# Bilinearization for Sine-Gordon-Type Equations

**I.A.Malloki and W.F.Al Rikabi**

**Department of Mathematics, College of Science, Al- Mustansiriyah University, Baghdad, Iraq**

**Abstract**

In this paper, we consider Heiternita classification of nonlinear partial differential equations, to put an explicit form of definition for sG-type equation and then adopt it to investigate a class of nonlinear partial differential equations. We have two goals: The first is to find some equations that satisfy the definition of sG type equation, while the second is to find the more general transformation for sG equation to be bilinearizable.

**الثنائية الخطية لمعادلات نوع ساين جوردن**

**انعام ماوكي** **وفاء فايق**

**الخلاصة**

في هذا البحث قمنا باتخاذ تصنيف هيترنيتا للمعادلات التفاضلية الجزئية غير الخطية وذلك بوضع شكل صريح لتعريف معادلات نوع sG ثم استخدامه لبحث احد أصناف المعادلات التفاضلية غير الخطية. لدينا هدفان: الأول لإيجاد المعادلات التي تحقق تعريف معادلة نوع sG والثاني لإيجاد التحويل الأكثر عموميا لجعل معادلة sG ثنائية الخطية.

## (1) Introduction

Recently, there are many introduced active and nonlinear wave system (in electronics and applied physics) which require a nonlinear theory of waves for their complete description.

One of important nonlinear partial differential equations in this field is sine- Gordon (sG). It is known to possess solutions that correspond to "solitons" , that is localized entities that maintain their shape after collisions and have certain properties characteristic of elementary particle [1].

While sG equation can't be solved in general, several classes of solutions can be found by making Lamb's transformations. In 1971 Hirota introduced a powerful method to construct N-soliton solution to integrable nonlinear evolution equation [2]. Accordingly, if one just wants to find soliton solutions, Hirota's method is the fastest in producing results [3]. In recent paper, Hietarinta [4] has searched for bilinearized equations of special type, which have three soliton solutions.

This paper will be essentially devoted to study the class of equations

$$u_{tt} - u_{xx} = F(u) \qquad \text{……………..(1)}$$

where F is an unknown function. We derive many equations and their transformations to Hirota's bilinearization .

**(2) Extension of Lamb's transformation:**

**Definition (1) [2]:-** A symbol $D_x$ is called the Hirota's derivative with respect to the variable $x$ and defined to act on a pair of functions $(f, g)$ as follows :

$$(D_x(f.g))(x) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x^{\cdot}}\right) f(x) g(x^{\cdot}) \Big|_{x^{\cdot}=x}$$

**Definition (2):** An equation (1) is called of sG-type equations (in the Hirota's sense) iff there exists a nontrivial dependent variable transformation

$$u(x,t) = G(g(x,t)/f(x,t))$$

and two nontrivial Hirota's polynomials $P_1$ and $P_2$ such that:

$$\left.\begin{aligned} &P_1(D_x, D_t)g.f = 0 \\ \text{and} \quad & \\ &P_2(D_x, D_t)(f.f - g.g) = 0 \end{aligned}\right\} \qquad \text{…………(2)}$$

For sG equation the dependent variable transformation which has been used is Lamb's transformation and it is found that, this transformation leads to Hirota's bilineariztion of the equation.

It seems reasonable to extend the dependent variable Lamb's transformation to

$$u(x,t) = G(\psi(x,t)) \qquad \text{……………….(3)}$$

such that the following assumptioms are valid

**(1)** The new dependent variable $\psi$ is expressed with the ratio of two functions $g$ and $f$ , i.e

$$\psi(x,t)=g/f \qquad \text{..................(4)}$$

**(2)** The equation will be transformed by $G$ into a homogenous equation of $f$ and $g$ .

**(3)** The homogenous equation must be splitted into a pair of bilinear equations (2).

These restrictions, will help us to find the function (transformation) $G$ and, then to identity the class of equation (1) (i.e to find the function F).

To start with, we find $u_{xx}$ , $u_{tt}$ in terms of $\psi$ and its derivatives, using properties of the bilinear derivatives:

If $$u=G(\psi) \qquad \text{..................(5)}$$

then

$$u_x=G'\psi_x=G'\frac{D_x g.f}{f^2}$$

$$u_{xx}=G'\psi_{xx}+G''\psi_x^2$$

$$=G'\left(\frac{D_x^2 g.f}{f^2}-\frac{g}{f}\frac{D_x^2 f.f}{f^2}\right)+G''\left(\frac{g}{f}\frac{D_x^2 g.f}{f^2}-\frac{1}{2}\frac{g^2}{f^2}\frac{D_x^2 f.f}{f^2}-\frac{1}{2}\frac{D_x^2 g.g}{f^2}\right) \qquad \text{......(6)}$$

