Bayesian Estimation of the Logormal Distrbution Mean Using Ranked SET Sampling

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Abstract

Bayes estimation for Lognormal mean using Ranked Set Sampling RSS is considered in this paper and compared to that using Simple Random Sampling SRS. It was show that the RSS Bayes estimator is better than SRS Bayes estimator in terms of the Bayes risk. Also the ratio between SRS Bayes risk to RSS Bayes risk is shown to be always greater than one.

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الملخص

لقد تم تقدير بييز للوسط الحسابي في حالة توزيع اللوغارتمي الطبيعي باستخدام طريقـة المعاينـة المرتبة RSS ومقارنته بتقدير بييز بطريقة المعاينة العشوائية البسيطة SRS. وقد وجد ان تقدير بييز للمعاينة المرتبة أفضل منه للمعاينة العشوائية البسيطة من حيث الخطورة في تقدير بييز. وقـد وجـد ان النسبة بين خطورة تقدير بييز للمعاينة العشوائية البسيطة الى خطورة بييز للمعاينة المرتبة دائما اكبر من الواحد.

Introduction

Fei, Sinha and Wu, (1994), investigated the performance of RSS in the estimation of other parameters (not essentially the mean) when the population is partially known. The shape and the scale parameter of the two parameters Weibull distribution are considered in their study. Stokes (1995) considered the location scale family $F(\frac{X-\mu}{\sigma})$ with F known, studied MLE from both SRS and RSS for two cases, single parameter is unknown and both parameters are unknown. Bimal, Sinha and Sumitra (1995), investigated some improvements of $\hat{\mu}_{RSS}$ with suitable modifications of RSS for estimation of μ when $F(X)$ is normal and exponential. In this paper we assume that the population under study is a Lognormal population with parameter mean θ unknown and variance σ^2 known, and θ has a prior density function as $\pi(\theta)$. As an attempt to estimate θ using Bayesian estimation, a simple random sample (SRS) of size m is taken from the population, on the other hand, a simple random sample of size m^2 is also taken from the population to obtain the rank set sample (RSS). For each sample the Bayesian estimator is found with respect to the Conjugate and Jeffery priors. The performance of the two estimators are compared using their Bayes risks if the two estimators are depend on the conjugate prior or compared using their risk function if the two estimators are depend on the Jeffery prior. And note that the squared error loss function $(SELF)$ will be used in this paper that means $L(\theta,a)=L(\theta-a)^2$ represent the loss that result from estimator θ by a.

The Bayes estimators with respect to the Conjugate prior:

The conjugate priors have the intuitively appealing feature of allowing one to begin with a certain functional form for the prior and end up with a posterior of the same functional form, but with updated parameters by the sample information ***(James, O.,Berger)***. The conjugate prior in this case is normal distribution, and without loss of generality we will assume that it is $\pi(\theta) \sim N(0,1)$.

The Bayes estimators depend on SRS .

Let $X_1, X_2, \cdots, X_m$ be a SRS from $\log N(\theta,\sigma^2)$ where σ^2 known, then without loss of generality take $\sigma^2 = 1$,then the probability density function is.

$$X_i \sim f(x_i|\theta) = \frac{1}{\sqrt{2\pi}\,x_i} e^{-\frac{(\log x_i - \theta)^2}{2}} \;,\; x_i > 0 \,, \quad -\infty < \theta < \infty \,, \quad \forall\; i = 1, \cdots m$$

and the joint density function is.

$$f(\underline{x}|\theta)=\prod_{i=1}^{m} f(x_i|\theta)=\left(\frac{1}{\sqrt{2\pi}}\right)^m \frac{1}{\prod_{i=1}^{m} x_i}\, e^{-\frac{\sum_{i=1}^{m}(\log x_i-\theta)^2}{2}}$$

and the density function of the conjugate prior is

$$\pi(\theta)=\frac{1}{\sqrt{2\pi}}e^{-\frac{\theta^2}{2}} \quad ,-\infty<\theta<\infty$$

