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Solution of Delay Fractional Differential Equations by Using Linear Multistep Method

حل المعادلات التفاضلية الكسرية التباطؤية باستخدام طريقة متعدد الخطوات الخطية

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Abstract:

The objective of this paper to find the solution of delay fractional differential equations as well as, their numerical solution. The modification was mode by introducing the new method which is linear multistep methods, which is solving numerically ordinary differential equations, delay differential equations and delay fractional differential equations. Furthermore this paper presents the statement and proof of the fundamental theorem of convergent and stability of delay fractional differential equations, this had been done with some modification.

الخلاصة

الهدف من هذا البحث هوِ لإيجاد حَلِّ المعادلات التفاضليةِ الكسورية التباطؤية بالإضافة إلى حَلّهم العددي. التعديل كَانَ نمطاً بتَقديم الطريقةِ الجديدةِ التي هي طرقُ متعددةُ الخطوات الخطيّةَ، التي تَحْلُّ معادلات تفاضليةَ عاديةَ بشكل عددي، المعادلات التفاضلية التباطؤية والمعادلات التفاضليةَ الكسرية التباطؤية. اضافة ان هذا البحث يبين ويبرهنَ النظريةِ الأساسيةِ المتقاربهِ وإستقرار المعادلات التفاضليةِ الكسريةِ التباطؤية، هذا مع عُمِلَ بَعْض التعديلِ عليها.

1.Introduction

The delay fractional differential equations play an important role in the theory of functional differential equations.In recent year, the theory of this class of equations had become an independent trend and literature on this subject comprises over 1000 titles. delay fractional differential equations have been studied during the last two centuries. The significance of these equations lies in their ability to describe processes with retarded time. The importance of these equations in various branches of technology, economics, biology, medical science, physics and social science has been recognized recently and has caused mathematics to study them with increasing interest.One of the most important branches of fractional differential equations is the so called delay fractional differential equations, which is an fractional differential equation consist of one or some its arguments evaluated at a time which is differ by any fixed number or variable which is called the time or lag. Also , theory of delay fractional differential equation can be considered as a generalization to the theory of fractional differential equations and using Linear Multistep Methods (LMM's) to solve delay fractional differential equations of the retarded and deriving some of the well known numerical methods to solve delay fractional differential equations.

2.Method of solution delay fractional differential equations

Because of the initial condition which a function given for time step interval with length equals τ, must find this solution for $t \geq t_0$ divided in to steps with length τ also.

Hence, we must find the solution for each time step. The best well know method for solving delay fractional differential equations is the method of steps (solution of fractional differential equations), [Driver, 1977],[Halanay, 1966], [El'sgol'c and Norkin, 1964] [Basim 2004], which is for simplicity with out lost of generality and illustration purpose of solving delay fractional differential equations of the form:

$$x^q(t) = f(t, x(t), x(t-\tau), x'(t-\tau)) \quad \ldots\ldots(1)$$

for each equation the solution is constructed step by step as follows:

Given that a function $\varphi(t)$ continues on $[t_0-\tau,t_0]$, therefore one can form the solution in the next time step interval $[t_0,t_0+\tau]$, by solving the following equation:

$$x^q(t) = f(t, x(t), \varphi(t-\tau), \varphi'(t-\tau)), \quad \text{for } t_0 \leq t \leq t_0+\tau, \ 0\leq q<1$$

with initial condition $x(t_0)=\varphi(t_0)$. If we suppose that $\varphi_1(t)$ is the existing by virtue of continuity hypotheses. Similarly if $\varphi_1(t)$ is defined on the whole segment $[t_0,t_0+\tau]$, hence by forming the new equation:

$$x^q(t) = f(t, x(t), \varphi_1(t-\tau), \varphi_1'(t-\tau)) , \quad \text{for } t_0+\tau \le t \le t_0+2\tau, \ 0\le q<1$$

one can find the solution of this equation $\varphi_2(t)$ to the time step $[t_0+\tau,t_0+2\tau]$, with the initial condition $x(t_0+\tau)=\varphi_1(t_0+\tau)$.

