A Sixth Order Exponential Compact ADI Scheme for Solving Three-Dimensional Unsteady State Convection-Diffusion Equation.

A.J. Harfash

Department of Mathematics, College of Science, University of Basrah.

Abstract

In this paper, we derive a sixth order exponential compact finite difference scheme with alternating direction implicit (ADI) scheme for solving three-dimensional unsteady convection-diffusion equation. This scheme is sixth order for space and second order for time. The unconditionally stability of this new version of finite difference scheme is proved with respect to initial values. Numerical experiments are introduced to test the accuracy of the sixth order compact finite difference. And it has high accuracy compare with the fourth order Karaa scheme and exponential fourth order compact Mahdi scheme.

Key words: Alternating direction implicit (ADI), Sixth order compact, Convection-diffusion equation, Exponential scheme.

1- Introduction

We consider the initial boundary of unsteady three-dimensional 3-D convection-diffusion problem for transport variable u

$$\frac{\partial u}{\partial t}-a_x\frac{\partial^2 u}{\partial x^2}-a_y\frac{\partial^2 u}{\partial y^2}-a_z\frac{\partial^2 u}{\partial z^2}+c_x\frac{\partial u}{\partial x}+c_y\frac{\partial u}{\partial y}+c_z\frac{\partial u}{\partial z}=0, \quad (1)$$

$$u(x,y,z,t)=g(x,y,z,t), \quad (x,y,z)\in\partial\Omega, \quad t\in(0,T], \quad (2)$$

$$u(x,y,z,0)=u_0(x,y,z), \quad (x,y,z)\in\Omega, \quad (3)$$

where $\Omega\in\Re^3$ is a cubic domain, $(0,T]$ is the time interval, and g and u_0 are given functions of sufficiently smoothness. In (1) c_x, c_y and c_z are constant speeds of convection and $a_x>0$, $a_y>0$ and $a_z>0$ are constant diffusivities in the x, y, and $z-$ direction respectively. This equation may be seen in computational hydraulics and fluid dynamics to model convection-diffusion of quantities such as mass, heat, energy, vorticity, etc (Roach, 1976).

Various numerical finite difference schemes have been proposed to solve the convection-diffusion problem approximately. Noye and Tan in (Noye et al., 1988).

drive several high order implicit schemes for unsteady 1-D convection-diffusion equations, and in (Noye et al., 1989) they proposed a compact nine-point high order compact (HOC) implicit scheme for unsteady 2-D convection-diffusion equations. These schemes have large interval of stability and its third order accurate in space and second order accurate in time. In (Rigal, 1994, Rigal, 1999) Rigal derived two classes of compact difference schemes of order 2 in time and 4 in space with different choices of weighting parameters.

Spotz and Carye (Spotz et al., 2001) extended the 2-D HOC scheme in (Gupta et al., 1984) for solving steady state equations to solve unsteady state 1-D convection-diffusion equations with variable coefficients and 2-D diffusion equations. Also, in (Kalita et al., 2002) and (Karaa) derived a HOC schemes with weight time discretization to solve the unsteady 2-D and 3-D convection-diffusion equations, respectively.

To obtain satisfactory higher order numerical results with reasonable computational cost, there have been attempts to develop higher order compact ADI methods. A high order split formula was first obtained by Michell and Faireweather (Mitchell et al., 1964) and later in (Dai et al., 2002) for 2-D diffusion problems. Karaa and Zhang (Karaa et al., 2004) derived a higher order compact ADI for solving 2-D convection-diffusion equations of order 2 in time and 4 in space and then they extended this scheme for 3-D convection-diffusion equations in (Karaa, 2006).

A class of HOC compact exponential finite difference methods is proposed for solving 1-D and 2-D steady state convection-diffusion equations with variable coefficients from Tian and Dai in (Tain et al., 2007(a)) and then Tian and Ge extended this work in (Tain et al., 2007(b)) to 2-D unsteady state convection-diffusion equations with ADI scheme. Shatha in (Mahdi) derived a fourth order compact exponential ADI finite difference scheme for 3-D unsteady state convection-diffusion equations.

In this paper, we derived a sixth compact exponential ADI finite difference scheme for 3-D unsteady state convection-diffusion equations for all points except the points that have location near the boundary in the network, for these points we use a fourth order scheme only.

Due to, if we use a sixth order scheme for all points we have a point out of the network. We prove that this new scheme is unconditionally stable with initial values. Numerical experiments are conducted to test this new ADI sixth order scheme. And to compare it with Mahdi ADI fourth order compact scheme and Karaa ADI fourth order compact scheme.

2. Sixth order exponential compact discretization

2.1 O(h^2) compact exponential FD schemes

Consider the steady state one-dimensional convection–diffusion equation

$$-au_{xx} + cu_x = f(x),\ x \in [a,b], \qquad (4)$$

where a is the positive constant conductivity, c is the constant convective velocity, f is a sufficiently smooth function of x, and u may represent heat, vorticity, etc. This equation is consistent with singular-perturbation problem as a is a small parameter. Firstly we divide $[a, b]$ into N equal parts with $x_i = ih_x$, $h_x = x_{i+1} - x_i$, $u_i = u(x_i)$, and $i = 0,\dots, N$. For a sufficiently smooth solution, derivative in (4) at interior grid points, can be defined using Taylor's expansion as

$$u_{x_i} = D_{\Delta x}u_i - \sum_{n=1}^{\infty} \frac{(h_x)^{2n}}{(2n+1)!} D_x^{2n+1} u_i,$$
$$u_{xx_i} = D_{\Delta x}^2 u_i - \sum_{n=1}^{\infty} \frac{2(h_x)^{2n}}{(2n+2)!} D_x^{2n+2} u_i,$$

where $D_{\Delta x}u_i = (u_{i+1} - u_{i-1})/2h_x$ and $D_{\Delta x}^2 u_i = (u_{i+1} - 2u_i + u_{i-1})/h_x^2$ are the central difference approximations for the first and second derivatives and D_x^n is the nth-order exact derivative operator at any interior x.

In this section, we develop HOC exponential FD scheme for solving the convection–diffusion equation (4). To formulate this scheme, we first introduce an $O(\Delta x^2)$ compact exponential FD method for Eq.(2). In the sub domain $[x_{i-1}, x_{i+1}]$, let us rewrite Eq. (4) as

$$-ae^{\frac{cx}{a}}(e^{-\frac{cx}{a}}u_x)_x = f_i,$$

and then

$$-a(e^{-\frac{cx}{a}}u_x)_x = e^{-\frac{cx}{a}} f_i \qquad (7)$$

Integration of Eq. (7) over a spaced interval from $x_{i-1/2}$ to $x_{i+1/2}$ gives

$$-c(e^{-\frac{ch_x}{2a}}(u_x)_{i+1/2} - e^{\frac{ch_x}{2a}}(u_x)_{i-1/2}) = (e^{-\frac{ch_x}{2a}} - e^{\frac{ch_x}{2a}})f_i \qquad (8)$$

Using the central difference to approximate u_{x_i}, we get

$$u_{xi} = \frac{u_{i+1/2} - u_{i-1/2}}{\Delta x} + O(\Delta x^2)$$

and thus, Eq.(8) can be written

$$.-\frac{c}{2h}(e^{-\frac{ch_x}{2a}}(2u_{i+1} - 2u_i) - e^{\frac{ch_x}{2a}}(2u_i - 2u_{i-1})) = (e^{-\frac{ch_x}{2a}} - e^{\frac{ch_x}{2a}})f_i \qquad (9)$$

