Adomian decomposition method for solving some models of nonlinear partial differential equations

A.H. Ali and A.S.J. Al-Saif
Department of Mathematics, College of Education, University of Basrah, Basrah, Iraq

Abstracts

By considering the Adomian decomposition method, we solve some models of nonlinear partial differential equations. We found the existence of exact solutions for those models. The numerical results show the efficiency and accuracy of this method.

طريقة أدومين للفصل لحل بعض نماذج المعادلات التفاضلية الجزئية اللاخطية

عبدالنبي حسين علي   عبدالستارجابرعلي السيف
قسم الرياضيات/كلية التربية/جامعة البصرة

الخلاصة

تركزت الدراسة في هذا البحث على استخدام طريقة أدومين للفصل لحل بعض نماذج المعادلات التفاضلية الجزئية اللاخطية، تم الحصول على الحل التحليلي لهذه النماذج وعلى نتائج عددية تثبت دقة وفعالية هذه الطريقة.

Keywords: Adomian decomposition method, nonlinear partial differential equations.

Introduction

Nonlinear partial differential equations can be found in wide variety scientific and engineering applications. Many important mathematical models can be expressed in terms of nonlinear partial differential equations. The most general form of nonlinear partial differential equation is given by:

$$F(u, u_t, u_x, u_y, x, y, t) = 0 \quad \textbf{(1a)}$$

with initial and boundary conditions

$$u(x, y, 0) = \phi(x, y), \forall x, y \in \Omega, \Omega \in R^2 \quad \textbf{(1b)}$$

$$u(x, y, t) = f(x, y, t), \forall x, y \in \partial\Omega \quad \textbf{(1c)}$$

where Ω is the solution region and $\partial\Omega$ is the boundary of Ω.

In recent years, much research has been focused on the numerical solution of nonlinear partial equations by using numerical methods and developing these methods (Al-Saif, 2007; Leveque, 2006; Rossler & Husner, 1997; Wescot & Rizwan-Uddin, 2001). In the numerical methods, which are commonly used for solving these kind of equations large size or difficult of computations is appeared and usually the round-off error causes the loss of accuracy.

The Adomian decomposition method which needs less computation was employed to solve many problems (Celik et al., 2006; Javidi & Golbabai, 2007). Therefore, we applied the Adomian decomposition method to solve some models of nonlinear partial equation, this study reveals that the Adomian decomposition method is very efficient for nonlinear models, and it results give evidence that high accuracy can be achieved.

The decomposition method

The principle of the Adomian decomposition method (ADM) when applied to a general nonlinear equation is in the following form (Celik et al., 2006; Seng et al., 1996):

$$Lu + Ru + Nu = g \qquad \textbf{(2)}$$

The linear terms decomposed into $Lu + Ru,$ while the nonlinear terms are represented by Nu, where L is an easily invertible linear operator, R is the remaining linear part. By inverse operator L, with $L^{-1}(.) = \int_0^t (.)dt$. Equation (2) can be hence as;

$$u = L^{-1}(g) - L^{-1}(Ru) - L^{-1}(Nu) \qquad \textbf{(3)}$$

The decomposition method represents the solution of equation (3) as the following infinite series:

$$u = \sum_{n=0}^{\infty} u_n \qquad \textbf{(4)}$$

The nonlinear operator $Nu = \Psi(u)$ is decomposed as:

$$Nu = \sum_{n=0}^{\infty} A_n \qquad \textbf{(5)}$$

where A_n are Adomian's polynomials, which are defined as (Seng et al., 1996):

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [\Psi(\sum_{i=1}^{n} \lambda^i u_i)]_{\lambda=0} \qquad n = 0,1,2,... \qquad \textbf{(6)}$$

Substituting equations (4) and (5) into equation (3), we have

$$u = \sum_{n=0}^{\infty} u_n = u_0 - L^{-1}(R(\sum_{n=0}^{\infty} u_n)) - L^{-1}(\sum_{n=0}^{\infty} A_n) \qquad \textbf{(7)}$$

