On Artin Cokernel of Dihedral Group D_n When n is an even Number

حول النواة المشارك - آرتن لزمرة ثنائي السطوح D_n عندما n عدد زوجي

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Abstract :

In this paper, we find the general form of Artin characters table $Ar(D_n)$, when n is an even number and the cyclic decomposition of Artin Cokernel $AC(D_n)$, when n is an even number such that ;

$n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots\cdots p_m^{\alpha m} \cdot 2^{\beta}$, p_i are distinct primes for all i =1 ,2 ,3,….,m , $p_i \neq 2$, then $AC(D_n) = \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_i+1)(\beta+1)-1} C_2$

الخلاصة :

وجدنا في هذا البحث ,الصيغة العامة لجدول شواخص آرتن $Ar(D_n)$ عندما n عدد زوجي و التجزئة الدائرية لزمرة النواة المشارك-آرتن $AC(D_n)$ عندما يكون n عدد زوجي بحيث إن $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots\cdots p_m^{\alpha m} \cdot 2^{\beta}$ وإن $p_1 , p_2 ,\cdots\cdots, p_m$ أعداد أولية مختلفة لكل i =1 ,2 ,3,….,m وان $p_i \neq 2$ فان

$AC(D_n) = \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_i+1)(\beta+1)-1} C_2$

Introduction :

The abelian group of all Z-valued characters of a finite group G under the operation of pointwise addition over the group of induced unit characters form all cyclic subgroups of the group G (Artin characters) , $\overline{R}$(G)/ T(G)) form a finite abelian group which is called Artin Cokernel of the group G ,denoted by AC(G) .The problem of determining the cyclic decomposition of AC(G) seem to be untouched .In this work, G is considered to be the dihedral group D_n when n is an even number .To do this work we must do the following steps

1- we must know the rational valued characters table of the group D_n , $\equiv^*(D_n)$.
2- we must find Artin characters table of the group D_n , $Ar(D_n)$.
3- we must find the matrix which expresses the Artin characters of the group D_n in terms of rational valued characters , $M(D_n)$.
4- From (3) we must find the invariant factors matrix $M(D_n)$.
5- From (4) we can find the cyclic decomposition of $AC(D_n)$.

The exponent of AC(G) is called the Artin exponent of the group G, denoted by A (G) .
In 1967 T.Y Lam [14] defined AC(G) and he studied AC(G) , when G is acyclic group.
In 1970 K . Yamacchi [11]studied 2-parts of A(G) . In 1976 G.David [4] studied A(G) of arbitrary characters of cyclic subgroups .In 1996 K.Knwabuez [10] studied A(G) of p-group .

In 2000 H.R.Yassien [6] studied the cyclic decomposition of AC(G) when G is an elementary abelian group .In 2002 H.H.Abbass [5] found $\equiv^*(D_n)$.In 2006 A.S.Abed [2] found $Ar(C_n)$ when C_n is the cyclic group of order n .In this paper , we find Ar (D_n) and we study $AC(D_n)$ of the non abelian group D_n ,when is an even number.

1.Some Basic Concepts:-

In this section , we shall give basic concepts , notations and theorems about matrix representation ,characters and Artin characters , which will be used in the next sections .

Definition (1.1):[2]

The general Linear group GL(n,F) is a multiplicative group of all non-singular $n\times n$ matrices over the field F .

Example (1.2) :

Consider the field of complex numbers C ,

GL(2,C)= { A: A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$; a,b,c,d $\in C$ and A is a non – singular }

Definition (1.3):[3]

A matrix representation of a group G is a homomorphism of G into GL(n,F), n is called the degree of matrix representation T . In particular ,T is called a unit representation (principal) if T(g)=1,for all $g\in G$.

Example (1.4):

Consider the symmetric group S_3 , define T: $S_3 \to$ GL(2,C) as follows :

T((1))=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, T((1 2))=$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, T((1 3))=$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$, T((2 3)) =$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$,

T((1 2 3))=$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, T((1 3 2))=$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$. T is a matrix representation of S_3 of degree 2.