By adding and subtracting u_{i-1}, u_{i+1} to the first and second term of right hand of Eq. (9), respectively, and then rearrangement of this equation we get

$$-\frac{c}{2\Delta x}((e^{-\frac{ch_x}{2a}} + e^{\frac{ch_x}{2a}})u_{i+1} - 2(e^{-\frac{ch_x}{2a}} + e^{\frac{ch_x}{2a}})u_i + (e^{-\frac{ch_x}{2a}} + e^{\frac{ch_x}{2a}})u_{i-1} \qquad (10)$$
$$+(e^{-\frac{ch_x}{2a}} - e^{\frac{ch_x}{2a}})u_{i+1} - (e^{-\frac{ch_x}{2a}} - e^{\frac{ch_x}{2a}})u_{i-1}) = (e^{-\frac{ch_x}{2a}} - e^{\frac{ch_x}{2a}})f_i.$$

Divide both sides of Eq. (10) by and after rearrangement, we get

$$-\alpha D_{\Delta x}^2 u_i + cD_{\Delta x}u_i = f_i, \qquad (11)$$

where

$$\alpha = \begin{cases} -\frac{ch_x}{2}\coth(ch_x/2a), & c \neq 0 \\ a, & c = 0 \end{cases} \qquad (12)$$

Eq. (11) is called a second-order exponential FD scheme for the convective diffusion model equation (4), which is nodally exact and gives rise to a tri-diagonal system of equations. In addition, scheme (11) provides the exact solution for the 1D convection–diffusion equation with constant convection coefficient in the absence of a source term. Scheme (11) and its some variants have been proposed via other approaches (Bear, 1972, Tain et al., 2003, Thomas, 1995, Zhang, 1998].

It is easily found that the second-order exponential FD scheme (11) applied to Eq. (4) is equivalent to the standard second-order central FD formula applied to the following equation:

$$-\frac{ch_x}{2}\coth\left(ch_x/2a\right)u_{xx}+cu_x=f. \qquad (13)$$

Eq. (13) also shows that, when the second-order exponential FD scheme (11) is used, an artificial diffusion coefficient $a\left[(ch/2a)\coth(ch/2a)-1\right]$ is perturbed to Eq. (4).

2.2 O(h⁶) compact exponential FD schemes

Consider the FD scheme for Eq. (4) with constant convection coefficient at a grid point x_i as

$$-\alpha D_{\Delta x}^2 u_i+cD_{\Delta x}u_i=\alpha_0 f_i+\alpha_1 f_{xi}+\alpha_2 f_{xxi}+\alpha_3 f_{xxxi}+\alpha_4 f_{xxxxi}, \qquad (14)$$

Where

$$\alpha=\begin{cases}-\dfrac{ch_x}{2}\coth\left(ch_x/2a\right), & c\neq 0\\ a &, \quad c=0\end{cases} \qquad (15)$$

$D_{\Delta x}^2$ and $D_{\Delta x}$ are as defined previously. In order to determine the parameters $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ and α_4, let us rewrite Eq. (14) as

$$-\alpha D_{\Delta x}^2 u_i+cD_{\Delta x}u_i=\alpha_0\left(-au_{xx}+cu_x\right)_i+\alpha_1\left(-au_{xx}+cu_x\right)_{xi}+\alpha_2\left(-au_{xx}+cu_x\right)_{xxi}$$
$$+\alpha_3\left(-au_{xx}+cu_x\right)_{xxxi}+\alpha_4\left(-au_{xx}+cu_x\right)_{xxxxi}. \qquad (16)$$

Straightforwardly calculating the right-hand side of Eq. (16), and substituting (5) and (6) into (16) and rearranging it, we obtain the following modified differential equation corresponding to the scheme(16):

$$(\alpha_0-1)cu_{xi}+(\alpha-a\alpha_0+c\alpha_1)u_{xxi}+(c\alpha_2-a\alpha_1-\frac{h_x{}^2c}{6})D_x^3u_i+(c\alpha_3-a\alpha_2+\frac{h_x{}^2\alpha}{12})D_x^4u_i$$
$$(c\alpha_4-a\alpha_3-\frac{h_x{}^4c}{120})D_x^5u+(-a\alpha_4+\frac{h_x{}^4\alpha}{360})D_x^6u_{i_i}+o(h_x{}^6)=0. \qquad (17)$$

Letting

$$(\alpha_0-1)=0,\quad (\alpha-a\alpha_0+c\alpha_1)=0,\quad (c\alpha_2-a\alpha_1-\frac{h_x{}^2c}{6})=0, \qquad (18)$$
$$(c\alpha_3-a\alpha_2+\frac{h_x{}^2\alpha}{12})=0,\ (c\alpha_4-a\alpha_3-\frac{h_x{}^4c}{120})=0,\quad (-a\alpha_4+\frac{h_x{}^4\alpha}{360})=0.$$

and solving the above resulting equations, we get the parameters

$$\alpha_0 = 1$$

$$,\alpha_1 = \begin{cases} \dfrac{a-\alpha}{c}, & ,c \neq 0 \\ 0 & ,\ c = 0 \end{cases} \qquad \alpha_2 = \begin{cases} \dfrac{a}{c}\left(\dfrac{a-\alpha}{c}\right) + \dfrac{h_x^2}{6} & ,c \neq 0 \\ \dfrac{h_x^2}{12}\ ,. & ,\ c = 0 \end{cases} \tag{19}$$

$$\alpha_3 = \begin{cases} \dfrac{a}{c}\left(\dfrac{a(a-\alpha)}{c^2} + \dfrac{h_x^2}{6}\right) - \dfrac{\alpha h_x^2}{12c} & ,c \neq 0 \\ 0 & ,\ c = 0 \end{cases}$$

$$\alpha_4 = \begin{cases} \dfrac{a^2}{c^2}\left(\dfrac{a(a-\alpha)}{c^2} + \dfrac{h_x^2}{6}\right) - \dfrac{a\alpha h_x^2}{12c^2} + \dfrac{h_x^4}{120} & ,c \neq 0 \\ \dfrac{h_x^4}{360} & ,\ c = 0 \end{cases}$$

The Taylor-series truncation error analysis shows that Eq. (14) with (19) for solving the model problem (4) is an $O(h_x^6)$ compact FD scheme. Notice that second-order central differences of the derivatives of f may be used in Eq. (4) while still maintaining overall $O(h_x^6)$ accuracy on 3-point stencil.