The posterior density function of $\theta \mid \underline{x}$ is

$$\pi(\theta|\underline{x})=\frac{\pi(\theta)\, f(\underline{x}|\theta)}{\int_{-\infty}^{\infty}\pi(\theta)\, f(\underline{x}|\theta)\, d\theta} \quad \frac{1}{\sqrt{2\pi\frac{1}{m+1}}}\, e^{-\frac{m+1}{2}\left(\theta-\frac{\sum_i^m \log x_i}{m+1}\right)^2}$$

then $\pi(\theta|\underline{x}) \sim N\left(\frac{\sum_i^m \log x_i}{m+1}, \frac{1}{m+1}\right)$

$$r(\hat{\theta}_{SRS}(\underline{x}),\pi)=E_{\pi(\theta)}(Var(\theta|\underline{x}))=\frac{1}{m+1}$$

The Bayes estimator of θ is $\hat{\theta}_{SRS}(\underline{x})=E(\theta|\underline{x})$ because the Bayes estimate with respect to $(SELF)$ is the posterior mean then

$$\hat{\theta}_{SRS}(\underline{x})=\frac{\sum_i^m \log x_i}{m+1}$$

and the Bayes risk of the estimator is

The Bayes estimators depend on RSS.

We assume that have m random samples each of size m since.

$$\begin{matrix} X_{11} & X_{12} & \cdots & X_{1m} \\ X_{21} & X_{22} & \cdots & X_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ X_{m1} & X_{m2} & \cdots & X_{mm} \end{matrix}$$

where $X_{ij} \sim \log N(\theta,1), \forall\, i, j = 1, 2, \cdots, m$.

Now let $Y_1, Y_2, \cdots, Y_m$ be the RSS ,where $Y_1 = \min(X_{11}, X_{12}, \cdots, X_{1m})$, $Y_2 = 2^{nd} \min(X_{21}, X_{22}, \cdots, X_{2m})$ $, \cdots, Y_m = \max(X_{m1}, X_{m2}, \cdots, X_{mm})$,i.e., Y_i is the $i-th$ order statistic of the set i. Then the pdf of $y_i \mid \theta$ is given by.

$$g(y_i \mid \theta) = \frac{m!}{(i-1)!(m-i)!}[F(y_i)]^{i-1}[1-F(y_i)]^{m-i} f(y_i) \qquad ;0 < y_i < \infty$$

where $f(y_i) = \frac{1}{\sqrt{2\pi}\, y_i} e^{-\frac{(\log y_i - \theta)^2}{2}}, \; y_i > 0, \quad -\infty < \theta < \infty$

Using the transformation $w_i = \log y_i$ then $w_i \sim N(\theta,1)$ .Then pdf of $y_i \mid \theta$ is became as.

$$g(w_i \mid \theta) = \frac{m!}{(i-1)!(m-i)!}[\Phi(w_i-\theta)]^{i-1}[1-\Phi(w_i-\theta)]^{m-i}\phi(w_i-\theta)$$

where $\phi(w_i - \theta)$ is a pdf of $N(\theta,1)$ and $\Phi(w_i - \theta)$ is a CDF of $N(\theta,1)$,since $w_1, \cdots w_m$ are independent their joint density is.

$$g(\underline{w} \mid \theta) = \prod_i^m g(w_i \mid \theta) = m(1-F(w_1))^{m-1} f(w_1) \times m(m-1)F(w_2)(1-F(w_2))^{m-2} f(w_2)$$
$$\times \cdots \times m(F(w_2))^{m-1} f(w_m)$$

$$= \prod_i^m \frac{m!}{(i-1)!(m-1)!} f(w_i) \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}\left[\binom{m-1}{k_1}\binom{m-2}{k_2}\cdots\binom{0}{k_m}(-1)^{k_1+k_2+\cdots+k_m}\right.$$
$$\left.(F(w_1))^{k_1}(F(w_2))^{k_2+1}\cdots(F(w_m))^{k_m+m-1}\right]$$