In general, by assuming that $\varphi_{k-1}(t)$ is defined on the interval $[t_0+(k-2)\tau, t_0+(k-1)\tau]$, $\forall$ k=1,2,3,…, the following equation is constructed:

$$x^q(t) = f(t, x(t), \varphi_{k-1}(t-\tau), \varphi_{k-1}'(t-\tau)), \text{ for } t_0+(k-1)\tau\le t \le t_0+k\tau, \ 0\le q<1$$

and consider a solution of this equation with the initial condition:

$$x(t_0+(k-1)\tau) = \varphi_{k-1}(t_0+(k-1)\tau),$$

which is denoted by $\varphi_k(t)$ for all k=1,2…. Now, consider the example to explain the method of steps.

Example (2.1):

Consider the retarded delay fractional differential equations:

$$x^{\frac{1}{2}}(t) = t^2 + tx(t-1) \quad \text{...}(2)$$

with initial condition $x(t)=\varphi_0(t)=t$, for $-1\le t\le 0$, and to find the solution in the first step interval [0, 1], we have to solve the following equation:

$$x^{\frac{1}{2}}(t) = t^2 + t\varphi_0(t-1), \quad \text{for } 0 \le t \le 1 \quad \text{..}(3)$$

then, we have:

$$x^{\frac{1}{2}}(t) = 2t^2 - t, \quad \text{for } 0 \le t \le 1 \quad \text{...}(4)$$

by using the solution of fractional differential equations, [Basim, 2004], applying $\frac{d^{-\frac{1}{2}}}{dt^{-\frac{1}{2}}}$ to both sides of equation (4), we get:

$$x(t) = \frac{4}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} - \frac{1}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + c_1 t^{-\frac{1}{2}} , \ 0 \le t \le 1$$

in order to find the solution in the second step interval, suppose that:

$$\varphi_1(t) = x_1(t) = \frac{4}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} - \frac{1}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + c_1 t^{-\frac{1}{2}} , \ 0 \le t \le 1$$

is the initial condition. Since $\varphi_1(t)$ is defined on the whole segment [0, 1], hence by forming a new equation:

$$x^{\frac{1}{2}}(t) = t^2 + t\varphi_1(t-1) , \ \text{for } 1 \le t \le 2 \quad \text{..}(5)$$

with initial condition,

$$\varphi_1(t) = \frac{4}{\Gamma(\frac{7}{2})} t^{\frac{5}{2}} - \frac{1}{\Gamma(\frac{5}{2})} t^{\frac{3}{2}} + c_1 t^{-\frac{1}{2}} , \ \text{for } 0 \le t \le 1$$

one can find the solution in the next step interval [1, 2], and solving equation (5), for x(t) we have:

$$\begin{aligned} x^{\frac{1}{2}}(t) &= t^2 + t\varphi_1(t-1) \\ &= t^2 + \frac{4}{\Gamma(\frac{7}{2})} t(t-1)^{\frac{5}{2}} - \frac{1}{\Gamma(\frac{5}{2})} t(t-1)^{\frac{3}{2}} + c_1 t(t-1)^{-\frac{1}{2}}, \ 1 \le t \le 2 \text{................}(6) \end{aligned}$$

by using fractional differential equation applying $\frac{d^{-\frac{1}{2}}}{dt^{-\frac{1}{2}}}$ to both sides of equation (6), we get:

$$x(t)=\frac{2}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{2}{3}t(t-1)^3-\frac{2}{\Gamma(\frac{9}{2})}(t-1)^4-\frac{1}{2}(t-1)^2+\frac{1}{12}(t-1)^3$$
$$+c_1\Gamma(\tfrac{1}{2})-c_1\Gamma(\tfrac{1}{2})(t-1)\ +c_2t^{-\frac{1}{2}}\ \ldots\ldots\ldots\ldots(7)$$

for all, $1 \le t \le 2$.
therefore x(t) is the desired second step solution. similarly, we proceed to the next intervals.
3.The Existence and Uniqueness Theorem
As in ordinary and fractional differential equations the existence and uniqueness delay fractional differential equations could also be established for this purpose, consider the delay fractional differential equations:

$y^q(t)=f(t, y(t), y(t-w(t))$

with initial condition,

$y(t)=\varphi(t),\quad t_0-w(t) \le t \le t_0$

and suppose that R denoted the rectangular region defined by:

