# Solving Euler′s Equation by Using New Transformation

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## Abstract

In this paper, we introduce definition of new transformation which we call it Temimi transformation .Also, we introduce properties ,theorems, proofs and transformations of the polynomials functions ,logarithms functions and other functions .Also ,we introduce how we can use this transformation and it's inverse to solve the Euler's equation [2 ] .

## المستخلص

في هذا البحث قدمنا تعريف لتحويل جديد أسميناه تحويل التميمي كذلك قدمنا خصائص ، نظريات ، براهين وتحويلات لمتعددات الحدود والدوال اللوغارتيمية ودوال أخرى لأجل اعتماد هذا التحويل ومعكوسه في حل معادلة أويلر [ ٢ ] .

## Introduction

Laplace transformation[1]is considered as one of the important transformations which is known to solve the L.O.D.E.with constants coefficients and which has the general formula

.Where $a_0,\dots,a_n$ are constants. $$a_0\frac{d^n y}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}\frac{dy}{dx}+a_n y=f(x)$$

With one condition that Laplace transformation of the function f(x) is defined .In this paper, we define a new transformation which is work to solve the L.O.D.E with variables coefficients( Euler's equation) which has the general form ,where $a_0,\dots,a_1$ are constants. $$a_0x^n\frac{d^n y}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}x\frac{dy}{dx}+a_0 y=f(x)$$

This transformation is defined for some functions for example constant functions, logarithm functions , polynomial functions and other functions .

## Definition 1

The Al-Temimi transformation for the function f(x) (x >1) is defined by the following integral :

$$,T(f(x))=\int_1^\infty x^{-p}f(x)dx=F(p)$$

such that this integral is convergent , P is constant .

**Property of this transformation**

This transformation is characterized by the linear property ,that is

, T{Af(x) + Bg(x)}=AT{f(x)} + BT{g(x)}

where A,B are constants ,the functions f(x),g(x) are defined when x>1 .

**Proof :**

$$T\{Af(x)+Bg(x)\}=\int_1^{\infty}(Af(x)+Bg(x))x^{-p}dx = A\int_1^{\infty}x^{-p}f(x)dx + B\int_1^{\infty}x^{-p}g(x)dx = AT\{f(x)\}+BT\{g(x)\}$$

## Transformations for some functions

We are going to find the Temimi transformation for some functions ,like the fixed functions ,logarithm functions ,polynomial functions and other functions .

; p >1 $T\{1\}=\dfrac{1}{p-1}$ 1- If f(x) = 1 , then

$$T\{1\}=\int_1^{\infty}x^{-p}dx=\frac{x^{-p=1}}{-p+1}]_1^{\infty}=\frac{1}{p-1}$$ proof :

;p>1 , and k is constant . $T\{k\}=\dfrac{k}{p-1}$ 2- If f(x) = k , x > 1 then

$$T\{k\}=\int_1^{\infty}kx^{-p}dx=k\int_1^{\infty}x^{-p}dx=k\,\frac{x^{-p+1}}{-p+1}]_1^{\infty}=\frac{k}{p-1}$$ proof :

p > n+1 ; $\dfrac{1}{p-(n+1)}$ = T{$x^n$} 3- If f(x) = $x^n$ , $n \in R$ then

$$T\{x^n\}=\int_1^{\infty}x^{-p}x^n dx=\int_1^{\infty}x^{-p+n}dx=\frac{x^{-p+n+1}}{-p+n+1}]_1^{\infty}=\frac{1}{p-(n+1)}$$ proof :

$T\{\ln x\}=\dfrac{1}{(p-1)^2}\,;\, p>1$ 4- If f(x) = ln x then

$$T\{\ln x\}=\int_1^{\infty}x^{-p}\ln x dx=\frac{x^{-p+1}\ln x}{-p+1}]_1^{\infty}-\int_1^{\infty}\frac{1}{-p+1}x^{-p+1}\frac{dx}{x}$$ proof :

$$=-\frac{1}{(-p+1)}\cdot\frac{x^{-p+1}}{(-p+1)}]_1^{\infty}=\frac{-1}{(-p+1)}\cdot\frac{-1}{(-p+1)}=\frac{1}{(p-1)^2}$$