We define two finite difference operators

$$\mathrm{A}_x = -\alpha D_{\Delta x}^2 + cD_{\Delta x}, \qquad \ell_x = \alpha_0 + \alpha_1 D_{\Delta x} + \alpha_2 D_{\Delta x}^2 + \alpha_3 D_{\Delta x}^3 + \alpha_4 D_{\Delta x}^4,$$

Eq.(11) can then be formulated symbolically as

$$\ell_x^{-1} \mathrm{A}_x u_i = f_i + O(h_x^6)$$

This symbolic construction can be used to derive high order compact schemes for higher dimensional problems. When applied to the steady state 3-D convection diffusion equation

$$-a_x \frac{\partial^2 u}{\partial x^2} - a_y \frac{\partial^2 u}{\partial y^2} - a_z \frac{\partial^2 u}{\partial z^2} + c_x \frac{\partial u}{\partial x} + c_y \frac{\partial u}{\partial y} + c_z \frac{\partial u}{\partial z} = f, \tag{20}$$

yield the following sixth order approximation

$$(\ell_x^{-1} \mathrm{A}_x + \ell_y^{-1} \mathrm{A}_y + \ell_z^{-1} \mathrm{A}_z) u_{ijk} = f_{ijk} + O(h_x^6) + O(h_y^6) + O(h_z^6)\,. \tag{21}$$

And

$$\alpha=\begin{cases}-\dfrac{c_x h_x}{2}\coth\left(c_x h_x/2a_x\right), & c_x\neq 0\\ a_x, & c_x=0\end{cases} \qquad \alpha_1=\begin{cases}\dfrac{a_x-\alpha}{c_x}, & c_x\neq 0\\ 0, & c_x=0\end{cases}$$

, ,

$$\alpha_4=\begin{cases}\dfrac{a_x^2}{c_x^2}\left(\dfrac{a_x(a_x-\alpha)}{c_x^2}+\dfrac{h_x^2}{6}\right)-\dfrac{a_x\alpha h_x^2}{12c_x^2}+\dfrac{h_x^4}{120}, & c_x\neq 0\\ \dfrac{h_x^{\,4}}{360}, & c_x=0\end{cases}$$

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$$\alpha_2=\begin{cases}\dfrac{a_x}{c_x}\left(\dfrac{a_x-\alpha}{c_x}\right)+\dfrac{h_x^2}{6}, & c_x\neq 0\\ \dfrac{h_x^{\,2}}{12}, & c_x=0\end{cases} \qquad \alpha_3=\begin{cases}\dfrac{a_x}{c_x}\left(\dfrac{a_x(a_x-\alpha)}{c_x^2}+\dfrac{h_x^2}{6}\right)-\dfrac{\alpha h_x^2}{12c_x}, & c_x\neq 0\\ 0, & c_x=0\end{cases}$$

$$\beta=\begin{cases}-\dfrac{c_y h_y}{2}\coth\left(c_y h_y/2a_y\right), & c_y\neq 0\\ a_y, & c_y=0\end{cases} \qquad \beta_1=\begin{cases}\dfrac{a_y-\beta}{c_y}, & c_y\neq 0\\ 0, & c_y=0\end{cases}$$

, ,

$$\beta_2=\begin{cases}\dfrac{a_y}{c_y}\left(\dfrac{a_y-\beta}{c_y}\right)+\dfrac{h_y^2}{6}, & c_y\neq 0\\ \dfrac{h_y^2}{12}, & c_y=0\end{cases} \qquad \beta_3=\begin{cases}\dfrac{a_y}{c_y}\left(\dfrac{a_y(a_y-\beta)}{c_y^2}+\dfrac{h_x^2}{6}\right)-\dfrac{\beta h_y^2}{12c_y}, & c_y\neq 0\\ 0, & c_y=0\end{cases}$$

'

$$\beta_4=\begin{cases}\dfrac{a_y^2}{c_y^2}\left(\dfrac{a_y(a_y-\beta)}{c_y^2}+\dfrac{h_y^2}{6}\right)-\dfrac{a_y\beta h_y^2}{12c_y^2}+\dfrac{h_y^4}{120}, & c_y\neq 0\\ \dfrac{h_y^4}{360}, & c_y=0\end{cases} \qquad \mu=\begin{cases}-\dfrac{c_z h_z}{2}\coth\left(c_z h_z/2a_z\right), & c_z\neq 0\\ a_y, & c_z=0\end{cases}$$

,

$$\mu_1=\begin{cases}\dfrac{a_z-\mu}{c_z}, & c_z\neq 0\\ 0, & c_z=0\end{cases} \qquad \mu_2=\begin{cases}\dfrac{a_z}{c_z}\left(\dfrac{a_z-\mu}{c_z}\right)+\dfrac{h_z^2}{6}, & c_z\neq 0\\ \dfrac{h_z^2}{12}, & c_z=0\end{cases}$$

,

$$\mu_3 = \begin{cases} \dfrac{a_z}{c_z}\left(\dfrac{a_z\left(a_y-\mu\right)}{c_z^2}+\dfrac{h_z^2}{6}\right)-\dfrac{\mu h_z^2}{12c_z} & ,c_z \neq 0 \\ 0 & , \quad c_z = 0 \end{cases}$$

.

$$\mu_4 = \begin{cases} \dfrac{a_z^2}{c_z^2}\left(\dfrac{a_z\left(a_z-\mu\right)}{c_z^2}+\dfrac{h_y^2}{6}\right)-\dfrac{a_z\mu h_z^2}{12c_z^2}+\dfrac{h_z^4}{120} & ,c_z \neq 0 \\ \dfrac{h_z^4}{360} & , \quad c_z = 0 \end{cases}$$

Applying to both sides of eq.(21) with the operator $\ell_x\ell_y\ell_z$, we obtain

$$(\ell_y\ell_z\mathrm{A}_x+\ell_x\ell_z\mathrm{A}_y+\ell_x\ell_y\mathrm{A}_z)u_{ijk}=\ell_x\ell_y\ell_z f_{ijk}+O(\Delta^6), \quad (22)$$

where $O(\Delta^6)$ denote to the $O(h_x{}^6)+O(h_y{}^6)+O(h_z{}^6)$. Notice that in deriving (22) we used the fact that the operators ℓ_x,ℓ_y and ℓ_z commute with each other, which is possible since the convection and diffusion terms are assumed constant.

$$\ell_x\ell_y\ell_z\frac{\partial u^n}{\partial t}=-(\ell_y\ell_z\mathrm{A}_x+\ell_x\ell_z\mathrm{A}_y+\ell_x\ell_y\mathrm{A}_z)u^n+O(\Delta^6)$$

4 . Sixth order compact alternating direction implicit (ADI)

A sixth order semi discrete approximation to the unsteady convection-diffusion equation in (1) can be obtained by replacing f with $-(\frac{\partial u}{\partial t})$ in (22)

Now by using Crank-Nicholson time discretization, we have

$$\ell_x\ell_y\ell_z(\frac{u^{n+1}-u^n}{\Delta t})=-\frac{1}{2}(\ell_y\ell_z\mathrm{A}_x+\ell_x\ell_z\mathrm{A}_y+\ell_x\ell_y\mathrm{A}_z)(u^n+u^{n+1})+O(\Delta^6)+O(\Delta t^2). \quad (23)$$

This discretization is second order in time and sixth order in space. After rearrangement Eq.(23), we have

$$\left(\ell_x\ell_y\ell_z+\frac{\Delta t}{2}(\ell_y\ell_z\mathrm{A}_x+\ell_x\ell_z\mathrm{A}_y+\ell_x\ell_y\mathrm{A}_z)\right)u^{n+1} \quad (24)$$
$$=\left(\ell_x\ell_y\ell_z-\frac{\Delta t}{2}(\ell_y\ell_z\mathrm{A}_x+\ell_x\ell_z\mathrm{A}_y+\ell_x\ell_y\mathrm{A}_z)\right)u^n+O(\Delta t\Delta^4)+O(\Delta t^2).$$

To get a solution for our equation, we must solve each time step a sparse linear system arising from the implicit discretization (21). A way around in developing an efficient solution method to our problem is to solve a perturbed problem that has the same order of accuracy as (24) and which allows to reduce the 3-D problem to a succession of many 1-D problem. To accomplish this, we add the terms

$$\frac{\Delta t^2}{4}(\ell_y A_x A_z + \ell_x A_y A_z + \ell_z A_x A_y)u^{n+1} + \frac{\Delta t^3}{8} A_x A_y A_z u^{n+1}$$