Consequently, it can be written as:

$$\left.\begin{aligned} u_0 &= \phi + L^{-1}(g) \\ u_1 &= -L^{-1}(R(u_0)) - L^{-1}(A_0) \\ u_2 &= -L^{-1}(R(u_1)) - L^{-1}(A_1) \\ &\vdots \\ u_n &= -L^{-1}(R(u_{n-1})) - L^{-1}(A_{n-1}) \end{aligned}\right\} \quad \textbf{(8)}$$

where ϕ is the initial condition,

Hence all the terms of u are calculated and the general solution obtained according to ADM as $u = \sum_{n=0}^{\infty} u_n$. The convergent of this series has been proved in (Seng et al., 1996). However, for some problems (Celik et al., 2006) this series can't be determined, so we use an approximation of the solution from truncated series

$$U_M = \sum_{n=0}^{M} u_n \text{ with } \lim_{M \to \infty} U_M = u \quad \textbf{(9)}$$

Applications

In this section, application of Adomian decomposition method for nonlinear models is given as in the illustrative examples; we consider the following three problems:

Problem 1

Consider the following hyperbolic nonlinear problem [2]:

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} \quad in \quad 0 < x \le 1, \quad 0 \le t \le 1 \quad \textbf{(10a)}$$

with the initial condition

$$u(x,0) = \frac{x}{10}, \qquad 0 < x \le 1 \quad \textbf{(10b)}$$

Eq. (10) has the exact solution (Bellman et al., 1972):

$$u(x,t) = \frac{-x}{(t-10)} \quad \textbf{(11)}$$

Now, we use ADM to solve Eq. (10). In this problem, we have

$$Nu = \Psi(u) = u\frac{\partial u}{\partial x}, g(x,t) = 0, Ru = 0, Lu = \frac{\partial u}{\partial t} \quad and \quad \phi = u(x,0) = \frac{x}{10}.$$

By using Eq. (6) Adomian's polynomials can be derived as follows:

$$\left.\begin{aligned} A_0 &= u_0 \frac{\partial u_0}{\partial x} \\ A_1 &= u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} \\ A_2 &= u_2 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial u_2}{\partial x} \\ A_3 &= u_3 \frac{\partial u_0}{\partial x} + u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x} + u_0 \frac{\partial u_3}{\partial x} \\ &\vdots \end{aligned}\right\} \qquad (12)$$

and so on. The rest of the polynomials can be constructed in similar manner.
By using Eqs. (8), we have

$$\left.\begin{aligned} u_0 &= \frac{x}{10} \\ u_1 &= \frac{x}{10}\left(\frac{t}{10}\right) \\ u_2 &= \frac{x}{10}\left(\frac{t}{10}\right)^2 \\ u_3 &= \frac{x}{10}\left(\frac{t}{10}\right)^3 \\ &\vdots \\ u_n &= \frac{x}{10}\left(\frac{t}{10}\right)^n \end{aligned}\right\} \qquad (13)$$

Substituting these individual terms in Eq. (4) we obtain

$$u(x,t) = \frac{x}{10}\left[1 + \frac{t}{10} + \left(\frac{t}{10}\right)^2 + \left(\frac{t}{10}\right)^3 + \cdots + \left(\frac{t}{10}\right)^n + \cdots\right] \qquad (14)$$

This gives the exact solution (11). This result can be verified through substitution.