Now $\prod_i^m f(w_i) = \prod_i^m \phi(w_i - \theta) = (\frac{1}{\sqrt{2\pi}})^m e^{-\frac{\sum_{i=1}^m (w_i - \overline{w})^2}{2}} e^{-\frac{m}{2}(\overline{w}-\theta)^2}$

Let

$$(\frac{1}{\sqrt{2\pi}})^m e^{-\frac{\sum_{i=1}^m (w_i - \overline{w})^2}{2}} \prod_i^m \frac{m!}{(i-1)!(m-1)!} = A_1$$

& $\binom{m-1}{k_1}\cdots\binom{0}{k_m} = A(k_1,\cdots k_m) = A(\underline{k})$

Then

$$g(\underline{w}|\theta) = A_1\, e^{-\frac{m}{2}(\overline{w}-\theta)^2} \sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0} [A(\underline{k})(-1)^{\sum_{i=1}^{m} k_i} \prod_{i=1}^{m} (\Phi(w_i-\theta))^{k_i+i-1}]$$

The posterior density function of $\theta|\underline{w}$ is given by.

$$\pi(\theta|\underline{w}) = \frac{\phi[(\theta-\frac{m\overline{w}}{m+1})\sqrt{m+1}]\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]}{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{-\infty}^{\infty}\phi[(\theta-\frac{m\overline{w}}{m+1})\sqrt{m+1}]\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]\,d\theta}$$

The Bayes estimator of θ is $\hat{\theta}_{RSS}(\underline{w}) = E(\theta|\underline{w}) = \int_{-\infty}^{\infty}\theta\,\pi(\theta|\underline{w})\,d\theta$

$$= \frac{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{-\infty}^{\infty}\theta\,\phi[(\theta-\frac{m\overline{w}}{m+1})\sqrt{m+1}]\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]\,d\theta}{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{-\infty}^{\infty}\phi[(\theta-\frac{m\overline{w}}{m+1})\sqrt{m+1}]\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]\,d\theta}$$

And the posterior variance is

$$Var(\theta|\underline{w}) = \int_{-\infty}^{\infty}(\hat{\theta}_{RSS}(\underline{w})-\theta)^2\pi(\theta|\underline{w})\,d\theta$$

$$= \frac{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{-\infty}^{\infty}(\hat{\theta}_{RSS}(\underline{w})-\theta)^2\,\phi[(\theta-\frac{m\overline{w}}{m+1})\sqrt{m+1}]\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]\,d\theta}{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{-\infty}^{\infty}\phi[(\theta-\frac{m\overline{w}}{m+1})\sqrt{m+1}]\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]\,d\theta}$$

The Bayes risk $r(\hat{\theta}_{RSS}(\underline{w}),\pi) = E_{\pi(\theta)}(Var(\theta|\underline{w}))$

The generalized Bayes estimators with respect to the Jeffrey prior:

The Bayesian approach can be used even when no prior information is available, in such situations, a non-information prior is used can be called Jeffrey prior. The Bayes risk has no meaning here infinite, so we compared depend on the risk function.

The generalized Bayes estimators depend on SRS .

Let $X_1, X_2, \cdots, X_m$ be a SRS from $\log N(\theta,1)$, and let the Jeffery prior in this case is $\theta \sim \pi(\theta) = 1$. Because the θ is allocation parameter see ***(James, O.,Berger)*** then the posterior density $\pi(\theta \mid \underline{x})$ is given by.

$$\pi(\theta \mid \underline{x}) = \frac{f(\underline{x} \mid \theta)}{\int_{-\infty}^{\infty} f(\underline{x} \mid \theta)\, d\theta} = \sqrt{\frac{m}{2\pi}}\, e^{-\frac{m}{2}[\theta - \frac{\sum_{i=1}^{m} \log x_i}{m}]^2}$$

since $\pi(\theta \mid \underline{x}) \sim N(\frac{\sum_{i=1}^{m} \log x_i}{m}, \frac{1}{m})$