And

$$\frac{\Delta t^2}{4}(\ell_y A_x A_z + \ell_x A_y A_z + \ell_z A_x A_y)u^{n} - \frac{\Delta t^3}{8} A_x A_y A_z u^{n}$$

to the right and left hand sides of (24), respectively, so that Eq.(24) after dropping the error terms becomes

$$(\ell_x\ell_y\ell_z + \frac{\Delta t}{2}(\ell_y\ell_z A_x + \ell_x\ell_z A_y + \ell_x\ell_y A_z) + \frac{\Delta t^2}{4}(\ell_y A_x A_z + \ell_x A_y A_z + \ell_z A_x A_y)$$
$$+ \frac{\Delta t^3}{8} A_x A_y A_z)u^{n+1} = (\ell_x\ell_y\ell_z - \frac{\Delta t}{2}(\ell_y\ell_z A_x + \ell_x\ell_z A_y + \ell_x\ell_y A_z) + \frac{\Delta t^2}{4}(\ell_y A_x A_z$$
$$+ \ell_x A_y A_z + \ell_z A_x A_y) - \frac{\Delta t^3}{8} A_x A_y A_z)u^{n}.$$

which can be factored as

$$(\ell_x + \frac{\Delta t}{2}A_x)(\ell_y + \frac{\Delta t}{2}A_y)(\ell_z + \frac{\Delta t}{2}A_z)u^{n+1} = (\ell_x - \frac{\Delta t}{2}A_x)(\ell_y - \frac{\Delta t}{2}A_y)(\ell_z - \frac{\Delta t}{2}A_z)u^{n}. \qquad (25)$$

To achieve unconditionally stability, one may resort to a fully implicit or Crank-Nicholson method for time discretization of Eq. (25). This will result in a system of algebraic equations that is sparse, which may require a large amount of computational effort. One remedy is to use ADI method, which only required solving 1-D implicit problems for each time step. The details of the ADI methods can be found in (Thomas, 1995) .

Now, we introduce an exponential higher order ADI scheme and corresponding boundary conditions, which will be used in our numerical solutions for 3-D convection-diffusion problems.

The resulting approximation (25) is second-order accurate in time and sixth-order accurate in space. From formula (25), we can obtain the following exponential high order ADI scheme. Introducing an intermediate variables u^* and u^{**}, Eq. (25) can be solved in three steps as

$$(\ell_x + \frac{\Delta t}{2}A_x)u^{**} = (\ell_x - \frac{\Delta t}{2}A_x)(\ell_y - \frac{\Delta t}{2}A_y)(\ell_z - \frac{\Delta t}{2}A_z)u^{n}, \qquad (26a)$$

$$(\ell_y + \frac{\Delta t}{2} \mathrm{A}_y) u^* = u^{**}, \quad (26b)$$

$$(\ell_z + \frac{\Delta t}{2} \mathrm{A}_z) u^{n+1} = u^*. \quad (26c)$$

4. Stability analysis

In this section firstly we extend the proof of stability of Karaa scheme(Karaa, 2006) who prove the scheme is stable for $c_x = c_y = c_z = 0$, therefore, we prove that the scheme is stable for all values of c_x, c_y, c_z. Secondly, we prove that our scheme is stable for $c_x = c_y = c_z = 0$. In the further we can extension the prove of stability of our scheme for all values of c_x, c_y, c_z.

To study the stability, we use von Neumann linear stability analysis. If we let $u_{ijk}^n = \xi^n e^{I\sigma_x i} e^{I\sigma_y j} e^{I\sigma_z k}$, to be the value of u^n at node (i, j, k), where $I = \sqrt{-1}$, ξ^n is the amplitude at time level n, and $\sigma_x (= 2\pi h_x / \Lambda_1)$, $\sigma_y (= 2\pi h_y / \Lambda_2)$ and $\sigma_z (= 2\pi h_z / \Lambda_3)$ are phase angles with wavelengths Λ_1 , Λ_2 and Λ_3 respectively, then the amplification factor $H(\sigma_x, \sigma_y, \sigma_z) = \xi^{n+1} / \xi^n$, for stability, has to satisfy the relation $\left| H(\sigma_x, \sigma_y, \sigma_z) \right| < 1$, for all σ_x, σ_y and σ_z in $[-\pi, \pi]$.

4.1 The stability of Samir Karaa scheme

The final formula of Karaa scheme can be written in the following formula (Karaa, 2006):

$$(\ell_x + \frac{\Delta t}{2} \mathrm{A}_x)(\ell_y + \frac{\Delta t}{2} \mathrm{A}_y)(\ell_z + \frac{\Delta t}{2} \mathrm{A}_z) u^{n+1} = (\ell_x - \frac{\Delta t}{2} \mathrm{A}_x)(\ell_y - \frac{\Delta t}{2} \mathrm{A}_y)(\ell_z - \frac{\Delta t}{2} \mathrm{A}_z) u^n,$$

$$\mathrm{A}_x = -(a_x + \frac{c_x^2 h_x^2}{12 a_x}) D_{\Delta x}^2 + c_x D_{\Delta x}, \qquad \ell_x = 1 + \frac{h_x^2}{12}(D_{\Delta x}^2 - \frac{c_x}{a_x} D_{\Delta x}), \quad (27)$$

$$\mathrm{A}_y = -(a_y + \frac{c_y^2 h_y^2}{12 a_y}) D_{\Delta y}^2 + c_y D_{\Delta y}, \qquad \ell_y = 1 + \frac{h_y^2}{12}(D_{\Delta y}^2 - \frac{c_y}{a_y} D_{\Delta y}),$$

$$\mathrm{A}_z = -(a_z + \frac{c_z^2 h_z^2}{12 a_z}) D_{\Delta z}^2 + c_z D_{\Delta z}, \qquad \ell_z = 1 + \frac{h_z^2}{12}(D_{\Delta z}^2 - \frac{c_z}{a_z} D_{\Delta z}),$$

By substituting the expression of u_{ijk}^n and u_{ijk}^{n+1} in (27), the amplification factor is found to be $H(\sigma_x, \sigma_y, \sigma_z) = \psi_x(\sigma_x)\psi_y(\sigma_y)\psi_z(\sigma_z)$. To show how to derive the value of $\psi_x(\sigma_x)$, $\psi_y(\sigma_y)$ and $\psi_z(\sigma_z)$ we reduce Eq. (27) in 1-D case and then substituting the value of A_x and

$$\xi^{n+1}\left[1+\frac{h_x{}^2}{12}(D_{\Delta x}^2-\frac{c_x}{a_x}D_{\Delta x})+\frac{\Delta t}{2}\left(-(a_x+\frac{c_x^2 h_x{}^2}{12a_x})D_{\Delta x}^2+c_x D_{\Delta x}\right)\right]e^{I\sigma_x i}$$
$$=\xi^{n}\left[1+\frac{h_x{}^2}{12}(D_{\Delta x}^2-\frac{c_x}{a_x}D_{\Delta x})-\frac{\Delta t}{2}\left(-(a_x+\frac{c_x^2 h_x{}^2}{12a_x})D_{\Delta x}^2+c_x D_{\Delta x}\right)\right]e^{I\sigma_x i}, \quad (28)$$