<u>Problem 2</u>

Let us consider the Problem

$$\frac{\partial u}{\partial t} = x^2 - \frac{1}{4}\left(\frac{\partial u}{\partial x}\right)^2, \qquad 0 < x \le 1, \quad 0 \le t \le 1 \qquad (15)$$

with the initial condition

$$u(x,0) = 0, \qquad 0 \le x \le 1 \qquad (16)$$

Eq. (15) has the exact solution (Bellman et al., 1972):

$$u(x,t) = x^2 \tanh(t) \qquad (17)$$

In this problem we have

$$Nu = \Psi((u) = \left(\frac{\partial u}{\partial x}\right)^2, \ g(x,t) = x^2, \ Ru = 0, \ Lu = \frac{\partial u}{\partial t} \ and \quad \phi = u(x,0) = 0.$$

By using Eq. (6), we obtain

$$\left.\begin{aligned} A_0 &= \left(\frac{\partial u_0}{\partial x}\right)^2 \\ A_1 &= 2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} \\ A_2 &= \left(\frac{\partial u_1}{\partial x}\right)^2 + 2\frac{\partial u_0}{\partial x}\frac{\partial u_2}{\partial x} \\ A_3 &= 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + 2\frac{\partial u_0}{\partial x}\frac{\partial u_3}{\partial x} \\ A_4 &= \left(\frac{\partial u_2}{\partial x}\right)^2 + 2\frac{\partial u_1}{\partial x}\frac{\partial u_3}{\partial x} + 2\frac{\partial u_0}{\partial x}\frac{\partial u_4}{\partial x} \\ &\vdots \end{aligned}\right\} \quad \textbf{(18)}$$

By using Eq. (8), we have

$$\left.\begin{aligned} u_0 &= x^2 t \\ u_1 &= -\frac{1}{3}x^2 t^3 \\ u_2 &= \frac{2}{15}x^2 t^5 \\ u_3 &= -\frac{17}{315}x^2 t^7 \\ u_4 &= \frac{62}{2835}x^2 t^9 \\ u_5 &= -\frac{1382}{155925}x^2 t^{11} \\ &\vdots \end{aligned}\right\} \quad \textbf{(19)}$$

From Eq.(4) we have

$$u(x,t) = x^2[t - \frac{1}{3}t^3 + \frac{2}{15}t^5 - \frac{17}{315}t^7 + \frac{62}{2835}t^9 - \frac{1382}{155925}t^{11} + \cdots] \quad \textbf{(20)}$$

which gives the exact solution (17).

Problem 3

Consider the nonlinear system of equations

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial t} &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{aligned}\right\} \quad \textbf{(21)}$$

with the initial conditions

$$u(x,y,0) = v(x,y,0) = x + y \quad \textbf{(22)}$$

The exact solution is given by

$$u(x,y,t)=v(x,y,t)=\frac{(x+y)}{(1-2t)} \tag{23}$$

In this problem Eqs. (21) can be written as:

$$\left.\begin{aligned} u &= L^{-1}(Nu) \\ v &= L^{-1}(Nv) \end{aligned}\right\} \tag{24}$$

where $L(.)=\frac{\partial}{\partial t}$, $Nu=\Psi_1(u,v)=u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$ and $Nv=\Psi_2(u,v)=u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$

By using Eq. (4) the solutions can be written as:

$$\left.\begin{aligned} u(x,y,t) &= \sum_{n=0}^{\infty} u_n(x,y,t) \\ v(x,y,t) &= \sum_{n=0}^{\infty} v_n(x,y,t) \end{aligned}\right\} \tag{25}$$

The associated decomposition scheme is given by

$$\left.\begin{aligned} u_0 &= u(x,y,0), \quad u_{n+1}=L^{-1}(\Psi_1(u_n,v_n)) \\ v_0 &= v(x,y,0), \quad v_{n+1}=L^{-1}(\Psi_2(u_n,v_n)) \end{aligned} \quad ,n=0,1,\cdots \right\} \tag{26}$$