Definition (1.5):[3]

The trace of an $n\times n$ matrix A is the sum of the main diagonal elements, denoted by tr(A)

Example (1.6):

Let A =$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, tr (A)= 1+1=2 .

Definition (1.7):[3]

Let T be a matrix representation of degree n of a finite group G over the field F .The character χ of degree n of T is the mapping χ :G $\to$ F defined by $\chi(g)$= tr(T(g)) for all $g\in G$. In particular , the character of the principal representation ($\chi(g) = 1$, for all $g \in G$) is called the principal character .

Example (1.8):

The character χ of the matrix representation T in example (1.4) is of degree 2 and it is defined as follows:

χ((1))= 1+1=2 , χ((1 2))= 0+0 =0 , χ((1 3))= -1+1=0 , χ((2 3))=1+-1=0, χ((1 2 3))=-1 and

χ((1 3 2))=-1 .

Definition (1.9):[3]

Two elements g and h in G are said to be conjugate if h= x g x^{-1} , for some x $\in$ G .

the relation of conjugacy is an equivalence relation on G. The equivalence classes determined by this relation are referred to as the conjugate classes and CL_g , $g \in G$ is the conjugate class of the element g .

Example (1.10):

The two elements (1 2 3) and (1 3 2) are conjugate in the symmetric group S_3 because (1 2) (1 2 3) $(1\ 2)^{-1}$ = (1 3 2) .

Definition (1.11):[3]

The centeralizer of x in G is the subgroup

$C_G(x) = \{a \in G : a x a^{-1} = x \}$.

Example (1.12):

The centeralizer of (1 2 3) in S_3 is the subgroup C_{S_3}((1 2 3))={ (1), (1 2 3), (1 3 2)

Definition (1.13):[3]

Let H be a subgroup of G and ϕ be a character of H ,the induced character on G is given by

$$\phi^{\uparrow G}(g) = \frac{1}{|H|}\sum_{x \in G} \phi^{\circ}(xgx^{-1})$$, where $g \in G$ and ϕ° is defined by

$$\phi^{\circ}(h) = \begin{cases} \phi(h) & if \quad h \in H \\ 0 & if \quad h \notin H \end{cases}$$

Example (1.14):

Consider the subgroup H={ (1), (1 2 3), (1 3 2)}of the symmetric group S_3 , let ϕ be the principal character of H ,i.e ϕ(g)=1 ,for all $g \in G$ .We calculate $\phi^{\uparrow S_3}$ as follows:

$$\phi^{\uparrow S_3}((1)) = \frac{1}{3}\sum_{x \in S_3} \phi^{\circ}(x(1)x^{-1}) = \frac{1}{3}\sum_{x \in S_3} \phi(1) = \frac{1}{3}\ 6 = 2 .$$

$$\phi^{\uparrow S_3}((12)) = \frac{1}{3}\sum_{x \in S_3} \phi^{\circ}(x(12)x^{-1}) = \frac{1}{3}[\phi^{o}((1)(12)(1) + \phi^{\circ}((12)(12)(12)) + \phi^{o}((13)(12)(13)) + \phi^{o}((23)(12)(23)) +$$

$$\phi^{o}((123)(12)(132)) + \phi^{o}((132)(12)(123))] = \frac{1}{3}[\phi^{o}((12) + \phi^{o}((12)) + \phi^{o}((23)) + \phi^{o}((13)) + \phi^{o}((23)) + \phi^{o}((13))]$$

$= \frac{1}{3}[0+0+0+0+0+0] = 0$, similarly , we calculate the other values of $\phi^{\uparrow S_3}$.

The following theorem is used to find the induced characters of the cyclic subgroups .

