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Euclidean Space , Vector permutation And Vec Operator

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Abstract

In this project , we defined and proofed some properties of the vec and vech operators for matrices by using the standard bases of n-Eclidean space . we connected this standard bases with each of the Kronecker product operation of matrices and the permutation matrix.

1- Introduction

1-1 Kronecker product of matrices[2]
Let $A=(a_{ij})$ be matrix of degree $m\times n$, B Matrix of degree $p\times q$ we define Kronecker product of two matrices A,B is of degree $mp\times nq$

$$A\otimes B=[a_{ij}B]$$

1-2 Vector of matrix (vec)[1]
Let $A=(a_{ij})$ be matrix of degree $m\times n$ we define vec(A) is vector of degree $mn\times 1$ write it

$$vec(A)=\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ . \\ . \\ . \\ a_{m1} \\ a_{12} \\ a_{22} \\ . \\ . \\ . \\ a_{m1} \\ . \\ . \\ . \\ a_{1n} \\ a_{2n} \\ . \\ . \\ . \\ a_{mn} \end{bmatrix}_{mn\times 1}$$

1-3 Vector half of matrix (vech)[1]

Let $A=(a_{ij})$ be symmetric matrix of degree $n\times n$ we define $vech(A)$ is vertical vector contains elements equal $n(n+1)/2$ and write :-

For upper part of matrix with diagonal

$$vech(A)=\begin{bmatrix} a_{11} \\ a_{12} \\ a_{22} \\ a_{13} \\ a_{23} \\ a_{33} \\ . \\ . \\ . \\ a_{nn} \end{bmatrix}_{nn\times 1}$$

For below part of matrix with diagonal

$$vech(A)=\begin{bmatrix} a_{11} \\ a_{21} \\ . \\ . \\ . \\ a_{n1} \\ a_{22} \\ a_{32} \\ . \\ . \\ . \\ a_{n2} \\ . \\ . \\ . \\ a_{nn} \end{bmatrix}_{nn\times 1}$$

1- 4 Some the principle basis of n Euclidian space

We can use the basis of n Euclidian space to write matrices and some other operation up on them . To prove mane of features which we are going to mention them and refer to later as proceed ;

$$e_i^s=[0\cdots 010\cdots 0]^T$$

Since e_i^s is vector of degree $s\times 1$ and that all it’s elements are zero except the one in position i as it's one .

2- Properties of vec and vech operator for matrices [3]

Before talking about some properties of vec operator and its proof , its important to know how to write vec operator by using the standard bases of n-Euclidean space with each matrix , multiplication of two matrices and Kronecker product operation of two matrices

Let $A=(a_{ij})$ be a matrix of degree $m\times n$, and $B=(b_{ij})$ be matrix of degree $n\times q$

$$A=\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}(e_i^m\otimes e_j^{n'})$$

$$vec(A)=vec(\sum_{i=1}^{m}\sum_{j=1}^{n}a_{ij}(e_i^m\otimes e_j^{n'})=\sum_{j=1}^{n}\sum_{i=1}^{m}a_{ij}(e_j^n\otimes e_i^m)$$

$$AB=\sum_{i=1}^{m}\sum_{j'=1}^{q}(\sum_{k=1}^{n}a_{ik}b_{kj'})e_i^m\otimes e_{j'}^{q'}$$

$$vec(AB)=vec\left[\sum_{i=1}^{m}\sum_{j'=1}^{q}(\sum_{k=1}^{n}a_{ik}b_{kj'})e_i^m\otimes e_{j'}^{q'}\right]$$

$$=\sum_{j'=1}^{q}\sum_{i=1}^{m}(\sum_{k=1}^{n}a_{ik}b_{kj'})e_j^q\otimes e_i^m$$

$$A\otimes B=\sum_{i=1}^{m}\sum_{i'=1}^{p}\sum_{j=1}^{n}\sum_{j'=1}^{q}a_{ij}b_{i'j'}(e_i^m\otimes e_{i'}^p)(e_j^{n'}\otimes e_{j'}^{q'})$$

$$vec(A\otimes B)=vec[\sum_{i=1}^{m}\sum_{i'=1}^{p}\sum_{j=1}^{n}\sum_{j'=1}^{q}a_{ij}b_{i'j'}(e_i^m\otimes e_{i'}^p)(e_j^{n'}\otimes e_{j'}^{q'})]$$

$$=\sum_{j=1}^{n}\sum_{j'=1}^{q}\sum_{i=1}^{m}\sum_{i'=1}^{p}a_{ij}b_{i'j'}(e_j^n\otimes e_{j'}^q)\otimes(e_i^m\otimes e_{i'}^p).$$

Also , its possible to express vech operator by using some standard bases of n-Euclidean space and for each of the cases the upper triangle or the lower triangle of the matrix and as proceed :
a-the upper triangle of the matrix

$$vech(A)=\sum_{j=1}^{m}\sum_{i=1}^{m}a_{ij}(e_{i+[(j-1)j/2}^{m(m+1)/2})$$

b-the lower triangle of the matrix

$$vech(A)=\sum_{j=1}^{m}\sum_{i=1}^{m}a_{ij}(e_{i+(j-1)(m-j/2)}^{m(m+1)/2})$$

Vec Permutation matrices[3]

The vectors $vec(A), vec(A')$ have the same elements for each , but with different sequence . If $I_{(m,n)}$ is permutation matrix
$vec(A)=I_{(m,n)}vec(A')$.