Rearrangement of Eq. (28), we get

$$\xi^{n+1}\left[1+\left(\frac{h_x{}^2}{12}-\frac{a_x\Delta t}{2}-\frac{c_x^2 h_x{}^2\Delta t}{24a_x}\right)D_{\Delta x}^2+\left(\frac{c_x\Delta t}{2}-\frac{h_x{}^2 c_x}{12a_x}\right)D_{\Delta x}\right]e^{I\sigma_x i}$$
$$=\xi^{n}\left[1+\left(\frac{\Delta x^2}{12}+\frac{a_x\Delta t}{2}+\frac{c_x^2 h_x{}^2\Delta t}{24a_x}\right)D_{\Delta x}^2-\left(\frac{c_x\Delta t}{2}+\frac{h_x{}^2 c_x}{12a_x}\right)D_{\Delta x}\right]e^{I\sigma_x i}, \quad (29)$$

Substituting the value of $D_{\Delta x}^2$ and $D_{\Delta x}$ in Eq. (29), and then simplification we get,

$$\xi^{n+1}\left[1+\left(\frac{1}{12}-\frac{a_x\Delta t}{2h_x{}^2}-\frac{c_x^2\Delta t}{24a_x}\right)\left(e^{I\sigma_x}-2+e^{-I\sigma_x}\right)+\left(\frac{c_x\Delta t}{4h_x}-\frac{h_x c_x}{24a_x}\right)\left(e^{I\sigma_x}-e^{-I\sigma_x}\right)\right]$$
$$=\xi^{n}\left[1+\left(\frac{1}{12}+\frac{a_x\Delta t}{2h_x{}^2}+\frac{c_x^2\Delta t}{24a_x}\right)\left(e^{I\sigma_x}-2+e^{-I\sigma_x}\right)-\left(\frac{c_x\Delta t}{4h_x}+\frac{h_x c_x}{24a_x}\right)\left(e^{I\sigma_x}-e^{-I\sigma_x}\right)\right],$$

By rearrangement of the above equation, we have

$$\xi^{n+1}[1-\frac{1}{6}+\frac{a_x\Delta t}{h_x{}^2}+\frac{c_x^2\Delta t}{12a_x}+(\frac{1}{12}-\frac{a_x\Delta t}{2h_x{}^2}-\frac{c_x^2\Delta t}{24a_x}+\frac{c_x\Delta t}{4h_x}-\frac{h_x c_x}{24a_x})e^{I\sigma_x}+(\frac{1}{12}-\frac{a_x\Delta t}{2h_x{}^2}-\frac{c_x^2\Delta t}{24a_x}$$
$$-\frac{c_x\Delta t}{4h_x}+\frac{h_x c_x}{24a_x})e^{-I\sigma_x}]=\xi^{n}[1-\frac{1}{6}-\frac{a_x\Delta t}{h_x{}^2}-\frac{c_x^2\Delta t}{12a_x}+(\frac{1}{12}+\frac{a_x\Delta t}{2h_x{}^2}+\frac{c_x^2\Delta t}{24a_x}-\frac{c_x\Delta t}{4h_x}-\frac{h_x c_x}{24a_x})e^{I\sigma_x}$$
$$+(\frac{1}{12}+\frac{a_x\Delta t}{2h_x{}^2}+\frac{c_x^2\Delta t}{24a_x}+\frac{c_x\Delta t}{4h_x}+\frac{h_x c_x}{24a_x})e^{-I\sigma_x}].$$

Transformation the value of exponential function in to (Sine) and (Cosine) functions, yield the following

$$\xi^{n+1}[1-\frac{1}{6}+\frac{a_x\Delta t}{\Delta x^2}+\frac{c_x^2\Delta t}{12a_x}+\frac{1}{6}\cos(\sigma_x)-\frac{a_x\Delta t}{\Delta x^2}\cos(\sigma_x)-\frac{c_x^2\Delta t}{12a_x}\cos(\sigma_x)+\frac{I\,c_x\Delta t}{2\Delta x}\sin(\sigma_x)$$
$$-\frac{I\Delta x c_x}{12a_x}\sin(\sigma_x)]=\xi^{n}[1-\frac{1}{6}-\frac{a_x\Delta t}{\Delta x^2}-\frac{c_x^2\Delta t}{12a_x}+\frac{1}{6}\cos(\sigma_x)+\frac{a_x\Delta t}{\Delta x^2}\cos(\sigma_x)+\frac{c_x^2\Delta t}{12a_x}\cos(\sigma_x) \quad (30)$$
$$-\frac{I\,c_x\Delta t}{2\Delta x}\sin(\sigma_x)-\frac{I\Delta x c_x}{12a_x}\sin(\sigma_x)].$$

By simplification Eq. (30) we get

$$\xi^{n+1}[1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2})+\frac{2a_x\Delta t}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2})+\frac{2c_x^2\Delta t}{12a_x}\sin^2(\frac{\sigma_x}{2})+\frac{I\,c_x\Delta t}{2h_x}\sin(\sigma_x)-\frac{Ih_xc_x}{12a_x}\sin(\sigma_x)]$$

$$=\xi^{n}[1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2})-\frac{2a_x\Delta t}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2})-\frac{2c_x^2\Delta t}{12a_x}\sin^2(\frac{\sigma_x}{2})-\frac{I\,c_x\Delta t}{2h_x}\sin(\sigma_x)-\frac{Ih_xc_x}{12a_x}\sin(\sigma_x)].$$

then we can find the value of $\psi_x(\sigma_x)$ as follows

$$\frac{\xi^{n+1}}{\xi^n}=\psi_x(\sigma_x)=\frac{1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2})-2\Delta t(\frac{a_x}{h_x^{\ 2}}+\frac{c_x^2}{12a_x})\sin^2(\frac{\sigma_x}{2})-\frac{I\,c_x\Delta t}{2h_x}\sin(\sigma_x)-\frac{Ih_xc_x}{12a_x}\sin(\sigma_x)}{1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2})+2\Delta t(\frac{a_x}{h_x^{\ 2}}+\frac{c_x^2}{12a_x})\sin^2(\frac{\sigma_x}{2})+\frac{I\,c_x\Delta t}{2\Delta x}\sin(\sigma_x)-\frac{Ih_xc_x}{12a_x}\sin(\sigma_x)}.$$

which can be written as follows

$$\psi_x(\sigma_x)=\frac{\gamma_1-\gamma_2-I(\gamma_3+\gamma_4)}{\gamma_1+\gamma_2+I(\gamma_3-\gamma_4)},\qquad (31)$$

where

$$\gamma_1=1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2}),\qquad \gamma_2=2\Delta t(\frac{a_x}{h_x^{\ 2}}+\frac{c_x^2}{12a_x})\sin^2(\frac{\sigma_x}{2}),$$

$$\gamma_3=\frac{c_x\Delta t}{2h_x}\sin(\sigma_x),\qquad \gamma_4=\frac{h_xc_x}{12a_x}\sin(\sigma_x).$$

and the similar expression for $\psi_y(\sigma_y)$ and $\psi_z(\sigma_z)$ may be written by replacing σ_x by σ_y and σ_z, Δx by Δy and Δz, c_x by c_y and c_z, a_x by a_y and a_z. For stability it is sufficient that $|\psi_x(\sigma_x)|^2\le 1$, $|\psi_y(\sigma_y)|^2\le 1$ and $|\psi_z(\sigma_z)|^2\le 1$. Imposing this condition directly on (31) yield $\gamma_1\gamma_2\ge\gamma_3\gamma_4$ as a necessary and sufficient condition for $|\psi_x(\sigma_x)|^2\le 1$. A simple calculation shows that

$$\gamma_1\gamma_2=2\Delta t(1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2}))(\frac{a_x}{h_x^{\ 2}}+\frac{c_x^2}{12a_x})\sin^2(\frac{\sigma_x}{2})\ge c_x^2\frac{\Delta t}{6a_x}(1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2}))\sin^2(\frac{\sigma_x}{2})$$

and

$$\gamma_3\gamma_4=c_x^2\frac{\Delta t}{24a_x}\sin^2(\sigma_x)=c_x^2\frac{\Delta t}{24a_x}(2\sin(\frac{\sigma_x}{2})\cos(\frac{\sigma_x}{2}))^2=c_x^2\frac{\Delta t}{6a_x}(1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2}))\sin^2(\frac{\sigma_x}{2})$$