Theorem (1.15):[6]

Let H be a cyclic subgroup of G and $h_1, h_2, \ldots\ldots, h_m$ are chosen representatives for the m-conjugate classes of H contained in CL_g , $g \in G$, then

$$\phi^{\uparrow G}(g)=\frac{|C_G(g)|}{|C_H(g)|}\sum_{i=1}^{m}\phi(h_i) \quad \text{if } h_i \in H \cap CL_g$$

$$\phi^{\uparrow G}(g)=0 \quad \text{if } H \cap CL_g = \Phi$$

For the proof , see [6]

Definition (1.16):[6]

Let G be a finite group, any character induced from the principal character of cyclic subgroup of G is called Artin character of G .

Example (1.17):

The character in example (1.14) is Artin character of the symmetric group S_3 .

Definition (1.18):[9]

Two elements of the group G are said to be Γ-conjugate if the cyclic subgroups they generate are conjugate in G ,this defines an equivalence relation on G .Its classes are called Γ-classes .

Example (1.19):

The two elements (1 2 3) and (1 3 2) are Γ - conjugate in the symmetric group S_3 because (1 2) < (1 2 3) > $(1\ 2)^{-1}$ = < (1 3 2 > .

Proposition (1.20):[14]

The number of all distinct Artin characters on a group G is equal to the number of Γ-classes on G .

For the proof , see [14] .

Definition (1.21):[2]

The information about Artin characters of a finite group G is displayed in a table called Artin characters table of G , denoted by Ar(G) which is $l \times l$ matrix whose columns are Γ-classes and rows the values of all Artin characters on G ,where l is the number of Γ-classes

Example (1. 22):

Given the cyclic group C_3 = < r > of order 3 ,the Γ-classes on C_3 [1]={ 1} and [r]={ r , r^2 }.

The Artin characters table of C_3 , Ar (C_3)=

	[1]	[r]
ϕ_1	3	0
ϕ_2	1	1

, where $\phi_1 and \phi_2$ are the Aritn characters of C_3 .

Definition (1.23):[3]

A rational valued character θ of G is a character whose values are in the set of integer numbers Z ,which is θ(g)∈ Z ,for all g∈ G .

Example (1.24):

The principal character on a finite group G is a rational valued character of G .

Proposition (1.25):[12]

The number of all distinct rational valued characters of a finite group G equals to the number of

Γ-classes on G .
For the proof , see [12] .

Definition (1.26):[12]
The information about rational valued characters of a finite group G is displayed in a table called the rational valued characters table of G ,denoted by $\equiv^{*}$(G) which is $l \times l$ matrix whose columns are Γ-classes and rows are the values of all rational valued characters of G ,where l is the number of Γ-classes .
Example (1.27):

$\equiv^{*}(C_3) =$

	[1]	[r]
θ_1	1	1
θ_2	2	-1

, where $\theta_1 and \theta_2$ are the rational valued characters of C_3 .

Theorem [Artin] (1.28):[9]
Every rational valued character of a finite group G can be written as a Linear combination of Artin's characters with coefficient rational numbers .
For the proof , see [9] .

2 . The Factor Group AC(G) :-
The definition of the factor group Ac (G) was introduced by T.Y Lam [14] in 1967 . The applications of
The factor group AC(G) not only in the mathematics but also in physics and chemistry .
In this section we shall study AC(G) , dihedral group D_n and $\equiv^{*}(D_n)$, when n is an even number .

Definition (2.1):[14]
Let $\overline{R}$(G) be the group of Z- valued generalized characters of G under the operation pointwise addition and T(G) is a normal subgroup of $\overline{R}$(G) generated by Artin's characters. The abelian factor group $\overline{R}$(G)/T(G) is called Artin's Cokernel of G , denoted by AC(G) .

Definition (2.2):[12]
Let M be a matrix with entries in a principle domain R . A K-minor of M is the determinant of $K \times K$
Submatrix preserving row and column order .

Definition (2.3):[12]
A K-th determinant divisor of M is the greatest common divisor (g.c.d) of all K-minor ,denoted by $D_K(M)$.