$R=\{(t, y):t_0-a-w \le t \le t_0+a,\ |y-y_0| \le b\}$

3.1. Existence theorem
consider the initial value problem of delay fractional differential equations:

$y^q(t)=f(t, y(t), y(t-w(t)))$ ....(8)

with initial condition,

$y(t)=\varphi(t),\quad t_0-w(t) \le t \le t_0$

and suppose that f, $\frac{\partial f}{\partial y(t)}$ and $\frac{\partial f}{\partial y(t-w(t))}$ are continuous on a region R, and f satisfies Lipschitz constant condition. Then equation (8) has a solution on R . one hypothesis of the existence in the rectangle R . It is follows that f must be bonded in R. let M>0 be number such that $|f(t, y(t), y(t-w(t)))| \le M$, for every point in R.
if we now take $\alpha=\min\{a, \frac{b}{M}\}$, and define the rectangle R_1 to $R_1=\{(t, y):|t-t_0| \le \alpha,\ |y-y_0| \le b\}$ clearly R_1 is a subset of R. If we now consider the sequence of functions:

$$\varphi_n(t)=\varphi_0+\int_{t_0}^{t} f(s,\varphi_{n-1}(s),\varphi_{n-1}(s-w(s)))ds$$

thus, we have the sequence of solutions will converge.

3.2. The uniqueness theorem
The solution of the delay fractional differential equation:

$y^q(t)=f(t, y(t), y(t-w(t)))$

with initial condition $y(t)=\varphi(t)$, $t_0-w(t) \le t \le t_0$, unique.
Proof:
Suppose that $\varphi_1(t)$ and $\varphi_2(t)$ are any two solution, then:

$$\varphi_1-\varphi_2=\int_{t_0}^{t}[f(s,\varphi_1(s),\varphi_1(s-w(s))-f(s,\varphi_2(s),\varphi_2(s-w(s))]ds$$

and suppose that K is the Lipschitz constant condition of the function f in a compact subset R_1 containing $(t, \varphi_1(t), \varphi_1(t-w(t)))$ and $(t, \varphi_2(t), \varphi_2(t-w(t)))$, for each $t\in R_1$. then:

$$|\varphi_1-\varphi_2|=\left|\int_{t_0}^{t}[f(s,\varphi_1(s),\varphi_1(s-w(s))-f(s,\varphi_2(s),\varphi_2(s-w(s))]ds\right|$$

$$\leq \int_{t_0}^{t} \left|[f(s,\varphi_1(s),\varphi_1(s-w(s)) - f(s,\varphi_2(s),\varphi_2(s-w(s))]\right| ds$$

$$\leq K\int_{t_0}^{t}\left[\left|\varphi_1(s)-\phi_2(s)\right| + \left|\varphi_1(s-w(s))-\varphi_2(s-w(s))\right|\right] ds$$

$$\leq K\alpha \sup_{a\leq s\leq b}\left[\left|\varphi_1(s)-\phi_2(s)\right| + \left|\varphi_1(s-w(s))-\varphi_2(s-w(s))\right|\right]$$

taking $0< K\alpha <1$, then we have:

$$[\left|\varphi_1(s)-\phi_2(s)\right| + \left|\varphi_1(s-w(s))-\varphi_2(s-w(s))\right|] < \varepsilon$$

and hence:

$$|\varphi_1(t)-\varphi_2(t)| = 0$$

therefore $\varphi_1(t) = \varphi_2(t)$, foe each $t \in R_1$.

4.Numerical Solution of Delay Fractional Differential Equations Using Linear Multistep Methods (LMM's).

The present section consists of a generalization and modification of the general LMM to solve DFDE. For this purpose, well consider, for simplicity and without lose of generality the retarded DFDE. , which has the form :

$$y^{(q)}(t) = f(t, y(t), y(t-\tau)), \qquad \text{.......(9)}$$

with initial condition:

$$y(t) = \varphi(t),\ t_0-\tau \leq t \leq t_0,\ 0 \leq q < 1,$$

where τ is a fixed number and $y(t_0) = y_0 = \varphi(t_0)$.

