Also Mushtaq in [8] has explained a suitable optimization method to find the solution of this problem by using the optimization method.

In this paper, we describe the basics concepts of two-dimensional IFS fractals including their generation, the forward (direct) problem, and their encoding, the inverse problem. The results of the computer program will be listed in an ordered tables.

2- Iterated Function System

The iterated function system (IFS) was introduced in 1988 [2] as an applications of the theory of discrete dynamical systems and useful tools to build fractals and other self similar sets. The mathematical theory of IFS is one of the basis for modeling techniques of fractals and is a powerful tool for producing mathematical fractals such as Cantor set, Sirpinski gasket, etc, as well as real word fractals representing such as clouds, trees, faces, etc. For more details, one can see [6]. IFS is defined through a finite set of affine counteractive mapping mostly of the form:

$$f_i : R^n \to R, i = 1,2,...,m, \quad n \in N$$

$$f(x) = L(x) + C$$

for each $x \in R^n$. Where L is an invariable linear map on R^n. C is vector in R^n. That is a composite of linear mapping L and translation C.

In particular case, two-dimensional affine maps have the following form:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

Where the linear mapping L on R² is represented by 2×2 matrix, and C is a translation (vector) in R². This map could be characterized by the six constants a, b, c, d, e and f, which establish the code of f.

3- The Collage Theorem:

A question may arise; how do we find the IFS associated to a certain image? The key ingredient in Barnsley's approach is **the Collage Theorem** [3]**.**

Collage Theorem [2]:

Let {S: f_i, i = 1, 2, …, m} be an IFS of contractive affine maps with contractivity factors $\{s_i\}_{i=1}^m$.

Let F = $\bigcup_{i=1}^m f_i$ be a Hutchinson operator with contractivity factor $s = \max_{1 \leq i \leq m}\{s_i\}$. Then for any set B $\in$ H (S),

$$D (A, B) < \frac{D(\mathcal{F}(B),B)}{1-s}. \text{ for } s \neq 1$$

This number is called the collage tolerance or union of the images for the set B under F, where D Hausdorff distance, A denotes the attractor of the IFS. Proof: in [3].

It is not easy to understand this applied theorem, but it can be by a simple example.

Mahmood in [7] explained this theorem by example of ivy leaf and Ovanessian in [9] by example of fern, while in [٨] explained this theorem by a simple example in R.

4- An Minimization Problem Using Least Square Method

The minimization is due to the application of the discrete least square method which is the difference between the calculated and exact set of points constituting the attractor of the IFS, this will be done on minimizing the Hausdorff distance between these two sets.

Considering that, the fractal shape is given in advance, and the problem is to find the affine maps that constructing the IFS, the procedure is as follows:

1. Select n number of data points from the fractal shape after scaling the figure using a prespecified coordinate axis.
2. Arrange the obtained data points from step (1) into either ascending or descending order.
3. Introducing an initial guess to the coefficients corresponding to the affine maps.
4. Applying the direct problem using the random iteration algorithm. The results of this program will produce a different shape, but has the property that their results depend on the initial parameters obtained from the inverse program.
5. Select, also n-number of data points resulted by applying the direct problem on the produced results of the inverse problem and rearranging these points either in ascending or descending order as step (2).
6. Evaluating the objective function using the discrete least square method. This function is given by:

$$d(A_e, A_a) = \sum_{i=1}^n (x_{e_i} - x_{a_i})^2$$

Where x_{e_i} is the x-coordinate of the original fractal shape, and x_{a_i} is the x-coordinate corresponding to the approximate fractal shape.

7. Minimize the objective function proposed in step (6) using any optimization method.

The direct optimization used in step (7) is the method of Hooke and Jeeves [4].

5- Collecting between the Collage Theorem and the optimization method

In the last section we applied the optimization method to find the inverse of fractal set by using Hooke and Jeves method with initial point which is differently according to each example therefore engross the long time speeded to get the results, it seems reliable to notice the all of the results are obtained with uncompleted time which is interrupter by the researchers which has its reassume of long time period. More accurate results could be obtained, but with long time period which may be for several days and even so for a week.

Now we applied the collage theorem in each example we can take in, and find the parameters of the affine mappings that constitute the Iterated Function System then depended this result the initial point to the optimization method which is given speeder and more accuracy to find the parameter of IFS.

6- Numerical Example

In order to check and examine the proposed approach in solving the inverse problem of fractal sets.

Example (Tree IFS): (2 – dimensional Example)

In this example, consider the IFS corresponding to the Tree, which is introduced and described by Barnsley in [2]. A graphical representation to this IFS, is as follows

Figure (3.3) The Tree

The IFS resembling this example which has to affine maps, are given in the following table:

F	a	B	c	d	e	f	P
1	-.4	-.4	-.9	-.4	-.4	-.2	0.34
2	-.7	0	-.5	.8	.6	-.2	0.66

Exact IFS codes related to the Tree

In Collage theorem got the following results:

F	a	B	c	d	e	f	P
1	-.31	-.62	-1.9	-.4	-.4	-.2	0.34
2	-.7	0	-.5	.8	.6	-.2	0.66

Approximate results corresponding to the IFS in Tree

In optimization method using the following initial values for a, b, c, d, e and f to be:

F	a	B	c	d	e	f	P
1	0	-1	-1	1	-1	-1	0.34
2	-1	0	-1	1	1	0	0.66

Initial values in the IFS related to Tree

Caring the computer program based minimizing the difference in x-coordinate values between the exact and approximate shapes, we get the following results:

F	a	B	c	d	e	f	P
1	-.208	-.581	-.877	.282	-.6998	-.288	0.34
2	-.79	.0079	-.299	.602	.582	-.6	0.66

Approximate results corresponding to the IFS in Tree

Now, we use the results of the Collage theorem as initial values to the optimization method, then we get the following results:

F	a	B	c	d	e	f	P
1	-.319	-.409	-.884	.364	-.501	-.189	0.34
2	-.709	.002	-.509	.703	.642	-.102	0.66

Approximate results corresponding to the IFS in Tree

The short time speeded and more accuracy to get this results with respect to above two methods.

7- Conclusion Remark

Due to the Simplest of direct optimization problem especial Hook & Jeeves method, the approximate method of "Hook & Jeeves" [4] will be used here in this work to find the numerical solution of the inverse problem..

8- Future Work

1. The most obvious next step would be the extension of this work to systems in three dimensions. This technique is not limited to two dimensions. There are really problems associated with interaction 3D environments and the read-time display of 3D fractal objects.
2. Fractal dimension has been used to characterize data texture in a large number of physical and biological sciences. In medical imaging, Alan and Murray are presented in [1] a new method of estimating fractal dimension which is applicable to data-limited medical images and which shows promise of improving the reparability classes of images. On of the open future work to find relation ship between dimension of image in this method and the dimension of fractal shape in our work.
3. Using this subject in coding theory, so as to use a fractal shape as a password to the systems.

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