Theorem (2.4):[12]
Let M be an $n \times n$ matrix with entries in a principle domain R ,then there exist matrices P and W such that

1- P and W are invertables .
2- P. M.W = D.
3- D is a diagonal matrix .
4- If we denote D_{jj} by d_j then there exists a natural number m ; $0 \le m \le n$ such that j > m implies d_j =0 and j < m implies $d_j \neq 0$ and 1 < j < m implies d_j/d_{j+1} .

For the proof , see [12]

Definition (2.5):[12]

Let M be a matrix with entries in a princpal domain R , and equivalent to matrix D={ $d_1,d_2,..,d_m$,0,0 …,0}, Such that d_j/d_{j+1} for $1 < j < m$, D is called the invariant factor matrix of M and $d_1,d_2,..,d_m$ the invariant factors of M .

Remark (2.6) :-

According to the Artin theorem (1.28) there exists an invertible matrix $M^{-1}(G)$ with entries in the field of rational Q such that

$\equiv^{*}(G) = M^{-1}(G) . Ar(G)$

and this implies

$M(G) = Ar(G).(\equiv^{*}(G))^{-1}$

By theorem (2.4) there exists two matrices P(G) and W(G) such that

P(G) . M(G) . W(G) = diag ={ $d_1,d_2,..,d_l$ }=D(G) , where $d_j = \pm D_j(M(G))/ D_{j-1}(M(G))$ and l is the number of Γ-classes .

Theorem (2.7):[6]

AC(G) = $\bigoplus_{j=1}^{l} C_{d_j}$ where $d_j = \pm D_j(M(G))/ D_{j-1}(M(G))$, and l is the number of all distinct Γ-classes and C_{d_j} is cyclic subgroup of order d_i .

For the proof , see [6]

Proposition (2.8):[12]

Let P be a prime number, then the rational valued characters table of cyclic group C_{p^s} =<r> is given by

Γ-Classes	[1]	$[r^{P^{S-1}}]$	$[r^{P^{S-2}}]$	$[r^{P^{S-3}}]$	…	$[r^{P^2}]$	$[r^P]$	[r]
θ_1	$p^{S-1}(p-1)$	$-p^{S-1}$	0	0	…	0	0	0
θ_2	$p^{S-2}(p-1)$	$p^{S-2}(p-1)$	$-p^{S-2}$	0	…	0	0	0
θ_3	$p^{S-3}(p-1)$	$p^{S-3}(p-1)$	$p^{S-3}(p-1)$	$-p^{S-3}$	…	0	0	0
:	:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:	:
θ_{s-1}	p(p-1)	p(p-1)	p(p-1)	p(p-1)	…	p(p-1)	-p	0
θ_s	p-1	p-1	p-1	p-1	…	p-1	p-1	-1
θ_{s+1}	1	1	1	1	…	1	1	1

$\equiv^{*}(C_{p^s}) =$

For the proof , see [12] .

Remark (2.9) :- In general if $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdot \cdots \cdots p_m^{\alpha m}$ where $p_1, p_2, \ldots, p_m$ are distinct primes ,then

$$\equiv^{*}(C_n) = \equiv^{*}(C_{p_1}^{\alpha 1}) \otimes \equiv^{*}(C_{p_2}^{\alpha 2}) \otimes \cdots \cdots \otimes \equiv^{*}(C_{p_m}^{\alpha m})$$ where $\otimes$ is the tensor product .

Definition (2.10):[9]

The dihedral group D_n is a certain non-abelian group of order 2n ,it is usually thought as a group of transformations of Euclidean plane of regular n-polygon consisting of rotation (r^k) (about the origin)with angle $2\pi k/n$ and reflections sr^k (a cross lines through the origin).

In general it can be written as

$D_n=\{S^i r^k : 0 \le k \le n-1, 0 \le j \le 1\}$,where $r^n=1, S^2=1$, $S r^k S = r^{-k}$.

The cyclic group of order n , $C_n=\langle r\rangle$ is a normal subgroup of D_n.