Similarly ,

$$u_t=G'\psi_t=G'\frac{D_t g.f}{f^2}$$

$$u_{tt}=G'\psi_{tt}+G''\psi_t^2$$

$$=G'\left(\frac{D_t^2 g.f}{f^2}-\frac{g}{f}\frac{D_t^2 f.f}{f^2}\right)+G''\left(\frac{g}{f}\frac{D_t^2 g.f}{f^2}-\frac{1}{2}\frac{g^2}{f^2}\frac{D_t^2 f.f}{f^2}-\frac{1}{2}\frac{D_t^2 g.g}{f^2}\right) \qquad \text{......(7)}$$

For details see [5]. Substituting (5), (6) and (7) into (1), we get

$$\left(G'+\frac{g}{f}G''\right)\left(\frac{D_x^2-D_t^2}{f^2}\right)(g.f)-\left(\frac{g}{f}G'+\frac{g^2}{2f^2}G''\right)\left(\frac{D_x^2-D_t^2}{f^2}\right)(f.f)$$

$$-\frac{G''}{2}\left(\frac{D_x^2-D_t^2}{f^2}\right)(g.g)+F(G)=0 \qquad \text{......(8)}$$

If we denote $\left(D_x^2 - D_t^2\right)g.f$ , $\left(D_x^2 - D_t^2\right)f.f$ and $\left(D_x^2 - D_t^2\right)g.g$ by $D_1$, $D_2$, and $D_3$ respectively then equation (8) will be written as,

$$\left(\frac{1}{f^2}G' + \frac{g}{f^3}G''\right)D_1 - \left(\frac{g}{f^3}G' + \frac{g}{2f^4}G''\right)D_2 - \frac{G''}{2f^2}D_3 + F(G) = 0$$

This equation can be rearranged into the following form:

$$\frac{D_1}{g^2}\left(\frac{g^2}{f^2}G' + \frac{g^3}{f^3}G''\right) - \frac{D_2}{f^2}\left(\frac{g}{f}G' + \frac{g^2}{2f^2}G''\right) - \frac{G''}{2f^2}D_3 + F(G) = 0 \qquad \text{........(9)}$$

and since , $\frac{g}{f} = \psi$ hence equation (9) will be

$$\frac{D_1}{g^2}\left(\psi^2 G' + \psi^3 G''\right) - \frac{D_2}{f^2}\left(\psi G' + \frac{\psi^2}{2}G''\right) - \frac{D_3}{2f^2}G'' + F(G) = 0 \qquad \text{......(10)}$$

For simplicity, let

$$I_1 = \psi^2 G' + \psi^3 G'' \;,\; I_2 = \psi G' + \frac{\psi^2}{2}G'', \; I_3 = G'' \;,\; I_4 = F(G) \qquad \text{..........(11)}$$

then equation (10) becomes

$$\frac{D_1}{g^2}I_1 - \frac{D_2}{f^2}I_2 - \frac{D_3}{2f^2}I_3 + I_4 = 0 \qquad \text{.........(12)}$$

**(3) Constructing the Transformation:**

In the following, we will investigate six cases. In each case an ordinary differential equation is created to obtain the transformation (3), we classify these six-cases into two families according to the linearity of the ordinary differential equation.

In this section, we assume that $I_i \neq 0$, *i=1,2,3,4*. The two cases $I_1 = 0$ or $I_2 = 0$ do not lead to the pair of bilinear equations (2), so they are excluded although these cases may lead to identification of the function F. If $I_3 = 0$ we get trivial transformation while if $I_4 = 0$ then the partial differential equation (1) becomes a linear one.

**(A) Linear Cases:-** These cases arise when,

**A1:** $I_2 = \alpha I_3$ for some nonzero constant $\alpha$

In this case we have
$$\left(\frac{\psi^2}{2} - \alpha\right)G'' + \psi G' = 0 \qquad \text{….(13)}$$

This is a second order linear homogenous ordinary differential equation with variable coefficients, it's general solution is:

$$G = \begin{cases} c_1 \tanh^{-1}\dfrac{\psi}{\sqrt{2\alpha}} + c_2 & if \quad |\psi| < \sqrt{2\alpha} \\ \\ c_1 \coth^{-1}\dfrac{\psi}{\sqrt{2\alpha}} + c_2 & if \quad |\psi| > \sqrt{2\alpha} \end{cases} \qquad \text{……….(14)}$$

where $c_1$ and $c_2$ are constants of integration .