Hence the generalized Bayes estimator of θ is

$$\hat{\theta}^{J}_{SRS}(\underline{x}) = E(\theta \mid \underline{x}) = \frac{\sum_{i=1}^{m} \log x_i}{m}$$

The risk function is $R(\hat{\theta}^{J}_{SRS}(\underline{x}), \theta) = Var\left(\frac{\sum_{i=1}^{m} \log x_i}{m}\right) + (bias)^2$

The generalized Bayes estimators depend on RSS .

Let $Y_1, Y_2, \cdots, Y_m$ be the RSS from $\log N(\theta,1)$, and let the Jeffery prior in this case is also $\theta \sim \pi(\theta) = 1$.

Using the transformation $w_i = \log y_i$ then $w_i \sim N(\theta, 1)$. Then pdf of $y_i \mid \theta$ is became as.

$$g(w_i \mid \theta) = \frac{m!}{(i-1)!(m-i)!}[\Phi(w_i - \theta)]^{i-1}[1 - \Phi(w_i - \theta)]^{m-i}\phi(w_i - \theta)$$

Then the joint density function of w's is

$$g(\underline{w}|\theta)=A_1\, e^{-\frac{m}{2}(\overline{w}-\theta)^2}\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}\,[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]$$

Then the posterior density $\pi\,(\theta\,|\,\underline{w})$ is given by. therefore,

$$\pi\,(\theta\,|\,\underline{w})=\frac{g\,(\underline{w}|\theta)}{\int_{-\infty}^{\infty}g\,(\underline{w}|\theta)\,d\theta}$$

$$\pi(\theta|\underline{w})=\frac{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}\,[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]e^{-\frac{m}{2}(\overline{w}-\theta)^2}}{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}\,[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{-\infty}^{\infty}\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]e^{-\frac{m}{2}(\overline{w}-\theta)^2}\,d\theta}$$

Thus, the generalized Bayes estimator $\hat{\theta}^{J}_{RSS}(\underline{w})=\int_{-\infty}^{\infty}\theta\,\pi(\theta|\underline{w})\,d\theta$

$$\hat{\theta}^{J}_{RSS}(\underline{w})=\frac{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}\,[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{-\infty}^{\infty}\theta\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]e^{-\frac{m}{2}(\overline{w}-\theta)^2}\,d\theta}{\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}\,[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{-\infty}^{\infty}\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]e^{-\frac{m}{2}(\overline{w}-\theta)^2}\,d\theta}$$

The risk function is given by

$$R(\hat{\theta}^{J}_{RSS}(\underline{w}),\theta)=\int_{\underline{w}}(\hat{\theta}^{J}_{RSS}(\underline{w})-\theta)^2\,g(\underline{w}|\theta)\,d\,\underline{w}$$

$$=A_1\sum_{k_1=0}^{m-1}\sum_{k_2=0}^{m-2}\cdots\sum_{k_m=0}^{0}\,[A(\underline{k})(-1)^{\sum_{i=1}^{m}k_i}\int_{\underline{w}}(\hat{\theta}^{J}_{RSS}(\underline{w})-\theta)^2\prod_{i=1}^{m}(\Phi(w_i-\theta))^{k_i+i-1}]e^{-\frac{m}{2}(\overline{w}-\theta)^2}\,d\,\underline{w}$$

Properties of the generalized Bayes estimator using RSS .