$T\{x^n\ln x\}=\dfrac{1}{(p-(n+1))^2}\,;\, P>(n+1)$ 5-If f(x) = $x^n$ ln x , $n \in R$ then

$$T\{x^n\ln x\}=\int_1^{\infty}x^{-p}x^n\ln x dx=\int_1^{\infty}x^{-p+n}\ln x dx=[\frac{x^{-p+n+1}}{-p+n+1}\ln x]_1^{\infty}-\int_1^{\infty}\frac{x^{-p+n}}{-p+n+1}dx$$ proof :

$$=\frac{-1}{-p+n+1}[\frac{x^{-p+n+1}}{-p+n+1}]_1^{\infty}=\frac{1}{p-(n+1)}[\frac{-1}{-p+n+1}]=\frac{1}{(p-(n+1))^2}$$

,and a be constant. $\mathrm{T}\{\sin(a\ln x)\}=\dfrac{a}{(p-1)^2+a^2}\,;\, p>1$ 6- If f(x) = sin(a ln x) then

$$=[\frac{x^{-p+1}}{-p+1}\sin(a\ln x)]_1^{\infty}-\int_1^{\infty}\frac{a}{-p+1}x^{-p}\cos(a\ln x)dx \quad \mathrm{T}\{\sin(a\ln x)\}=\int_1^{\infty}x^{-p}\sin(a\ln x)dx$$ **proof :**

$$(1+\frac{a^2}{(p-1)^2})\int_1^\infty x^{-p}\sin(a\ln x)dx=\frac{a}{(p-1)^2}$$

$$=\frac{a}{(p-1)^2}-\frac{a^2}{(p-1)^2}\int_1^\infty x^{-p}\sin(a\ln x)dx\Rightarrow$$

$$\therefore\int_1^\infty x^{-p}\sin(a\ln x)dx=\frac{a}{(p-1)^2+a^2}$$

;p>1 , and a be constant $T\{\cos(a\ln x)\}=\dfrac{(p-1)}{(p-1)^2+a^2}$ 7- If f(x) = cos (a lnx) then

$=[\dfrac{x^{-p+1}}{-p+1}\cos(a\ln x)]_1^\infty-\displaystyle\int_1^\infty\frac{-a}{-p+1}x^{-p}\sin(a\ln x)dx$ $T\{\cos(alnx)\}=\displaystyle\int_1^\infty x^{-p}\cos(a\ln x)dx$ proof :

$$=\frac{-1}{-p+1}+\frac{a}{-p+1}\{[\frac{x^{-p+1}}{-p+1}\sin(a\ln x)]_1^\infty-\int_1^\infty\frac{a}{-p+1}x^{-p}\cos(a\ln x)dx\}$$

$$=\frac{1}{p-1}-\frac{a^2}{(p-1)^2}\int_1^\infty x^{-p}\cos(a\ln x)dx$$

$$\therefore\int_1^\infty x^{-p}\cos(a\ln x)dx=\frac{(p-1)}{(p-1)^2+a^2}$$

$$\Rightarrow(1+\frac{a^2}{(p-1)^2})\int_1^\infty x^{-p}\cos(a\ln x)dx=\frac{1}{(p-1)}\Rightarrow$$

;│p-1│> a $\dfrac{a}{(p-1)^2-a^2}$ T{sinh(a lnx)}= 8- If f(x) = sinh(a lnx) then

$\displaystyle\int_1^\infty x^{-p}\sinh(a\ln x)dx=\int_1^\infty x^{-p}(\frac{e^{a\ln x}-e^{-a\ln x}}{2})dx=\frac{1}{2}\int_1^\infty(x^{-p+a}-x^{-(p+a)})dx$ proof : T{sinh(alnx)}=

$$=\frac{1}{2}[\frac{1}{p-a-1}-\frac{1}{p+a-1}]=\frac{a}{(p-1)^2-a^2}$$

;│p-1│> a $\dfrac{(p-1)}{(p-1)^2-a^2}$ T{cosh(a lnx)= 9- If f(x) = cosh (alnx) then