Proposition (2.11):[5]

The rational valued characters table of D_n when n is an even number is given by:

$\equiv^*(D_n) =$

Γ-Classes	Γ-Classes of C_n $[r^k]$	[s]	[sr]
θ_1		0	0
θ_2		0	0
: :	$\equiv^*(C_n)$	: :	: :
θ_{l-1}		-1	1
θ_l	1 1 ………………………….. 1	1	1
θ_{l+1}	1 1 …………………………. ……1	-1	-1
θ_{l+2}		1	-1

Where $\theta_{l+2}(r^k) = 1$ if k is an even number .

and $\theta_{l+2}(r^k) = -1$ if k is an odd number.

l is the number of Γ-Classes of C_n .

For the proof , see [5] .

Theorem (2.12):[2]

Let p be a prime number, then the Artin characters table of C_{p^s} is given by :

Γ-Classes	[1]	$[r^{p^{s-1}}]$	$[r^{p^{s-2}}]$	$[r^{p^{s-3}}]$	…	[r]
φ'_1	p^S	0	0	0	…	0
φ'_2	p^{S-1}	p^{S-1}	0	0	…	0
φ'_3	p^{S-2}	p^{S-2}	p^{S-2}	0	…	0
:	:	:	:	:	:	:
:	:	:	:	:	:	:
φ'_s	p	p	p	p	…	0
φ'_{s+1}	1	1	1	1	…	1

For the proof , see [2] .

Remark (2.13) :-

Let n be any positive integer and

$n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots\cdots p_m^{\alpha m}$ where $p_1 , p_2 ,\dots, p_m$ are distinct primes ,then

$$Ar(C_n) = \quad Ar(C_{p_1}^{\alpha 1}) \otimes Ar(C_{p_2}^{\alpha 2}) \otimes \cdots\cdots \otimes Ar(C_{p_m}^{\alpha m})$$

Where $\otimes$ is the tensor product .

Proposition (2.14):[13]

If P be a prime number and S is a positive integer ,then

$$M\left(C_{p^s}\right) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \text{ and } P\left(C_{p^s}\right) = \begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

For the proof , see [13]

Remark (2.15) :-

In general if $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots p_m^{\alpha m} \cdot 2^{\beta}$ where $p_1 , p_2 ,\dots, p_m$ are distinct primes ,then

1- $P(C_n) = P(C_{p_1}^{\alpha 1}) \otimes P(C_{p_2}^{\alpha 2}) \otimes \cdots\cdots \otimes P(C_{p_m}^{\alpha m}) \otimes P\left(C_{2^{\beta}}\right)$

We can write

2-

$$M(D_n) = \left[\begin{array}{cccc|cc} & & & \beta.times & \left\{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}\right. & \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \\ & R_2(C_n) & & & 0 & 1 \\ & & & \beta.times & \left\{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}\right. & \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \\ & & & & 0 & 1 \\ & & & \beta.tims & \left\{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}\right. & \begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \\ 0 & 0 & \cdots & & & \\ 0 & 0 & \cdots & & 0 & 1 \end{array}\right]$$

Where $R_2(C_n)$ is the matrix obtainned by omitting the last two rows {0,0,…….,1,1}and{0,0,……..,0,1}and the last two columns {1,…..,1,0,1,…..,1,0,…..,1,0}and{1,1,…..,1}from the tensor product

$$M(C_{p_1}^{\alpha 1}) \otimes M(C_{p_2}^{\alpha 2}) \otimes \cdots\cdots \otimes M(C_{p_m}^{\alpha m}) \otimes M\left(C_{2^{\beta}}\right).$$

3 . The Main Results:-

This section is devoted to sudy the Artin characters Ar (D_n) and the cyclic decomposition of the factor group Ac(D_n) , when n is an even number .