We decompose Ψ_1 and Ψ_2 according to the series $\sum_{n=0}^{\infty} A_n$ and $\sum_{n=0}^{\infty} B_n$ respectively, where A_n and B_n are calculated by the Adomian's polynomials which are defined in Eq. (6) then we obtain

$$\left.\begin{aligned} A_0 &= u_0\frac{\partial u_0}{\partial x}+v_0\frac{\partial u_0}{\partial y} \\ A_1 &= u_0\frac{\partial u_1}{\partial x}+v_0\frac{\partial u_1}{\partial y}+u_1\frac{\partial u_0}{\partial x}+v_1\frac{\partial u_0}{\partial y} \\ A_2 &= u_0\frac{\partial u_2}{\partial x}+v_0\frac{\partial u_2}{\partial y}+u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}+u_2\frac{\partial u_0}{\partial x}+v_2\frac{\partial u_0}{\partial y} \\ &\vdots \end{aligned}\right\} \tag{27}$$

Similarly:

$$\left.\begin{aligned} B_0 &= u_0\frac{\partial v_0}{\partial x} + v_0\frac{\partial v_0}{\partial y} \\ B_1 &= u_0\frac{\partial v_1}{\partial x} + v_0\frac{\partial v_1}{\partial y} + u_1\frac{\partial v_0}{\partial x} + v_1\frac{\partial v_0}{\partial y} \\ B_2 &= u_0\frac{\partial v_2}{\partial x} + v_0\frac{\partial v_2}{\partial y} + u_1\frac{\partial v_1}{\partial x} + v_1\frac{\partial v_1}{\partial y} + u_2\frac{\partial v_0}{\partial x} + v_2\frac{\partial v_0}{\partial y} \\ &\vdots \end{aligned}\right\} \tag{28}$$

By using Eq. (8) we have

$$\left.\begin{aligned} u_0 &= x+y \\ v_0 &= x+y \\ u_1 &= (x+y)(2t) \\ v_1 &= (x+y)(2t) \\ u_2 &= (x+y)(2t)^2 \\ v_2 &= (x+y)(2t)^2 \\ u_3 &= (x+y)(2t)^3 \\ v_3 &= (x+y)(2t)^3 \\ &\vdots \\ u_n &= (x+y)(2t)^n \\ v_n &= (x+y)(2t)^n \end{aligned}\right\} \tag{29}$$

From Eq.(4) we have

$$u(x,y,t) = v(x,y,t) = (x+y)[1+2t+(2t)^2+(2t)^3+\cdots+(2t)^n+\cdots] \tag{30}$$

which gives the exact solution (23).

Numerical results and discussion

In this section we show that the efficiency of ADM for solving nonlinear problems1,2 and 3 through comparing it with differential quadrature method (DQM) (Al-Saif, 2007; Bellman et al., 1972) in term of absolute errors.

Tables 1,2 and 3 show the errors obtained by the ADM (for different M) and DQM. We see, the values we obtained is better than the results obtained by DQM. From tables 1 and 2, it should be noted that when M<10 the number of terns are not enough to obtain the approximate accurate solutions, while from table 3, we see that about 30 terms we needed to obtain approximate accurate solutions. The overall errors can be made even much smaller by adding new terms of the decomposition. Moreover, the rate of convergence is increased with the increasing of the total number of terms for $0 \le t \le 1$and $0 \le t \le 0.5$ for problems 1, 2 and 3 respectively (see Fig. 1, 2 and 3).

Table 1: Comparison of ADM and DQM solutions for problem 1.

t	x	ADM			DQM
		$\lvert u-U_5\rvert$	$\lvert u-U_{10}\rvert$	$\lvert u-U_{15}\rvert$	$\lvert u-u*\rvert$
0.1	0.125	1.2626e-012	1.7347e-018	0	2.5252e-012
	0.5	5.0505e-012	6.9389e-018	0	1.5152e-011
	0.875	8.8384e-012	1.3878e-017	0	8.8384e-009
0.5	0.125	4.1118e-009	1.2802e-015	5.2042e-018	9.2106e-010
	0.5	1.6447e-008	5.1209e-015	2.0817e-017	1.0526e-009
	0.875	2.8783e-008	8.9651e-015	4.1633e-017	7.3684e-008
1	0.125	1.3889e-007	1.3889e-012	1.2143e-017	4.1667e-007
	0.5	5.5556e-007	5.5556e-012	4.8572e-017	1.1111e-008
	0.875	9.7222e-007	9.7222e-012	9.7145e-017	6.8055e-008