$\displaystyle\int_1^\infty x^{-p}\cosh(a\ln x)dx=\int_1^\infty\frac{1}{2}[x^{-p}(e^{(a\ln x)}+e^{-(a\ln x)})dx$ T{cosh(a lnx)= proof :

$$=\frac{1}{2}\int_1^\infty(x^{-p+a}+x^{-p-a})dx=\frac{(p-1)}{(p-1)^2-a^2}$$

**Theorem (1)**

$T\{x^{-a}f(x)\}=F(p+a)$ If T{f(x)} = F(p) and a is constant ,then

$$T\{x^{-a}f(x)\}=\int_1^\infty x^{-p}x^{-a}f(x)dx=\int_1^\infty x^{-(p+a)}f(x)dx=F(p+a)$$

## proof :

**Definition 2**

Let f(x) be a function where (x>1)and T{f(x)}=F(p) ,f(x) is said to be an inverse for the Temimi transformation and written as : $T^{-1}\{F(p)\}= f(x)$

where $T^{-1}$ returns the transformation to the original function .

; p>1 $T\{1\}=\frac{1}{p-1}$ since $T^{-1}\{\frac{1}{p-1}\}=1$ For example

; p>1 $T\{k\}=\frac{k}{p-1}$ since $T^{-1}\{\frac{k}{p-1}\}=k$ Also

;p>(n+1) $\frac{1}{p-(n+1)}$ since $T\{x^n\}=$ $T^{-1}\{\frac{1}{p-(n+1)}\}=x^n$ And

; p>1

$$T\{\ln x\}=\frac{1}{(p-1)^2} \quad \text{since} \quad T^{-1}\{\frac{1}{(p-1)^2}\}=\ln x$$

$$T\{x^n \ln x\}=\frac{1}{(p-(n+1))^2}; P>(n+1) \text{ since } T^{-1}\{\frac{1}{(p-(n+1))^2}\}=x^n \ln x$$

$$T\{\sin(a\ln x)\}=\frac{a}{(p-1)^2+a^2}; p>1 \text{ since } T^{-1}\{\frac{a}{(p-1)^2+a^2}\}=\sin(a\ln x)$$

; p>1 $T\{\cos(a\ln x)\}=\frac{(p-1)}{(p-1)^2+a^2}$ since $T^{-1}\{\frac{(p-1)}{(p-1)^2+a^2}\}=\cos(a\ln x)$

;|p-1|>a $\frac{a}{(p-1)^2-a^2}$ since T{sinh(a lnx)}= $T^{-1}\{\frac{a}{(p-1)^2-a^2}\}=\sinh(a\ln x)$

;|p-1|> a $\frac{(p-1)}{(p-1)^2-a^2}$ T{cosh(a lnx)= since $T^{-1}\{\frac{(p-1)}{(p-1)^2-a^2}\}=\cosh(a\ln x)$

A property of $T^{-1}$ is the linear property as it is for the transformation T .

If $T^{-1}\{F_1(p)\}=f_1(x),\dots,T^{-1}\{F_n(p)\}=f_n(x)$ and $a_1,\dots,a_n$ are constants then ,
$T^{-1}\{a_1F_1(p)+a_2F_2(p)+\dots+a_nF_n(p)\}=a_1f(x)+a_2f(x)+\dots+a_nf_n(x)$

$T^{-1}\{F(a + p)\}= x^{-a}\, T^{-1}\{F(p)\}$ If $T^{-1}\{F(p)\}=f(x)$ then **Theorem (2)**