Hence $\gamma_1\gamma_2 \geq \gamma_3\gamma_4$ and it follows that $|\psi_x(\sigma_x)|^2 \leq 1$. In the same way, we may find that $|\psi_y(\sigma_y)|^2 \leq 1$ and $|\psi_z(\sigma_z)|^2 \leq 1$. Thus we conclude that the Karaa scheme is unconditionally stable.

4.2 The stability of sixth order exponential compact scheme

Substituting the discrete Fourier mode (27) in (25), the amplification factor $H(\sigma_x,\sigma_y,\sigma_z) = \xi^{n+1}/\xi^n$, can be written as $H(\sigma_x,\sigma_y,\sigma_z) = \psi_x(\sigma_x)\psi_y(\sigma_y)\psi_z(\sigma_z)$. To show how to derive the value of $\psi_x(\sigma_x)$, $\psi_y(\sigma_y)$ and $\psi_z(\sigma_z)$ we reduce Eq. (25) in to 1-D case and then substituting the value of A_x and ℓ_x

$$\xi^{n+1}\left[1+\alpha_1 D_{\Delta x}+\alpha_2 D_{\Delta x}^2+\alpha_3 D_{\Delta x}^3+\alpha_4 D_{\Delta x}^4+\frac{\Delta t}{2}\left(-\alpha D_{\Delta x}^2+c_x D_{\Delta x}\right)\right]e^{I\sigma_x i}$$
$$=\xi^{n}\left[1+\alpha_1 D_{\Delta x}+\alpha_2 D_{\Delta x}^2+\alpha_3 D_{\Delta x}^3+\alpha_4 D_{\Delta x}^4-\frac{\Delta t}{2}\left(-\alpha D_{\Delta x}^2+c_x D_{\Delta x}\right)\right]e^{I\sigma_x i}, \quad (32)$$

Rearrangement Eq. (32) we get

$$\xi^{n+1}\left[1+\left(\alpha_1+\frac{c_x\Delta t}{2}\right)D_{\Delta x}+\left(\alpha_2-\frac{\alpha\,\Delta t}{2}\right)D_{\Delta x}^2+\alpha_3 D_{\Delta x}^3+\alpha_4 D_{\Delta x}^4\right]e^{I\sigma_x i}$$
$$=\xi^{n}\left[1+\left(\alpha_1-\frac{c_x\Delta t}{2}\right)D_{\Delta x}+\left(\alpha_2+\frac{\alpha\,\Delta t}{2}\right)D_{\Delta x}^2+\alpha_3 D_{\Delta x}^3+\alpha_4 D_{\Delta x}^4\right]e^{I\sigma_x i}, \quad (33)$$

Substituting the value of $D_{\Delta x}$, $D_{\Delta x}^2$, $D_{\Delta x}^3$ and $D_{\Delta x}^4$ in Eq. (33), and after simplification we get,

$$\xi^{n+1}[1+(\frac{\alpha_1}{2h_x}+\frac{c_x\Delta t}{4\Delta x})(e^{I\sigma_x}-e^{-I\sigma_x})+(\frac{\alpha_2}{h_x^{\,2}}-\frac{\alpha\,\Delta t}{2h_x^{\,2}})(e^{I\sigma_x}-2+e^{-I\sigma_x})+\frac{\alpha_3}{2h_x^{\,3}}(e^{2I\sigma_x}-2e^{I\sigma_x}$$
$$+2e^{-I\sigma_x}-e^{-2I\sigma_x})+\frac{\alpha_4}{h_x^{\,4}}(e^{2I\sigma_x}-4e^{I\sigma_x}+6-4e^{-I\sigma_x}+e^{-2I\sigma_x})]=\xi^n[1+(\frac{\alpha_1}{2h_x}-\frac{c_x\Delta t}{4h_x})(e^{I\sigma_x}-e^{-I\sigma_x})$$
$$+(\frac{\alpha_2}{h_x^{\,2}}+\frac{\alpha\,\Delta t}{2h_x^{\,2}})(e^{I\sigma_x}-2+e^{-I\sigma_x})+\frac{\alpha_3}{2h_x^{\,3}}(e^{2I\sigma_x}-2e^{I\sigma_x}+2e^{-I\sigma_x}-e^{-2I\sigma_x})$$
$$+\frac{\alpha_4}{h_x^{\,4}}(e^{2I\sigma_x}-4e^{I\sigma_x}+6-4e^{-I\sigma_x}+e^{-2I\sigma_x})].$$

Rearrangement the above equation, yields

the

$$\xi^{n+1}[1-\frac{2\alpha_2}{h_x^{\,2}}+\frac{\alpha\,\Delta t}{h_x^{\,2}}+\frac{6\alpha_4}{h_x^{\,4}}+(\frac{\alpha_2}{h_x^{\,2}}-\frac{\alpha\,\Delta t}{2h_x^{\,2}}+\frac{\alpha_1}{2h_x}+\frac{c_x\Delta t}{4h_x}-\frac{\alpha_3}{h_x^{\,3}}-\frac{4\alpha_4}{h_x^{\,4}})e^{I\sigma_x}+(\frac{\alpha_2}{h_x^{\,2}}-\frac{\alpha\,\Delta t}{2h_x^{\,2}}-\frac{\alpha_1}{2h_x}$$
$$-\frac{c_x\Delta t}{4h_x}+\frac{\alpha_3}{h_x^{\,3}}-\frac{4\alpha_4}{h_x^{\,4}})e^{-I\sigma_x}+(\frac{\alpha_4}{h_x^{\,4}}+\frac{\alpha_3}{2h_x^{\,3}})e^{2I\sigma_x}+(\frac{\alpha_4}{h_x^{\,4}}-\frac{\alpha_3}{2h_x^{\,3}})e^{-2I\sigma_x}]=\xi^n[1-\frac{2\alpha_2}{h_x^{\,2}}-\frac{\alpha\,\Delta t}{h_x^{\,2}}+\frac{6\alpha_4}{h_x^{\,4}}$$
$$+(\frac{\alpha_2}{h_x^{\,2}}+\frac{\alpha\,\Delta t}{2h_x^{\,2}}+\frac{\alpha_1}{2h_x}-\frac{c_x\Delta t}{4h_x}-\frac{\alpha_3}{h_x^{\,3}}-\frac{4\alpha_4}{h_x^{\,4}})e^{I\sigma_x}+(\frac{\alpha_2}{h_x^{\,2}}+\frac{\alpha\,\Delta t}{2h_x^{\,2}}-\frac{\alpha_1}{2h_x}+\frac{c_x\Delta t}{4h_x}+\frac{\alpha_3}{h_x^{\,3}}-\frac{4\alpha_4}{h_x^{\,4}})e^{-I\sigma_x}$$
$$+(\frac{\alpha_4}{h_x^{\,4}}+\frac{\alpha_3}{2h_x^{\,3}})e^{2I\sigma_x}+(\frac{\alpha_4}{h_x^{\,4}}-\frac{\alpha_3}{2h_x^{\,3}})e^{-2I\sigma_x}].$$