Theorem (3.1):

The Artin characters table of the dihedral group Dn when n is an even number $Ar(D_n)$ =

Γ-Classes	[1]	$[r^{\frac{n}{2}}]$	Γ-Classes of C_n				[S]	[Sr]
$\lvert CL_\alpha \rvert$	1	1	2	2	....	2	n/2	n/2
$\lvert C_{D_n}(CL_\alpha) \rvert$	2n	2n	n	n	....	n	2^2	2^2
Φ_1							0	0
Φ_2							0	0
:				2Ar(C_n)			:	:
:							:	:
Φ_l							0	0
Φ_{l+1}	n	0	…………			0	2	0
Φ_{l+2}	n	0	…………			0	0	2

Where l is the number of Γ-classes of C_n .

Proof:-

Let $g \in D_n$ and ϕ'_j is the Artin characters of C_n for all j=1,2,…, l

Case (I):

If H is a subgroup of C_n =<r> , $1 < j < l$ and the ϕ the principal character of H ,then applying theorem (1.15) yields

$$\Phi_j(g)= \frac{|C_{D_n}(g)|}{|C_H(g)|}\sum_{i=1}^{m}\phi(h_i)$$

(i) If g =1

$$\Phi_j(1)= \frac{|C_{D_n}(1)|}{|C_H(1)|}\cdot\phi(1) = \frac{2n}{|C_H(1)|}\cdot 1 = \frac{2|C_{C_n}(1)|}{|C_H(1)|} = 2.\phi'_1(1) \text{ since } H \cap CL(1) = \{1\}$$

(ii) If $g= r^{\frac{n}{2}}$,$g \neq 1$ and $g \in H$

$$\Phi_j(g)= \frac{|C_{D_n}(g)|}{|C_H(g)|}\cdot\phi(1) = \frac{2n}{|C_H(g)|}\cdot 1 \quad \text{since } H \cap CL(g) = \{g\} \text{ and } \phi(g) = 1$$

$$= \frac{2|C_{C_n}(g)|}{|C_H(g)|}\cdot\phi(g) = 2.\phi'_j(g)$$

(iii) If $g \neq r^{\frac{n}{2}}$ and $g \in H$

$$\Phi_j(g)= \frac{|C_{D_n}(g)|}{|C_H(g)|}\left(\phi(g)+\phi(g^{-1})\right)$$

$$= \frac{n}{|C_H(g)|}(1+1) \quad \text{since } H \cap CL(g) = \{g,g^{-1}\} \text{ and } \phi(g) = \phi(g^{-1}) = 1$$

$$= \frac{2|C_{C_n}(g)|}{|C_H(g)|}\cdot\phi(g) = 2.\phi'_j(g)$$

(iv) If $g \notin H$
$\Phi_j(g) = 0$ since $H \cap CL(g) = \Phi$
$= 2.\,0 = 2.\phi'_j(g)$.

Case (II):
If H = < S > ={1,S}
(i) If g = 1

$$\Phi_{l+1}(1) = \frac{|C_{D_n}(1)|}{|C_H(1)|} \cdot \phi(1) = \frac{2n}{2} \cdot 1 = n$$

(ii) If g = S

$$\Phi_{l+1}(S) = \frac{|C_{D_n}(S)|}{|C_H(S)|} \cdot \phi(1) = \frac{2^2}{2} \cdot 1 = 2$$

Otherwise
$\Phi_{l+1}(g) = 0$ since $H \cap CL(g) = \Phi$

Case (III):
If H = < Sr > ={1,Sr}
(i) If g = 1

$$\Phi_{l+2}(1) = \frac{|C_{D_n}(1)|}{|C_H(1)|} \cdot \phi(1) = \frac{2n}{2} \cdot 1 = n \qquad \text{since } H \cap CL(1) = \{1\}$$

(ii) If g = Sr

$$\Phi_{l+2}(Sr) = \frac{|C_{D_n}(Sr)|}{|C_H(Sr)|} \cdot \phi(1) = \frac{2^2}{2} \cdot 1 = 2 \qquad \text{since } H \cap CL(Sr) = \{Sr\}$$