## Proof :

$T^{-1}\{F(a + p)\}= x^{-a} f(x)=x^{-a} T^{-1}\{F(p)\}$

For example to find the inverse of the terms, we get

$$T^{-1}\{\frac{1}{p+1}\}=\frac{1}{x^2} \quad 1-$$

$$T^{-1}\{\frac{1}{(p-2)^2}\} = x\ln x\,2\text{-}$$

$$= T^{-1}\{\frac{(p-1)+2}{(p-1)^2+4}\} = T^{-1}\{\frac{p-1}{(p-1)^2+4}\} + T^{-1}\{\frac{2}{(p-1)^2+4}\}\ 3 - T^{-1}\{\frac{p+1}{(p^2-2p+5)}\}$$

= cos(2lnx) + sin(2lnx)

**Solving the Linear Differential Equations with Variable Coefficients**

One of the most important applications of the Temimi transformation is solving the linear differential equations with variables coefficients .Transforming the L.O.D.E.into algebraic equation in y (p) is done by transforming the derivations and their coefficients

and the function f(x) to the new formulas .After transforming the $x^n \frac{d^n y}{dx^n}, \ldots, x\frac{dy}{dx}$ equation into an algebraic equation ,we are going to find the inverse transformation for this algebraic equation and the result will be the solution of the differential equation .

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots . a_{n-1} x\frac{dy}{dx} + a_0 y = f(x)$$ **Definition 3 :**

**The equation**

where $a_0,\ldots,a_n$ are constants and f(x) is a function of x ,is called Euler's equation .

**Theorem (3)** :

If the function f(x) is defined for (x>1) and its derivatives $f^{(1)}(x), f^{(2)}(x), \ldots, f^{(n)}(x)$ are exist then $T\{x^n f^{(n)}(x)\} = -f^{(n-1)}(1) - (p-n)f^{(n-2)}(1) - \ldots - (p-n)(p-(n-1))\ldots(p-2)f(1) + (p-n)!F(p)$

$$= x^{-p+1} f(x)]_1^\infty - \int_1^\infty (-p+1)x^{-p} f(x)dx;\ p>1\ T\{xf^{(1)}(x) = \int_1^\infty x^{-p+1} f^{(1)}(x)dx$$ **proof** :

$$= -f(1) + (p-1)F(p)$$

$$= x^{-p+2} f^{(1)}(x)]_1^\infty - (-p+2)\int_1^\infty x^{-p+1} f^{(1)}(x)dx\ \ T\{x^2 f^{(2)}(x)\} = \int_1^\infty x^{-p+2} f^{(2)}(x)dx$$ and

$=-f^{(1)}(1)-(p-2)f(1)+(p-2)(p-1)F(p)$

$=-f^{(2)}(1)-(p-3)f^{(1)}(1)-(p-3)(p-2)f(1)+(p-3)(p-2)(p-1)F(p).$

$T\{x^3 f^{(3)}(x)\} = \int_1^\infty x^{-p+3} f^{(3)}(x)dx$ Also

Thus ,by repeating this method for n-times ,we get

$T\{x^n f(x)\} = -f^{(n)}(1) - (p-n)f^{(n-2)}(1) - (p-n)(p-(n-1))f^{(n-3)}(1) - \ldots - (p-n)(p-(n-1))(p-(n-2))\ldots(p-2)f(1) + (p-n)!F(p).$

**Example (1) :**

To find the solution of the differential equation $y(1) = y'(1) = 0$ ;

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x\log x$$

we take T to both sides of this equation , we get

$\frac{1}{(p-2)^2}$ -$f^{(1)}$(1) - (p-2)f(1) + (p-2)(p-1)F(p) + 4(p-1)F(p)-4f(1)+2F(p)=

…(1) $\frac{1}{p(p+1)(p-2)^2}$ F(p)= $\frac{1}{(p-2)^2} \Rightarrow$ F(p)[ $p^2$ + p ]=

By using partial fractions for the right side of the equation(1), we get

$$\frac{1}{p(p+1)(p-2)^2} = \frac{A}{(p-2)} + \frac{B}{(p-2)^2} + \frac{C}{(p+1)} + \frac{D}{p} \quad \ldots(2)$$

$\therefore$ A + B + D = 0 …(3)

-A + B - 4C - 3D = 0 …(4)

2A + B + 4C = 0 …(5)

4D = 1…(6)

By solving these four equations for getting the constants values ,we find that A = -5/36 , B = 1/6 , C = -1/9 , D = 1/4 .