above equation lead to the following equation

$$\xi^{n+1}[1-\frac{2\alpha_2}{h_x^{\ 2}}+\frac{\alpha\,\Delta t}{h_x^{\ 2}}+\frac{6\alpha_4}{h_x^{\ 4}}+\frac{2\alpha_2}{h_x^{\ 2}}\cos(\sigma_x)-\frac{\alpha\,\Delta t}{h_x^{\ 2}}\cos(\sigma_x)+\frac{I\alpha_1}{h_x}\sin(\sigma_x)+\frac{Ic_x\Delta t}{2h_x}\sin(\sigma_x)$$
$$-\frac{2I\alpha_3}{h_x^{\ 3}}\sin(\sigma_x)-\frac{8\alpha_4}{h_x^{\ 4}}\cos(\sigma_x)+\frac{2\alpha_4}{h_x^{\ 4}}\cos(2\sigma_x)+\frac{I\alpha_3}{h_x^{\ 3}}\sin(2\sigma_x)]=\xi^{n}[1-\frac{2\alpha_2}{h_x^{\ 2}}-\frac{\alpha\,\Delta t}{h_x^{\ 2}}+\frac{6\alpha_4}{h_x^{\ 4}} \tag{34}$$
$$+\frac{2\alpha_2}{h_x^{\ 2}}\cos(\sigma_x)+\frac{\alpha\,\Delta t}{h_x^{\ 2}}\cos(\sigma_x)+\frac{I\alpha_1}{h_x}\sin(\sigma_x)-\frac{Ic_x\Delta t}{2h_x}\sin(\sigma_x)-\frac{2I\alpha_3}{h_x^{\ 3}}\sin(\sigma_x)-\frac{8\alpha_4}{h_x^{\ 4}}\cos(\sigma_x)$$
$$+\frac{2\alpha_4}{h_x^{\ 4}}\cos(2\sigma_x)+\frac{I\alpha_3}{h_x^{\ 3}}\sin(2\sigma_x)].$$

By simplification Eq. (34), we get

$$\xi^{n+1}[1-\frac{4\alpha_2}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2})+\frac{2\alpha\,\Delta t}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2})+\frac{6\alpha_4}{h_x^{\ 4}}+\frac{I\alpha_1}{h_x}\sin(\sigma_x)+\frac{Ic_x\Delta t}{2h_x}\sin(\sigma_x)-\frac{2I\alpha_3}{h_x^{\ 3}}\sin(\sigma_x)$$
$$-\frac{8\alpha_4}{h_x^{\ 4}}\cos(\sigma_x)+\frac{2\alpha_4}{h_x^{\ 4}}\cos(2\sigma_x)+\frac{I\alpha_3}{h_x^{\ 3}}\sin(2\sigma_x)]=\xi^{n}[1-\frac{4\alpha_2}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2})-\frac{2\alpha\,\Delta t}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2})+\frac{6\alpha_4}{h_x^{\ 4}} \tag{35}$$
$$+\frac{I\alpha_1}{h_x}\sin(\sigma_x)-\frac{Ic_x\Delta t}{2h_x}\sin(\sigma_x)-\frac{2I\alpha_3}{h_x^{\ 3}}\sin(\sigma_x)-\frac{8\alpha_4}{h_x^{\ 4}}\cos(\sigma_x)+\frac{2\alpha_4}{h_x^{\ 4}}\cos(2\sigma_x)+\frac{I\alpha_3}{h_x^{\ 3}}\sin(2\sigma_x)].$$

Then we can find the value of $\psi_x(\sigma_x)$ from Eq. (35) as follows

$$\frac{\xi^{n+1}}{\xi^{n}}=\psi_x(\sigma_x)=\frac{\gamma_1-\gamma_2+\gamma_3+I(\gamma_4-\gamma_5+\gamma_6)}{\gamma_1+\gamma_2+\gamma_3+I(\gamma_4+\gamma_5+\gamma_6)}. \tag{36}$$

Where

$$\gamma_1=1-\frac{4\alpha_2}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2}),\qquad \gamma_2=\frac{2\alpha\,\Delta t}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2}),$$

$$\gamma_3=\frac{6\alpha_4}{h_x^{\ 4}}-\frac{8\alpha_4}{h_x^{\ 4}}\cos(\sigma_x)+\frac{2\alpha_4}{h_x^{\ 4}}\cos(2\sigma_x)=\frac{16\alpha_4}{h_x^{\ 4}}\sin^4(\frac{\sigma_x}{2}),$$

$$\gamma_4=\frac{\alpha_1}{h_x}\sin(\sigma_x),\qquad \gamma_5=\frac{c_x\Delta t}{2h_x}\sin(\sigma_x),\qquad \gamma_6=-\frac{2\alpha_3}{h_x^{\ 3}}\sin(\sigma_x)+\frac{\alpha_3}{h_x^{\ 3}}\sin(2\sigma_x).$$

and the similar expression for $\psi_y(\sigma_y)$ and $\psi_z(\sigma_z)$ may be written by replacing σ_x by σ_y and σ_z, Δx by Δy and Δz, c_x by c_y and c_z, a_x by a_y and a_z. For stability it is sufficient that $|\psi_x(\sigma_x)|^2\le 1$, $|\psi_y(\sigma_y)|^2\le 1$ and $|\psi_z(\sigma_z)|^2\le 1$. Imposing this condition directly on (36) yield $\gamma_1\gamma_2+\gamma_3\gamma_2+\gamma_5\gamma_4+\gamma_6\gamma_4\ge 0$ as a necessary and sufficient condition for $|\psi_x(\sigma_x)|^2\le 1$. A simple calculation shows that

$$\gamma_1\gamma_2+\gamma_3\gamma_2+\gamma_5\gamma_4+\gamma_6\gamma_4=\frac{2\alpha\,\Delta t}{h_x^{\ 2}}\left(1-\frac{4\alpha_2}{h_x^{\ 2}}\sin^2(\frac{\sigma_x}{2})\right)\sin^2(\frac{\sigma_x}{2})+\frac{32\alpha\,\Delta t\alpha_4}{h_x^{\ 6}}\sin^6(\frac{\sigma_x}{2}) \qquad (37)$$

$$+\frac{c_x\Delta t\alpha_1}{2h_x^{\ 2}}\sin^2(\sigma_x)-\frac{2c_x\Delta t\alpha_3}{h_x^{\ 4}}\sin^2(\sigma_x)\sin^2(\frac{\sigma_x}{2}).$$

we will verify $\gamma_1\gamma_2+\gamma_3\gamma_2+\gamma_5\gamma_4+\gamma_6\gamma_4\geq 0$. First assume $c_x=0,$ then

$$\alpha=a_x,\quad \alpha_1=0,\quad \alpha_2=\frac{\Delta x^2}{12},\quad \alpha_3=0,\quad \alpha_4=\frac{h_x^{\ 4}}{360}. \qquad (38)$$