4.1. General LMM's in DFDE's:

Consider IVP given be equation (9), where $y(t) = \varphi(t)$, $t_0-\tau \leq t \leq t_0$, and $y(t_0) = y_0 = \varphi(t_0)$. Hence the general form of LMM for solving equation (9) is given by:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f(x_{n+j}, y_{n+j}, y_{n+j-\tau}) \qquad \text{.......(10)}$$

where $x_{n+j}=x_0 + jh$, $y_{n+j} = y(x_n+jh)$ and $y_{n+j-\tau} = y(x_n+jh-\tau)$ and α_j, β_j are constant. Hence:

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f(x_{n+j}, y_{n+j}, \varphi_{n+j-\tau})$$

expanded this formulate, gives:

$$\alpha_0 y_n+\alpha_1 y_{n+1}+\ldots+\alpha_k y_{n+k} = h[\beta_0 f(x_n, y_n, \varphi_{n-\tau})+\beta_1 f(x_{n+1}, y_{n+1}, \varphi_{n+1-\tau})+\ldots+\beta_k f(x_{n+k}, y_{n+k}, \varphi_{n+k-\tau})]$$

which is general Delay Fractional Linear Multistep Method (DFLMM).

As a classification to the DFLMM, we say that the DFLMM is of explicit type if $\beta_k=0$ and it is implicit type if $\beta_k\neq 0$.

Also in some cases, implicit method may reduce to explicit methods depending on the function f whether it is linear or non-linear.

The local truncation error of DFLMM is defined to be,

$$T_n = \sum_{j=0}^{k}\alpha_j y(x_n + jh) - h\sum_{j=0}^{k}\beta_j f(x_n + jh, y(x_n + jh), y(x_n + jh - \tau)) \qquad \text{.......(11)}$$

or

$$T_n = \sum_{j=0}^{k} \alpha_j y(x_n + jh) - h\sum_{j=0}^{k} \beta_j y^{(q)}(x_n + jh)$$

after expanding the function $y(x_n+jh)$ and it's by using solution fractional differential equation applying $\frac{d^{q-1}}{dt^{q-1}}$ to both sides, and collocation terms in equation (11).

$$y(x_n + jh) = y(x_n) + \frac{y'(x_n)}{1!}(x_n + jh - x_n) + \frac{y''(x_n)}{2!}(x_n + jh - x_n)^2 + \ldots$$

$$= y(x_n) + \frac{y'(x_n)}{1!} jh + \frac{y''(x_n)}{2!}(jh)^2 + \ldots$$

and

$$y^{(q)}(x_n + jh) = y^{(q-1)}(x_n) + \frac{y^{(q)}(x_n)}{1!} jh + \frac{y^{(q+1)}(x_n)}{2!}(jh)^2 + \ldots + ax_n^{q-1}$$

and so on . by substituting in the local truncation error, we get:

$$T_n = \sum_{j=0}^{k} \alpha_j y_{n+j} - h\sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}, y_{n+j-\tau})$$

$$= \sum_{j=0}^{k} \alpha_j y_{n+j} - h\sum_{j=0}^{k} \beta_j f(x_{n+j}, y_{n+j}, \varphi_{n+j-\tau})$$

$=\alpha_0 y_n+\alpha_1 y_{n+1}+\ldots+\alpha_k y_{n+k}$ - $h[\beta_0 f(x_n, y_n, \varphi_{n-\tau})+\beta_1 f(x_{n+1}, y_{n+1}, \varphi_{n+1-\tau})+\ldots+\beta_k f(x_{n+k}, y_{n+k}, \varphi_{n+k-\tau})]$

$$= (\alpha_k + \alpha_{k-1} + \ldots + \alpha_2 + \alpha_1 + \alpha_0)y^{(q-1)}(x_n) + h(k\alpha_k + (k-1)\alpha_{k-1} + \ldots + 2\alpha_2$$

$$+ \alpha_1 - \beta_0 - \beta_1 \ldots - \beta_k)y^{(q)}(x_n) + h^2(\frac{k^2}{2}\alpha_k + \frac{(k-1)^2}{2}\alpha_{k-1} + \ldots + 2\alpha_2$$