Otherwise
$\Phi_{l+2}(g) = 0$ since $H \cap CL(g) = \Phi$

Theorem (3.2):

If n is an even number and $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots p_m^{\alpha m} \cdot 2^{\beta}$ where $p_1 , p_2 ,\ldots, p_m$ are distinct primes and $P_i \neq 2$ for all i=1,2,….,m ,then the cyclic decomposition of AC(D_n) is

$$AC(D_n) = \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)-1} C_2$$

Proof:-

By theorem (3.1) we obtained the Artin's characters table Ar(D_n) and from proposition (2.11) we can find the rational valued characters table $\equiv^*(D_n)$.
Thus,
by the definition of the matrix M(D_n) (Remark (2.6))
We have $M(D_n) = Ar(D_n).(\equiv^*(D_n))^{-1}$,then

$$M(D_n)=\begin{bmatrix} & & & & & & & & & \beta.times\left\{\begin{matrix}1&1&1&1\\1&1&1&1\\\vdots&\vdots&\vdots&\vdots\\1&1&1&1\end{matrix}\right\} \\ & R_2(C_n) & & & & & & & & \begin{matrix}0&1&1&0\end{matrix} \\ & & & & & & & & & \beta.times\left\{\begin{matrix}1&1&1&1\\1&1&1&1\\\vdots&\vdots&\vdots&\vdots\\1&1&1&1\end{matrix}\right\} \\ & & & & & & & & & \begin{matrix}0&1&1&0\end{matrix} \\ & & & & & & & & & \beta.times\left\{\begin{matrix}1&1&1&1\\1&1&1&1\\\vdots&\vdots&\vdots&\vdots\\1&1&1&1\end{matrix}\right\} \\ 0&0&\cdots&\cdots&\cdots&\cdots&0& & & \\ 0&0&\cdots&\cdots&\cdots&\cdots&0& & & \begin{matrix}0&1&1&0\end{matrix} \\ 1&1&\cdots&\cdots&\cdots&\cdots&1& & & \begin{matrix}0&1&0&1\end{matrix} \\ 1&1&\cdots&\cdots&\cdots&\cdots&1& & & \begin{matrix}1&1&0&0\end{matrix} \end{bmatrix}$$

Which is $[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)+2]\times[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)+2]$ square matrix .

By theorem (2.4) and remark (2.6) we can take

$$P(D_n)=\begin{bmatrix} & & & & 0 & 0 \\ & & & & 0 & 0 \\ P(C_n) & & & & \vdots & \vdots \\ & & & & 0 & 0 \\ & & & & -1 & 1 \\ & & & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$ and

$$W(D_n)=\begin{bmatrix} & & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 \\ & I_k & & & & \vdots & \vdots & \vdots \\ & & & & & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 0 \\ -1 & -1 & \cdots & -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Where $k=[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)-1]$ and I_k is the identity matrix of order $k\times k$.

Then

$P(D_n)\,.\,M(D_n)\,.W(D_n)=D(D_n)$ = diag {2,2,2,…..,2,1,1,1}}

= $\{d_1,d_2,\ldots\ldots\ldots..,d_{((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_i+1)(\beta+1)-1)}$, 1,1,1}

By using theorem (2.7)

$AC(D_n) = \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)-1} C_{d_i} = \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)-1} C_2$

since $d_i=2$ for all i=1,2,….., $[(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)(\beta+1)-1]$

Example (3.3):

To find $AC(D_{1800})$, $AC(D_{365904})$

$$AC(D_{1800}) = AC(D_{5^2\times 3^2\times 2^3}) = \bigoplus_{i=1}^{(2+1)\cdot(2+1)(3+1)-1} C_2 = \bigoplus_{i=1}^{35} C_2$$

$$AC(D_{365904}) = AC(D_{11^2\times 7\times 3^3\times 2^4}) = \bigoplus_{i=1}^{(2+1)\cdot(1+1)\cdot(3+1)(4+1)-1} C_2 = \bigoplus_{i=1}^{119} C_2 \ .$$

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