1) Let c be a constant, then $\hat{\theta}^{J}_{RSS}(\underline{w}+c)=\hat{\theta}^{J}_{RSS}(\underline{w})+c$

Proof:

$$\hat{\theta}^{J}_{RSS}(\underline{w}+c)=\int_{-\infty}^{\infty}\theta\,\pi(\theta|\underline{w}+c)\,d\theta=\frac{\int_{-\infty}^{\infty}\theta\,g(\underline{w}+c\,|\theta)\,d\theta}{\int_{-\infty}^{\infty}g(\underline{w}+c\,|\theta)\,d\theta}$$

$$= \frac{\int_{-\infty}^{\infty} \theta \prod_{i=1}^{m} [\Phi(w_i + c - \theta)]^{i-1}[1 - \Phi(w_i + c - \theta)]^{m-i} \phi(w_i + c - \theta)\, d\theta}{\int_{-\infty}^{\infty} \prod_{i=1}^{m} [\Phi(w_i + c - \theta)]^{i-1}[1 - \Phi(w_i + c - \theta)]^{m-i} \phi(w_i + c - \theta)\, d\theta}$$

Let $\theta - c = k \Rightarrow d\theta = dk$ then

$$\hat{\theta}^{J}_{RSS}(\underline{w} + c) = \frac{\int_{-\infty}^{\infty} (k + c) \prod_{i=1}^{m} [\Phi(w_i - k)]^{i-1}[1 - \Phi(w_i - k)]^{m-i} \phi(w_i - k)\, dk}{\int_{-\infty}^{\infty} \prod_{i=1}^{m} [\Phi(w_i - k)]^{i-1}[1 - \Phi(w_i - k)]^{m-i} \phi(w_i - k)\, dk}$$

$$= \frac{\int_{-\infty}^{\infty} k \prod_{i=1}^{m} [\Phi(w_i - k)]^{i-1}[1 - \Phi(w_i - k)]^{m-i} \phi(w_i - k)\, dk}{\int_{-\infty}^{\infty} \prod_{i=1}^{m} [\Phi(w_i - k)]^{i-1}[1 - \Phi(w_i - k)]^{m-i} \phi(w_i - k)\, dk} + c = \hat{\theta}^{J}_{RSS\,(\underline{w})} + c$$

2) The risk function of $\hat{\theta}^{J}_{RSS(\underline{w})}$ is free of θ.

Proof:

$$R(\hat{\theta}^{J}_{RSS}(\underline{w}), \theta) = E[(\hat{\theta}^{J}_{RSS}(\underline{w}) - \theta)^2 \,|\, \theta]$$

$$= \int_{\underline{w}} (\hat{\theta}^{J}_{RSS}(\underline{w}) - \theta)^2 g(\underline{w} \,|\, \theta)\, d\underline{w} = \int_{\underline{w}} \hat{\theta}^{J}_{RSS}(\underline{w} - \theta)^2 g(\underline{w} \,|\, \theta)\, d\underline{w}$$

Using property one.

$$= \int_{\underline{w}} \hat{\theta}^{J}_{RSS}(\underline{w} - \theta)^2 \prod_{i=1}^{m} \frac{m!}{(i-1)!(m-i)!} [\Phi(w_i - \theta)]^{i-1}[1 - \Phi(w_i - \theta)]^{m-i} \phi(w_i - \theta)\, d\underline{w}$$

let $\underline{w} - \theta = \underline{z} \Rightarrow d\underline{w} = d\underline{z}$

$$= \int_{\underline{z}} \hat{\theta}^{J}_{RSS}(\underline{z})^2 \prod_{i=1}^{m} \frac{m!}{(i-1)!(m-i)!} [\Phi(\underline{z})]^{i-1}[1 - \Phi(\underline{z})]^{m-i} \phi(\underline{z})\, d\underline{z}$$

$$=\int_{\underline{z}} \hat{\theta}^J_{RSS}(\underline{z})^2 g(\underline{z})\, d\underline{z}$$ which is free of θ.