Table 2: Comparison of ADM and DQM solutions for problem 2.

t	x	ADM			DQM
		$\lvert u-U_5\rvert$	$\lvert u-U_{10}\rvert$	$\lvert u-U_{15}\rvert$	$\lvert u-u*\rvert$
0.1	0.125	1.3793e-015	0	0	2.3920e-010
	0.5	2.2069e-014	0	0	1.2758e-009
	0.875	6.7599e-014	0	0	3.5880e-009
0.5	0.125	6.1400e-008	6.5565e-013	6.0715e-018	7.3940e-008
	0.5	9.8240e-007	1.0490e-011	9.7145e-017	6.6545e-009
	0.875	3.0086e-006	3.2127e-011	2.7756e-016	8.3180e-008
1	0.125	9.8548e-005	1.0776e-006	1.1783e-008	9.1392e-007
	0.5	1.5768e-003	1.7241e-005	1.8853e-007	2.4372e-007
	0.875	4.8289e-003	5.2801e-005	5.7736e-007	8.9192e-007

Table 3: Comparison of ADM and DQM solutions for problem 3.

t	y	x	ADM				DQM
			$\lvert u-U_5\rvert$	$\lvert u-U_{10}\rvert$	$\lvert u-U_{15}\rvert$	$\lvert u-U_{30}\rvert$	$\lvert u-u*\rvert$
0.1	0.125	0.125	1.0000e-004	3.2000e-008	1.0240e-011	0	2.8125e-008
		0.785	4.0000e-004	1.2800e-007	4.0960e-011	0	1.1250e-007
	0.5	0.5	4.0000e-004	1.2800e-007	4.0960e-011	0	1.1250e-007
	0.875	0.125	4.0000e-004	1.2800e-007	4.0960e-011	0	1.1250e-007
		0.785	7.0000e-004	2.2400e-007	7.1680e-011	0	1.9687e-007
0.2	0.125	0.125	4.2667e-003	4.3691e-005	4.4739e-007	4.8023e-013	2.5000e-007
		0.785	1.7067e-002	1.7476e-004	1.7896e-006	1.9209e-012	1.0000e-006
	0.5	0.5	1.7067e-002	1.7476e-004	1.7896e-006	1.9209e-012	1.0000e-006
	0.875	0.125	1.7067e-002	1.7476e-004	1.7896e-006	1.9209e-012	1.0000e-006
		0.785	2.9867e-002	3.0583e-004	3.1317e-006	3.3618e-012	1.7500e-006
0.3	0.125	0.125	4.8600e-002	3.7791e-003	2.9387e-004	1.3817e-007	1.8750e-006
		0.785	1.9440e-001	1.5117e-002	1.1755e-003	5.5268e-007	7.5000e-006
	0.5	0.5	1.9440e-001	1.5117e-002	1.1755e-003	5.5268e-007	7.5000e-006
	0.875	0.125	1.9440e-001	1.5117e-002	1.1755e-003	5.5268e-007	7.5000e-006
		0.785	3.4020e-001	2.6454e-002	2.0571e-003	9.6720e-007	1.3125e-005