Since A be a matrix of degree $m\times n$; $I_{(m,n)}$ is called the matrix of permutation . its clear that the symbol $I_{(m,n)}$ is a rearrangement of the matrix I_{mn} which is possible to obtain it by taking each row of order n beging from the first element, and each row of order n begins from the second element,and so on..
For example

$$I_{(2,3)}=\left[\begin{array}{ccc|ccc}1&0&0&0&0&0\\0&0&0&1&0&0\\\hline 0&1&0&0&0&0\\0&0&0&0&1&0\\\hline 0&0&1&0&0&0\\0&0&0&0&0&1\end{array}\right].$$

Also the row 6,4,2,5,3,1 sequentially for identity matrix I_6 .
When $B=B', n\times n, B=\sum_{i\le j}b_{ij}A_{ij}$, where

$$A_{11}=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad A_{12}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\quad A_{22}=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\quad A_{13}=\begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix}.$$

And so on ,all $n\times n$ with each 0 denoting a dimensionally appropriate matrix of zero , then

[2] 2.1-$vec(b)=\sum_{i\le j}b_{ij}vec(A_{ij})=L_{n'}vech(B)$,

where

$$L_{n'}=[vec(A_{11}),vec(A_{12}),vec(A_{22}),...,vec(A_{nn})]$$

Proof: we will prove this feature when $n = 3$ to avoid complicated symbols . The proof in the general case doesn't differ in case when $n \neq 3$

$$A_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$A_{33} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$vec(\mathbf{B}) = b_{11}vec(A_{11}) + b_{12}vec(A_{12}) + b_{13}vec(A_{13}) + b_{22}vec(A_{22}) + b_{23}vec(A_{23}) + b_{33}vec(A_{33})$$

$$= b_{11}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_{12}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_{13}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b_{22}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b_{23}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b_{33}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{12} \\ b_{22} \\ b_{23} \\ b_{13} \\ b_{23} \\ b_{33} \end{bmatrix}$$

$$L_{3'}vech(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{12} \\ b_{22} \\ b_{13} \\ b_{23} \\ b_{33} \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \\ b_{12} \\ b_{22} \\ b_{23} \\ b_{13} \\ b_{23} \\ b_{33} \end{bmatrix}$$

$$vec(B) = \sum_{i \le j} b_{ij} vec(A_{ij}) = L_{3'}vech(B)$$

[2] 2.2- $(L_n L_{n'})^{-1} L_n vec(B) = vech(B)$

By using the standard bases of n- Euclidean space

$$(L_n L_{n'})^{-1} L_n \left[\sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} (e_j^n \otimes e_i^n) or \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \le j}}^{n} b_{ij} (e_j^n \otimes e_i^n) + (e_i^n \otimes e_j^n) \right] = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} e_{i+(j-1)(m-j/2)}^{m(m+1)/2}$$

[2] 2.3 - $L_{n'}(L_n L_{n'})^{-1} L_n = \frac{1}{2}(I_{n^2} + I_{(n,n)})$

Proof: we will prove this feature when $n = 3$ to avoid complicated symbols . The proof in the general case doesn't differ in case when $n \neq 3$

$$L_{3'}(L_3L_{3'})^{-1}L_3=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0&0\\0&0&0&0&0&\frac{1}{2}&0&\frac{1}{2}&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0&0\\0&\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&\frac{1}{2}&0&\frac{1}{2}&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0&0\\0&0&0&0&0&\frac{1}{2}&0&\frac{1}{2}&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}$$

$$\tfrac{1}{2}(I_9+I_{(3,3)})=\tfrac{1}{2}\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}+\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0\\0&1&0&0&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0&0\\0&\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&\frac{1}{2}&0&\frac{1}{2}&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0&0\\0&0&0&0&0&\frac{1}{2}&0&\frac{1}{2}&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}$$