**A2:** $I_1 = \alpha I_2, \quad \alpha \neq 0$ then

$$\left(\psi^2 - \alpha\frac{\psi}{2}\right)G'' + (\psi - \alpha)G' = 0$$

This is a second order, variable coefficients equation. It's general solution is,

$$G = c_1\left(Ln\,\psi + \frac{\alpha}{2\psi}\right) + c_2 \qquad \text{………………(15)}$$

where $c_1$ and $c_2$ are constants of integration

**A3:** If $I_1 = \alpha I_3, \ \alpha \neq 0$ then

$$\left(\psi^3 - \frac{\alpha}{2}\right)G'' = -\psi^2 G'$$

This is a second order, variable coefficients equation. It's solution is obtained (using Maple package) and it is written as

$$G = \psi \quad hypergeom([1/3, 1/3], 4/3, \psi^3/\alpha)/\alpha^{1/3} \qquad \text{………..(16)}$$

**(B) Nonlinear Cases:-** These cases arise when,

**B1:** $I_3 = \alpha I_4, \ \alpha \neq 0,$ then

$$G'' = F(G) \qquad \text{…………………(17)}$$

Unfortunately, consideration of the ordinary differential equation (17) shows that, it is not an easy work to derive the transformation G unless F is known.

In 1960, the well known book of G. Murphy has contained a collection of more than two thousand equations with their solutions [6]. His tabulation of nonlinear second order ordinary differential equations had enable us to look for the solvable forms of equation (17). Table (1) shows these solvable cases.

| | **Equation *** | **Solution *** | **Equ. No.*** |
|---|---|---|---|
| 1 | $y'' = 0$ | $y = c_1 x + c_2$ | (1) |
| 2 | $y'' = a^2 y$ | $a(x+c_2) = \ln[ay + \sqrt{c_1 a^2 y^2}]$ | (2) |
| 3 | $y'' = 6y^2$ | $y = \wp(c_2 + x; 0, c_1)$ | (3) |
| 4 | $y'' = 2y^3$ | $\int dy \big/ \sqrt{y^4 + c_1} = x + c_2$ | (6) |
| 5 | $y'' = a + by + 2y^3$ | $\int dy \big/ \sqrt{2ay + by^2 + y^4 + c_1} = x + c_2$ | (7) |
| 6 | $y'' = a_0 + a_1 y + a_2 y^2 + a_3 y^3$ | $\int dy \big/ \sqrt{2a_0 y + a_1 y^2 + 2a_2 y^3/2 + a_3 y^4/2 + c_1}$ $= x + c_2$ | (12) |
| 7 | $y'' + a\sin y = 0$ | $y = 2\cos^{-1} sn[2a/k\,(x + c_2)]$ | (14) |
| 8 | $y'' + ae^y = 0$ | $\int dy \big/ i\sqrt{ae^y + c_1} = x + c_2$ | (15) |
| 9 | $2y'' = 1 + 12y^2$ | $y = \wp(x + c_2; 1, c_1)$ | (73) |
| 10 | $2y'' = y(a - y^2)$ | $y' = \sqrt{C + ay^2/2 - y^4/4}$ | (74) |

**Table (1)**

**The solvable cases**

***** As in the ref.[6]

**B2:** $I_2 = \alpha I_4$ , $\alpha \neq 0$ , in this case our equation will be

$$\psi G' + \frac{1}{2}\psi^2 G'' = \alpha F(G) \qquad \text{……………(18)}$$

This is an Euler equation [6] . It can be transformed to the following constant coefficients equation.

$$\dot{G} + \ddot{G} = 2\alpha F(G) \qquad \dot{G} = dG/ds \quad , \quad s = \ln\psi$$

This equation can be solved if $2\alpha F(G) = -\beta G$ for some constant $\beta$, and its general solution is as follows:

- If , $\beta = \frac{1}{4}$ then $G = \dfrac{(c_1 + c_2 \ln\psi)}{\sqrt{\psi}}$
- If , $\beta < \frac{1}{4}$ then $G = c_1\psi^{m_1} + c_2\psi^{m_2}$

  where $m_1, \ m_2$ are solutions to the equation $m^2 + m + \beta = 0$ .
- If , $\beta > \frac{1}{4}$ then
  $$G = \frac{1}{\sqrt{\psi}}[c_1 \cos(\sqrt{4\beta - \ln\psi}) + c_2 \sin(\sqrt{4\beta - \ln\psi})$$

where $c_1$ and $c_2$ are constants of integration .

If equation (18) is not linear, we can proceed as follows:

If we multiply both sides of the equation by $G'$ and integrate, we get

$$\frac{dG}{\sqrt{c + 2\alpha F\,(G)}} = \frac{d\psi}{\psi}$$

where c constant of integration.