3) $\hat{\theta}^J(-w_1,-w_2,\cdots,-w_m) = -\hat{\theta}^J(w_m, w_{m-1},\cdots,w_1)$

proof:

$$\hat{\theta}^J_{RSS}(-w_1,-w_2,\cdots,-w_m) = \frac{\int_{-\infty}^{\infty}\theta\, g(-w_1,-w_2,\cdots,-w_m \mid \theta)\, d\theta}{\int_{-\infty}^{\infty} g(-w_1,-w_2,\cdots,-w_m \mid \theta)\, d\theta}$$

$$= \frac{\int_{-\infty}^{\infty}\theta \prod_{i=1}^{m}[\Phi(-w_i-\theta)]^{i-1}[1-\Phi(-w_i-\theta)]^{m-i}\phi(-w_i-\theta)\, d\theta}{\int_{-\infty}^{\infty}\prod_{i=1}^{m}[\Phi(-w_i-\theta)]^{i-1}[1-\Phi(-w_i-\theta)]^{m-i}\phi(-w_i-\theta)\, d\theta}$$

But

$$\Phi(-w_i-\theta) = \Phi(-(w_i+\theta)) = 1-\Phi(w_i+\theta)\ \&\ \phi(-w_i-\theta) = \phi(w_i+\theta)$$

. Then

$$\hat{\theta}^J_{RSS}(-w_1,-w_2,\cdots,-w_m) = \frac{\int_{-\infty}^{\infty}\theta\prod_{i=1}^{m}[1-\Phi(w_i+\theta)]^{i-1}[\Phi(w_i+\theta)]^{m-i}\phi(w_i+\theta)\, d\theta}{\int_{-\infty}^{\infty}\prod_{i=1}^{m}[1-\Phi(w_i+\theta)]^{i-1}[\Phi(w_i+\theta)]^{m-i}\phi(w_i+\theta)\, d\theta}$$

Let $-\theta = k \Rightarrow d\theta = -dk$,and $j = m-(i-1)$,then

$$\hat{\theta}^J_{RSS}(-w_1,-w_2,\cdots,-w_m) = \frac{\int_{-\infty}^{\infty} k\prod_{i=1}^{m} g(w_{m-(j-1)} \mid k)\, dk}{\int_{-\infty}^{\infty}\prod_{i=1}^{m} g(w_{m-(j-1)} \mid k)\, dk} = -\hat{\theta}^J(w_m, w_{m-1},\cdots,w_1)$$

4) $\hat{\theta}^J(\underline{w})$ is unbiased estimator for θ; $E(\hat{\theta}^J_{Rss}(\underline{w}) \mid \theta) = \theta$.

proof:

$$E(\hat{\theta}^J_{Rss}(\underline{w})-\theta \mid \theta) = \int_{\underline{w}} \hat{\theta}^J_{Rss}(\underline{w}-\theta)\, g(\underline{w} \mid \theta)\, d\underline{w}$$

Using property one, and let $\underline{k} = \underline{w} - \theta$, then

$$E(\hat{\theta}^J_{RSS}(\underline{k}) = \int_{\underline{z}} \hat{\theta}^J_{RSS}(\underline{k}) \prod_{i=1}^{m} \frac{m!}{(i-1)!(m-i)!}[\Phi(k_i)]^{i-1}[1-\Phi(k_i)]^{m-i}\phi(k_i)\,dk_i$$

using the transformation $k_i = -u_i$, and let $j = m - i + 1$,and using the property three we have.

$$E(\hat{\theta}^J_{RSS}(-\underline{u})) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} -\hat{\theta}^J_{RSS}(u_m,\cdots,u_1)\prod_{j=1}^{m} g(u_{m-(j-1)})\,du_1,\cdots du_m$$

rename $u_1,\cdots,u_m$ respectively by $k_m,\cdots,k_1$ then

$$E(\hat{\theta}^J_{RSS}(\underline{k})) = -\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \hat{\theta}^J_{RSS}(k_1,\cdots,k_m)\prod_{j=1}^{m} g(k_j)\,dk_m,\cdots dk_1$$

$$= E(\hat{\theta}^J_{RSS}(\underline{k}))$$

Thus

$$E(\hat{\theta}^J_{RSS}(\underline{k})) = -E(\hat{\theta}^J_{RSS}(\underline{k})) \Rightarrow 2E(\hat{\theta}^J_{RSS}(\underline{k})) = 0 \Rightarrow E(\hat{\theta}^J_{RSS}(\underline{w}-\theta)) = 0$$

$$E(\hat{\theta}^J_{RSS}(\underline{w}-\theta)) = E(\hat{\theta}^J_{RSS}(\underline{w})\,|\,\theta) - \theta = 0$$

Thus $\hat{\theta}^J_{RSS}(\underline{w})$ is unbiased estimator for θ.