$$+ \tfrac{1}{2}\alpha_1 - \beta_1 - 2\beta_2 \ldots - k\beta_k)y^{(q+1)}(x_n) + h^3(\frac{k^3}{3!}\alpha_k - \frac{(k-1)^3}{3!}\alpha_{k-1}$$

$$+ \ldots + \tfrac{8}{3!}\alpha_2 + \tfrac{1}{3!}\alpha_1 - \tfrac{1}{2!}\beta_1 - 2\beta_2 \ldots - \tfrac{k^2}{2!}\beta_k)y^{(q+2)} + \ldots + ax^{q-1}$$

therefore

$T_n=C_0 y^{(q-1)}(x_n) + C_1 h y^{(q)}(x_n) + C_2 h^2 y^{(q+1)}(x_n)+\ldots+C_p h^p y^{(q+p)}(x_n)+\ldots+ax^{q-1}$ .... (12)

where C_0, C_1, …, C_p, a are constants, and a simple calculation yields the following formula for the constants C_i's, i=0,1,…,p, in terms of the coefficients α_j and β_j:

$$C_0 = \alpha_k + \alpha_{k-1} + \ldots + \alpha_2 + \alpha_1 + \alpha_0$$

$$C_1 = k\alpha_k + (k-1)\alpha_{k-1} + \ldots + 2\alpha_2 + \alpha_1 - \beta_0 - \beta_1 \ldots - \beta_k$$

$$C_2 = \frac{k^2}{2}\alpha_k + \frac{(k-1)^2}{2}\alpha_{k-1} + \ldots + 2\alpha_2 + \tfrac{1}{2}\alpha_1 - \beta_1 - 2\beta_2 \ldots - k\beta_k$$

$$C_3 = \frac{k^3}{3!}\alpha_k - \frac{(k-1)^3}{3!}\alpha_{k-1} + \ldots + \tfrac{8}{3!}\alpha_2 + \tfrac{1}{3!}\alpha_1 - \tfrac{1}{2!}\beta_1 - 2\beta_2 \ldots - \tfrac{k^2}{2!}\beta_k$$

$$\vdots \quad C_p = \tfrac{1}{p!}(k^p\alpha_k + (k-1)^p\alpha_{k-1} + \ldots + 2^p\alpha_2 + \alpha_1) - \frac{1}{(p-1)!}(\beta_1 + 2^{p-1}\beta_2 + \ldots + k^{p-1}\beta_k)$$

these formula can be used to derived a certain DFLMM of given structure.

5. Illustrative example

Example (5.1):

Consider the Delay fractional differential equation:

$$x^{\frac{1}{2}}(t) = t^2 + tx(t-1),$$

with initial condition:

$x(t)=\varphi(t)=t$, for $-1 \le t \le 0$,

and suppose that we want to find the numerical solution in first time step, with step length h=0.1 given by:

$y_{n+1}=y_n+hf(x_n, y_n, y_{n-\tau})$,

by using the computer program to find the solution to the first time step, get the results presented in table (5.1) and it's comparison with the exact solution.

Table (5.1).

x_i	Numerical solution	Exact solution	Absolute Error
0.1	0	-0.00433	0.00433
0.2	-0.007	-0.01466	0.00766
0.3	-0.017	-0.027	0.01
0.4	-0.026	-0.03733	0.01133
0.5	-0.030	-0.04166	0.01166
0.6	-0.025	-0.036	0.011
0.7	-0.0069	-0.01633	0.00933
0.8	0.028	0.02133	0.00666
0.9	0.084	0.081	0.00299
1	0.165	0.16666	0.00166
1.1	0.2662334	0.2768	1.060551×10^{-2}
1.2	0.38562	0.4081094	2.248937×10^{-2}
1.3	0.5263801	0.56182	3.544819×10^{-2}
1.4	0.69052	0.73976	0.0492388
1.5	0.8806	0.9442711	6.357765×10^{-2}
1.6	1.10029	1.17843	7.814145×10^{-2}
1.7	1.35368	1.44624	9.256709×10^{-2}
1.8	1.646307	1.75275	0.1064509
1.9	1.984887	2.104237	0.11935
2	2.377554	2.508334	0.1307805

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