Suppose that F(G) = -cos G , then

$$\int \frac{dG}{\sqrt{c - 2\alpha \cos G}} = \frac{d\psi}{\psi}$$

To find $I=\int\frac{dG}{\sqrt{c-2\alpha\cos G}}$ we proceed as in ref. [7] :

since $$\cos G=2\cos^2\frac{G}{2}-1$$

hence
$$I=\int\frac{dG}{\sqrt{c-2\alpha\left(2\cos^2 G/2-1\right)}}$$
or
$$I=\frac{1}{\sqrt{\alpha}}\int\frac{dG}{\sqrt{\left(c/\alpha+2\right)\left(1-\frac{4}{c/\alpha+2}\cos^2 G/2\right)}}$$
Now, let

$k^2=\frac{4}{c/\alpha+2}$ and $\cos G/2=v$ , then
$$I=\frac{k}{2\sqrt{\alpha}}\int\frac{-2dv}{\sqrt{1-v^2}\sqrt{1-k^2v^2}}$$

Thus the dependent-variable transformation which lead to bilinearization of sG equation are necessarily of the form
$$G=2\cos^{-1}sn\left(\frac{2\sqrt{\alpha}}{k}\left(\ln\psi+c\right)\right)$$
where *sn* is a Jacobian elliptic function.

**B3:** If $I_1=\alpha I_4$ , $\alpha\neq 0$ , in this case
$$\psi^3G''+\psi^2G'=\alpha F(G) \qquad \text{…………(19)}$$

One can see that this equation cannot be simplified unless *F=0* or *F=G*. If *F=0* then we get the transformation $G=c_1\ln\psi+c_2$ . When *F=G* , the following transformation is useful

$$\psi = \frac{-2}{z}, \quad G = u(z)$$

and equation (19) will be transformed to the equation

$$zu'' + u' + \frac{\alpha}{2}u = 0$$

The last equation is the confluent hypergeometric differential equation; also called the Kummer equation and the Pochhammer-Barnes equation. A solution of it is confluent hypergeometric function, a Kummer function, or a Pochhammer function [6].

## (4) Identifying the Function *F:*

For the linear cases, when the transformation *G* is known, one has to identify the function *F* in the equation. We illustrate that when

$$G = \tan^{-1}\psi \qquad \text{………………..(20)}$$

**Proposition**:- Equation (1) admits the transformation (20) iff

$$F(u) = (\alpha_1 + \alpha_2 \tan u + \alpha_3 \tan^2 u)\frac{(1-\tan^2 u)}{(1+\tan^2 u)^2}$$

proof :

Substituting G and its derivatives in(11), we get

$$I_1 = \psi^2 G' + \psi^3 G'' = \frac{g^2\left(f^2 - g^2\right)}{\left(f^2 + g^2\right)^2}$$

$$I_2 = \psi G' + \frac{\psi^2}{2} G' = \frac{-gf^3}{\left(f^2 + g^2\right)^2}$$

$$I_3 = 2I_2$$

$$I_4 = F(G)$$

and (12) will be

$$\left( D_1 \frac{\left(f^2 - g^2\right)}{(f^2+g^2)^2} + (D_2 + D_3)\frac{gf}{(f^2+g^2)^2} \right) + F(\tan^{-1}\psi) = 0 \quad \text{…..(21)}$$

This identity must be, homogenous in f and g ,splitted into a pair of equations, therefore

$$F\left(\tan^{-1}\psi\right) = \mu\left(f, g\right)\frac{f^2 - g^2}{\left(f^2+g^2\right)^2} \quad \text{……..(22)}$$

where $\mu = \alpha_1 f^2 + \alpha_2 fg + \alpha_3 g^2$ ,for some constants $\alpha_1$ , $\alpha_2$ and $\alpha_3$ . Now, if ,

$\tan^{-1}\psi = u$ then , $\psi = \frac{g}{f} = \tan u$ and F is identified.

We get special cases when $\mu = 1+\psi^2,\quad 1-\psi^2,\quad 4\psi$ .

## References

[1] Drazin P. G., "Soliton", Cambridge University press, Cambridge, London, New York ,1983..

[2] Caudrey P.J., "Hirota's Method of Solving Soliton-type Equations", Lecture given at NATO Advanced Study Institute on Nonlinear Equations in Physics and Mathematics, Istanbul, August 1977.

[3] Hietarinta J.: Introduction to the bilinear method, in ``Integrability of Nonlinear Systems", eds. Y. Kosman-Schwarzbach, B. Grammaticos and K.M. Tamizhmani, Lecture Notes in Physics Vol. 495 (Springer, 1997) pp. 95-103.

[4] Hietarinta J. "A search of bilinear equations passing Hirota's three-soliton condition: III. sine-Gordon-type bilinear equations", J. Math. Phys. 28, 2586-2592 1987.

[5] Al-Rikabi W.F. "Sine-Gordon-type bilinear equation" MSc. thesis, Al- Mustansiryah University, 2005

[6] Murphy G.M., "Ordinary Differential Equations and Their Solutions", D. Van Nostrand Company, Inc. New York, 1960.

[7] Osborne, A.D. and Stuart, A.E.G., "On the separability of the sine-Gordon equation and similar quasilinear partial differential equations" J. Math. Phys. 19(7), 1573- 1579, 1978.