Then

$$L_{3'}(L_3L_{3'})^{-1}L_3=\tfrac{1}{2}(I_9+I_{(3,3)})$$

$\sin ce K_n = L_{n'}(L_nL_{n'})^{-1}$

[2] 2.4 - $K_nL_n vec(B)=vec(B)$

Proof: we will prove this feature when $n=3$ to avoid complicated symbols . the proof in the general case doesn't differ in case when $n\neq 3$

$$K_3L_3vec(B)=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0&0\\0&\frac{1}{2}&0&\frac{1}{2}&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&0&\frac{1}{2}&0&\frac{1}{2}&0\\0&0&\frac{1}{2}&0&0&0&\frac{1}{2}&0&0\\0&0&0&0&0&\frac{1}{2}&0&\frac{1}{2}&0\\0&0&0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}b_{11}\\b_{12}\\b_{13}\\b_{12}\\b_{22}\\b_{23}\\b_{13}\\b_{23}\\b_{33}\end{bmatrix}=\begin{bmatrix}b_{11}\\b_{12}\\b_{13}\\b_{12}\\b_{22}\\b_{23}\\b_{13}\\b_{23}\\b_{33}\end{bmatrix}=vec(B)$$

[2] 2.5 - $tr(C'D)=[vec(C)]'vec(D)$

Proof: we will prove this feature when $n=3$ to avoid complicated symbols . the proof in the general case doesn't differ in case when $n\neq 3$

Let

$$C=\begin{pmatrix}c_{11}&c_{12}&c_{13}\\c_{21}&c_{22}&c_{23}\\c_{31}&c_{32}&c_{33}\end{pmatrix},D=\begin{pmatrix}d_{11}&d_{12}&d_{13}\\d_{21}&d_{22}&d_{23}\\d_{31}&d_{32}&d_{32}\end{pmatrix}$$

$$(C'D)=\begin{pmatrix}c_{11}&c_{21}&c_{31}\\c_{12}&c_{22}&c_{32}\\c_{13}&c_{23}&c_{33}\end{pmatrix}\begin{pmatrix}d_{11}&d_{12}&d_{13}\\d_{21}&d_{22}&d_{23}\\d_{31}&d_{32}&d_{33}\end{pmatrix}$$

$$=\begin{pmatrix}c_{11}d_{11}+c_{21}d_{21}+c_{31}d_{31}&&\\&c_{12}d_{12}+c_{22}d_{22}+c_{32}d_{32}&\\&&c_{13}d_{13}+c_{23}d_{23}+c_{33}d_{33}\end{pmatrix}$$

$$tr(C'D)=c_{11}d_{11}+c_{21}d_{21}+c_{31}d_{31}+c_{12}d_{12}+c_{22}d_{22}+c_{32}d_{32}+c_{13}d_{13}+c_{23}d_{23}+c_{33}d_{33}$$

$$(vec(c))'vec(D)=\begin{bmatrix}c_{11}&c_{21}&c_{31}&c_{12}&c_{22}&c_{32}&c_{13}&c_{23}&c_{33}\end{bmatrix}\begin{bmatrix}d_{11}\\d_{21}\\d_{31}\\d_{12}\\d_{22}\\d_{32}\\d_{13}\\d_{23}\\d_{33}\end{bmatrix}$$

$$=c_{11}d_{11}+c_{21}d_{21}+c_{31}d_{31}+c_{12}d_{12}+c_{22}d_{22}+c_{32}d_{32}+c_{13}d_{13}+c_{23}d_{23}+c_{33}d_{33}$$

[3] 2.6 - $vec(ABC)=(C'\otimes A)vec(B)$

By using the standard bases of n-Euclidean space

Let A be matrix of degree $m\times n$, B matrix of degree $n\times q$, and C matrix of degree $p\times q$

$$B = (b_{ij}) = \sum_{i=1}^{n}\sum_{j=1}^{p} b_{ij}(e_i^p e_j^{n'})$$

$$ABC = \sum_{i=1}^{n}\sum_{j=1}^{p} b_{ij}(Ae_i^p)(e_j^{n'})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{p} b_{ij}\left[(Ae_i^p)(C'e_j^n)'\right]$$

$$l.H.S = vec(ABC) = \sum_{i=1}^{n}\sum_{j=1}^{p} b_{ij} vec\left[(Ae_i^p)(e_j^{n'}C)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{p} b_{ij}(C' \otimes A) vec(e_i^p e_j^n)$$

$$= (C' \otimes A)\sum_{i=1}^{n}\sum_{j=1}^{p} b_{ij} vec(e_i^p e_j^{n'})$$

$$= (C' \otimes A)\sum_{i=1}^{n}\sum_{j=1}^{p} b_{ij}(e_j^n \otimes e_j^p)$$

$$= (C' \otimes A) vec(B)$$

$$= R.H.S$$

References

[1] Harold V.Henerson and S.R.Searle . (1979):vec and vech operators for matrices , with Some uses in jacobians and multivariate statistics . the Canadian jornal of statistics vol.7.no.1979.page 65-81.

[2] Graybill ,F.A .(1983).”Matrices with application in statistics “. Wadsorth International Group, Belmont, California.

[3] Alsharifi,H.2005.,”Using the n-Euclidean Space For vec and vech Operators For matrices With Applications” Msc thesis.