We substitute this values in equation (2), we get

$$F(p) = \frac{\frac{-5}{36}}{(p-2)} + \frac{\frac{1}{6}}{(p-2)^2} + \frac{\frac{-1}{9}}{(p+1)} + \frac{\frac{1}{4}}{p}$$

Now ,we can find the solution of the original equation by taking $T^{-1}$to both sides of the above equation ,we get

$$y = \frac{-5}{36}x + \frac{1}{6}x\ln x - \frac{1}{9x^2} + \frac{1}{4x}$$

**Example (2) :**

To find the solution of the differential equation

,y(1) = 2 , y′(1) = -4 $\quad x^2y'' + 6xy + 6y = \frac{1}{x^2}$

we take T to both sides of this equation and we get

$\frac{1}{(p+1)}$ -$f^{(1)}$(1) - (p-2)f(1)+(p-1)(p-2)F(p)+6(p-1)f(p)-6F(1)+6F(p)=

$\frac{2p^2+6p+5}{(p+1)}$ F(p) ($p^2$ + 3p + 2) = $\frac{1}{(p+1)} \Rightarrow$ F(p) ($p^2$ + 3p + 2) - 2p - 4 =

$$F(p) = \frac{2p^2+6p+5}{(p+1)^2(p+2)}$$

By using partial fractions for the right side of the above equation , we get

$$\frac{2p^2+6p+5}{(p+1)^2(p+2)} = \frac{A}{(p+1)} + \frac{B}{(p+1)^2} + \frac{C}{(p+2)}$$

A+ C = 2 …(7) So

3A + B + 2C = 6…(8)

2A + 2B + C = 5…(9)

From the above equations ,we get A = B = C = 1 .

$$F(p) = \frac{1}{(p+1)} + \frac{1}{(p+1)^2} + \frac{1}{(p+2)}$$ So

And by taking $T^{-1}$ to both sides of the above equation ,we get

$$T^{-1}\{F(p)\}=T^{-1}\{\frac{1}{(p+1)}+\frac{1}{(p+1)^2}+\frac{1}{(p+2)}\}$$

$$\therefore y=\frac{1}{x^2}+\frac{\ln x}{x^2}+\frac{1}{x^3}$$ So

**Example (3) :**

To solve the differential equation

xy′ + y = 16 sin (ln x)  y(1) = -7

we take T to both sides of the above equation, we get

$$-f(1)+(p-1)F(p)+F(p)=\frac{16}{(p-1)^2+1}\Rightarrow pF(p)=\frac{1}{(p-1)^2+1}-7=\frac{-7p^2+14p+2}{p^2-2p+2}$$

$$\Rightarrow F(p)=\frac{-7p^2+14p+2}{p(p^2-2p+2)}$$

By using partial fractions for the right side of the above equation , we get

$$F(p)=\frac{A}{p}+\frac{Cp+D}{p^2-2p+2} \quad \ldots(10)$$

A + C = -7 …(11)

-2A + D = 14…(12)

2A = 2 …(13)

After solving these equations ,we find  A = 1 , C = -8 , D = 16 .

We substitute this values in equation (10) we get

$$F(p)=\frac{1}{p}+\frac{8}{(p-1)^2+1}-\frac{8(p-1)}{(p-1)^2+1}$$

And by taking the inverse transformation of the last equation ,we get the solution of the required differential equation

$$y=\frac{1}{x}+8\sin(\ln x)-8\cos(\ln x)$$

## References

**[ 1 ]** Braun ,M.,"Differential Equation and their Application ", 4 th ed .New York :Spring - Verlag , 1993 .

**[ 2 ]** Coddington ,E.A.," An Introduction to Ordinary Differential Equations ",New York : Dover , 1989 .