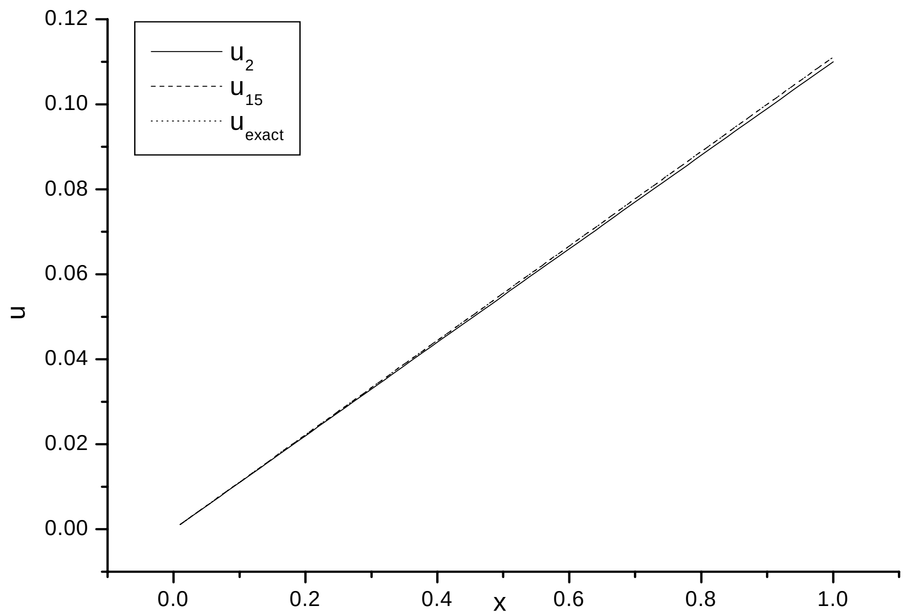

Figure 1:Values of exact solution and Adomian Approximations at t=1.

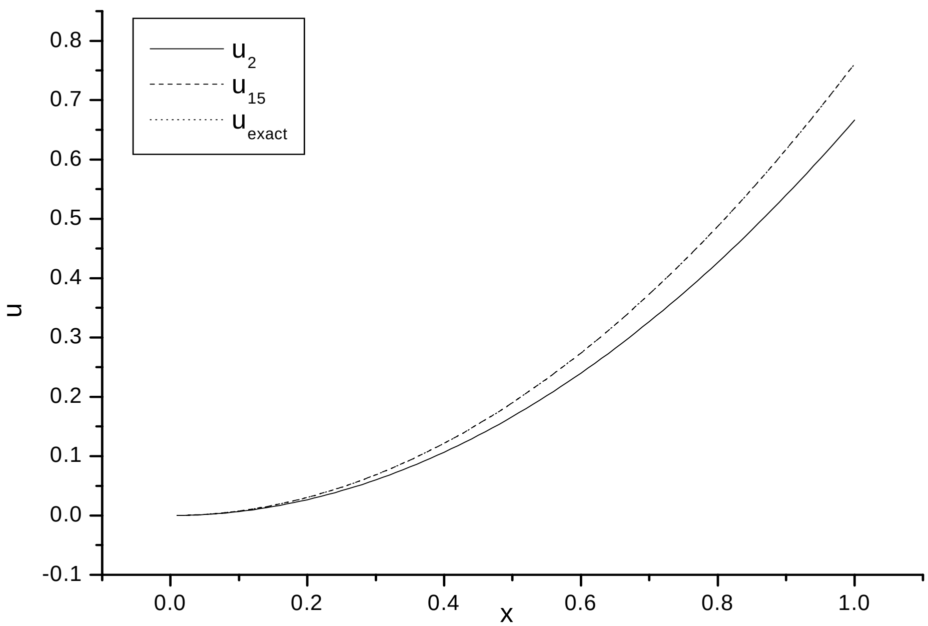

Figure 2:Values of exact solution and Adomian Approximations at t=1.