Numerical comparison.

In this section we will using the simulation to obtain the values of the Bayes risk $\{r(\hat{\theta}_{SRS},\pi), r(\hat{\theta}_{RSS},\pi)\}$ and risk function $\{R(\hat{\theta}^J_{SRS},\theta), R(\hat{\theta}^J_{RSS},\theta)\}$ depend on the conjugate prior and Jeffery prior respectively, and also we compared the two estimators by using efficiency since.

1) The efficiency of $\hat{\theta}_{RSS}$ with respect to $\hat{\theta}_{SRS}$ depend on the conjugate prior is defined by $eff = \frac{r(\hat{\theta}_{RSS},\pi)}{r(\hat{\theta}_{SRS},\pi)}$

2) The efficiency of $\hat{\theta}^J_{SRS}$ with respect to $\hat{\theta}^J_{SRS}$ depend on the Jeffery prior is defined by $eff = \frac{R(\hat{\theta}^J_{RSS},\theta)}{R(\hat{\theta}^J_{SRS},\theta)}$

Table one contains values for $r(\hat{\theta}_{SRS},\pi), r(\hat{\theta}_{RSS},\pi), R(\hat{\theta}^J_{SRS},\theta)$ & $R(\hat{\theta}^J_{RSS},\theta)$

along with the efficiency for selected values of r & m where r is the number of cycles and m is the set size.

Table (1):Bayes risk $\{r(\hat{\theta}_{SRS},\pi), r(\hat{\theta}_{RSS},\pi)\}$ and risk function $\{R(\hat{\theta}^{J}_{SRS},\theta), R(\hat{\theta}^{J}_{RSS},\theta)\}$ and efficiency with respect to the conjugate prior and Jeffery prior respectively

r	m	$r(\hat{\theta}_{SRS},\pi)$	$r(\hat{\theta}_{RSS},\pi)$	*eff*	$R(\hat{\theta}^{J}_{SRS},\theta)$	$R(\hat{\theta}^{J}_{RSS},\theta)$	*eff*
1	2	0.3365	0.24919	1.350375	0.5061	0.3308	1.529927
	3	0.2535	0.14319	1.770375	0.3317	0.1601	2.07183
	4	0.2119	0.09609	2.205224	0.2368	0.095	2.492632
	5	0.1757	0.06339	2.771731	0.189	0.059619	3.17013
2	2	0.2073	0.14269	1.4528	0.2398	0.1573	1.524476
	3	0.154	0.078398	1.964336	0.1591	0.076471	2.080527
	4	0.1201	0.047961	2.504118	0.1114	0.042164	2.642064
	5	0.100877	0.03267	3.087756	0.087333	0.023884	3.656548
3	2	0.1517	0.09989	1.518671	0.1528	0.0967	1.580145
	3	0.109767	0.052803	2.078802	0.0963	0.046175	2.085544
	4	0.088353	0.032856	2.689098	0.074234	0.025348	2.928594
	5	0.071756	0.021366	3.35842	0.055837	0.01241	4.499355

Conclusion.

1. The Bayes risk using RSS is always less than the Bayes risk using SRS for all the cases we considered.

2. The risk function using the generalized Bayes RSS estimator is always less than that using the generalized Bayes SRS estimator.
3. Both the Bayes risk using RSS and SRS decreases as $n = r\,m$ increases.
4. Both the risks function decreases as $n = r\,m$ increases.
5. The efficiency is always grater than one.
6. The efficiency is increasing within a cycle, as m increases.

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