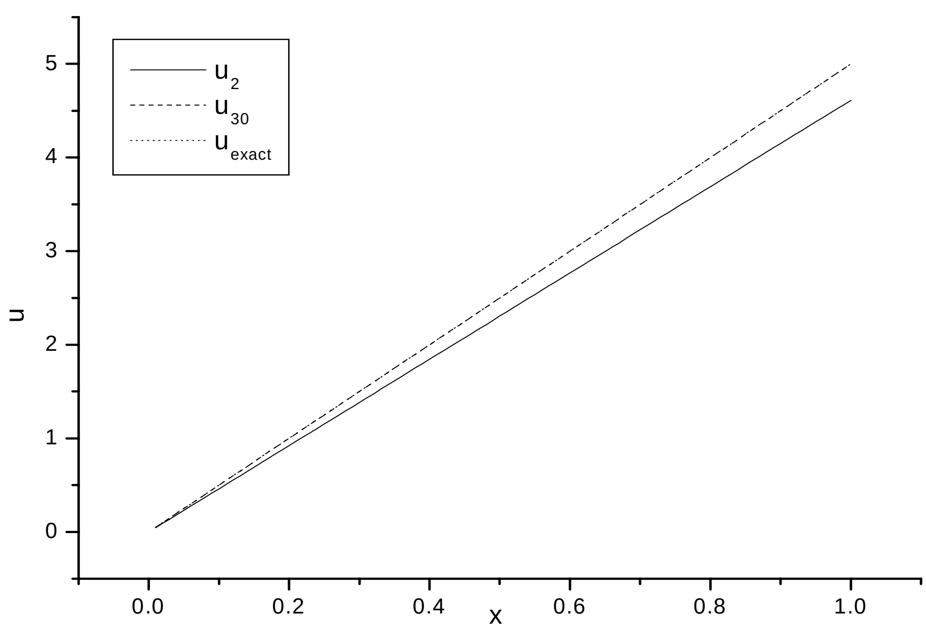

Figure 3: Comparsion between exact solution and ADM at the diagonal and t=0.3.

Conclusion

In this paper, we have applied the Adomian decomposition method for solving three problems of nonlinear partial equations. We demonstrated that the decomposition procedure is quite efficient to determine the exact solutions. However, the method gives a simple powerful tool for obtaining the solutions without a need for large size of computations. It is also worth noting that the advantage of this method sometimes displays a fast convergence of the solutions. In addition, the numerical results which obtained by this method indicate a high degree of accuracy.

References

[1]- Al-saif A.S.J., (2007), "Numerical study for convection motion stability of the ncompressible two-dimensional fluid flow by differential quadrature method", J.Basrah Researches (Sciences), Vol.33, No. 1. p.103-110.

[2]- Bellman R., Kashef B.G. and Casti J., (1972), "Differential quadrature: A technique for the rapid solution of nonlinear partial differential equations", J.comput. Phys., Vol.10,No.1, p. 40-52.

[3]- Celik E., Bayram M. Yeloglu T., (2006), "Solution of differential–algebra Equations by Adomian decomposition method", Inter. J. Pure and Appl. Math. Sciences, Vol.3, No.1, p. 93-100.

[4]- Javidi M. and Golbabai A., (2007), "Adomian decomposition method for approximating the solution of parabolic equations", J. Appl. Math. Sciences, Vol.1, No.5, p. 219-225.

[5]- Leveque R.J., (2006), "Finite difference methods for differential equations" A Math. 585 winters Quarter, University of Washington version of January.

[6]- Roessler J. and Husner H., (1997), "Numerical solution of 1+2 dimensional Fisher's equation by finite elements and Galerkin method", J. Math. Comput. Modeling, Vol. 25, No. 9, p. 57-67.

[7]- Seng V. Abbaoui K. Cherruault Y., (1996), "Adomian's polynomial for nonlinear operators", J. Math. Comput. Modeling, Vol. 24, No. 1, p.59-65.

[8]- Wescot B.L. and Rizwan-uddin, (2001), "An efficient formulation of modified nodal integral method and application to the two-dimension Burger's equations", J. Nucl. Sci. and Eng., Vol. 139, p. 293-305.