Substituting (38) into (37), we have

$$\gamma_1\gamma_2+\gamma_3\gamma_2+\gamma_5\gamma_4+\gamma_6\gamma_4=\frac{2a_x\,\Delta t}{h_x^{\ 2}}\left(1-\frac{1}{3}\sin^2(\frac{\sigma_x}{2})\right)\sin^2(\frac{\sigma_x}{2})+\frac{32a_x\,\Delta t}{360h_x^{\ 2}}\sin^6(\frac{\sigma_x}{2}).$$

Hence $\gamma_1\gamma_2+\gamma_3\gamma_2+\gamma_5\gamma_4+\gamma_6\gamma_4\geq 0$ and it follows that $|\psi_x(\sigma_x)|^2\leq 1$ if $c_x=0$. In the same way, we may find that $|\psi_y(\sigma_y)|^2\leq 1$ and $|\psi_z(\sigma_z)|^2\leq 1$. Thus we conclude that the scheme is unconditionally stable for $c_x=c_y=c_z=0$.

4.Numerical Results

We first examine a convection-diffusion problem in the cubic region with coefficients $a_x=2,\ a_y=a_z=1$ and $c_x=c_y=c_z=1$. The exact solution of this test problem given by

$$u(x,y,z)=e^{x+y+z+t},\ \ 0\leq x,\ y,\ z\leq 1,\ t>0.$$

The initial and boundary condition are taken from this solution. We consider uniform grids with different mesh sizes and different regions and compare the accuracy of the computed solutions from the present sixth order exponential and fourth order compact ADI scheme and fourth order ADI scheme of Karaa and exponential fourth order ADI scheme of Mahdi. The quantity that we compare is the L^2_norm errors of the computed solution with respect to the exact solution. In the first and second test we take $h=h_x=h_y=h_z$. In Fig. 1 and Fig. 2, we plot the L^2_norm errors for $0\leq t\leq 1$, and (a) $h=$ 0.1 and region=$[0,1]^3$, (b) $h=$ 0.2 and region=$[0,2]^3$, (c) $h=$ 0.3 and region=$[0,3]^3$, (d) h=0.4 region=$[0,4]^3$, (e) $h=$ 0.5 region=$[0,5]^3$, at Δt=0.002 and Δt =0.001, respectively. The figures show the superiority of the present sixth order exponential compact (ADI) scheme over the exponential fourth order ADI scheme of Mahdi. In Table 1, we show that, the new scheme is more effective in term of accuracy.

Also, we use another test problem which has the analytic solution given, as in (Karaa), by

$$u(x,y,z,t)=\frac{1}{(4t+1)^{3/2}}\exp\left[-\frac{(x-c_x t-0.5)^2}{a_x(4t+1)}-\frac{(y-c_y t-0.5)^2}{a_y(4t+1)}-\frac{(z-c_z t-0.5)^2}{a_z(4t+1)}\right],$$
$$0\le x,y,z\le 1,\ t>0.$$

The boundary and the initial conditions are directly taken from this solution, and we take $a_x = a_y = a_z = 0.01$ and $c_x = c_y = c_z = 0.8$. In Table 2 we show that, the L^2_norm of the new scheme is less from the other schemes.

5-Conclusions

We have introduced a coupled sixth order exponential and fourth order compact ADI scheme with Crank-Nicholson technique for solving three-dimensional unsteady convection-diffusion equation. The unconditionally stability of sixth order exponential and fourth order compact finite difference schemes is proved with respect to initial values. Our Numerical results showed that the coupled sixth order exponential and fourth order compact finite difference schemes is computationally more efficient and more accurate than the fourth order scheme of Karaa and exponential fourth order scheme of Mahdi.

TABLE 1. L^2 _norm errors at t = 1 computed by three different schemes.

For test problem1.

h	0.1	0.2	0.3	0.4	0.5
region	$[0,1]^3$	$[0,2]^3$	$[0,3]^3$	$[0,4]^3$	$[0,5]^3$
Δt =0.1					
Samir Karaa Fourth order ADI	2.02E-04	5.05E-03	7.61E-02	1.12723	18.345891
Exponential Fourth order ADI	2.01E-04	5.01E-03	7.36E-02	1.0180327	14.525646
Exponential sixth order ADI	7.70E-05	2.05E-03	2.62E-02	2.92E-01	3.1937764
Δt =0.01					
Samir Karaa Fourth order ADI	2.46E-06	8.57E-05	3.28E-03	1.21E-01	4.029996
Exponential Fourth order ADI	2.36E-06	5.14E-05	7.50E-04	1.06E-02	1.64E-01
Exponential sixth order ADI	1.05E-06	2.13E-05	2.73E-04	3.34E-03	4.83E-02
Δt =0.002					
Samir Karaa Fourth order ADI	1.93E-07	3.64E-05	2.56E-03	1.11E-01	3.8899696
Exponential Fourth order ADI	9.46E-08	2.08E-06	3.40E-05	7.34E-04	2.34E-02
Exponential sixth order ADI	4.23E-08	8.95E-07	1.69E-05	5.06E-04	1.75E-02
Δt =0.001					
Samir Karaa Fourth order ADI	1.22E-07	3.49E-05	2.54E-03	1.11E-01	3.8855934
Exponential Fourth order ADI	2.37E-08	5.39E-07	1.16E-05	4.26E-04	1.90E-02
Exponential sixth order ADI	1.06E-08	2.56E-07	8.87E-06	4.17E-04	1.65E-02

TABLE 2. L^2 _norm errors at t = 1 computed by three different schemes.

For test problem2.

h	0.1	0.2	0.3	0.4	0.5
region	$[0,1]^3$	$[0,2]^3$	$[0,3]^3$	$[0,4]^3$	$[0,5]^3$
Δt =0.1					
Samir Karaa Fourth order ADI	2.95E-05	2.74E-04	4.74E-04	1.00E-03	1.40E-03
Exponential Fourth order ADI	5.43E-07	3.73E-04	2.42E-04	1.67E-04	1.11E-03
Exponential sixth order ADI	3.06E-08	3.98E-05	2.46E-05	1.91E-05	7.13E-05
Δt =0.01					
Samir Karaa Fourth order ADI	2.46E-06	1.87E-04	5.51E-04	1.05E-03	1.41E-03
Exponential Fourth order ADI	4.04E-07	3.80E-04	2.43E-04	1.68E-04	1.10E-03
Exponential sixth order ADI	1.38E-08	4.04E-05	2.46E-05	1.91E-05	6.21E-05
Δt =0.002					
Samir Karaa Fourth order ADI	2.75E-06	1.88E-04	5.52E-04	1.05E-03	1.41E-03
Exponential Fourth order ADI	4.08E-07	3.80E-04	2.43E-04	1.68E-04	1.10E-03
Exponential sixth order ADI	1.38E-08	4.04E-05	2.46E-05	1.91E-05	6.20E-05
Δt =0.001					
Samir Karaa Fourth order ADI	2.76E-06	1.88E-04	5.52E-04	1.05E-03	1.41E-03
Exponential Fourth order ADI	4.09E-07	3.80E-04	2.43E-04	1.68E-04	1.10E-03
Exponential sixth order ADI	1.38E-08	4.04E-05	2.46E-05	1.91E-05	6.20E-05

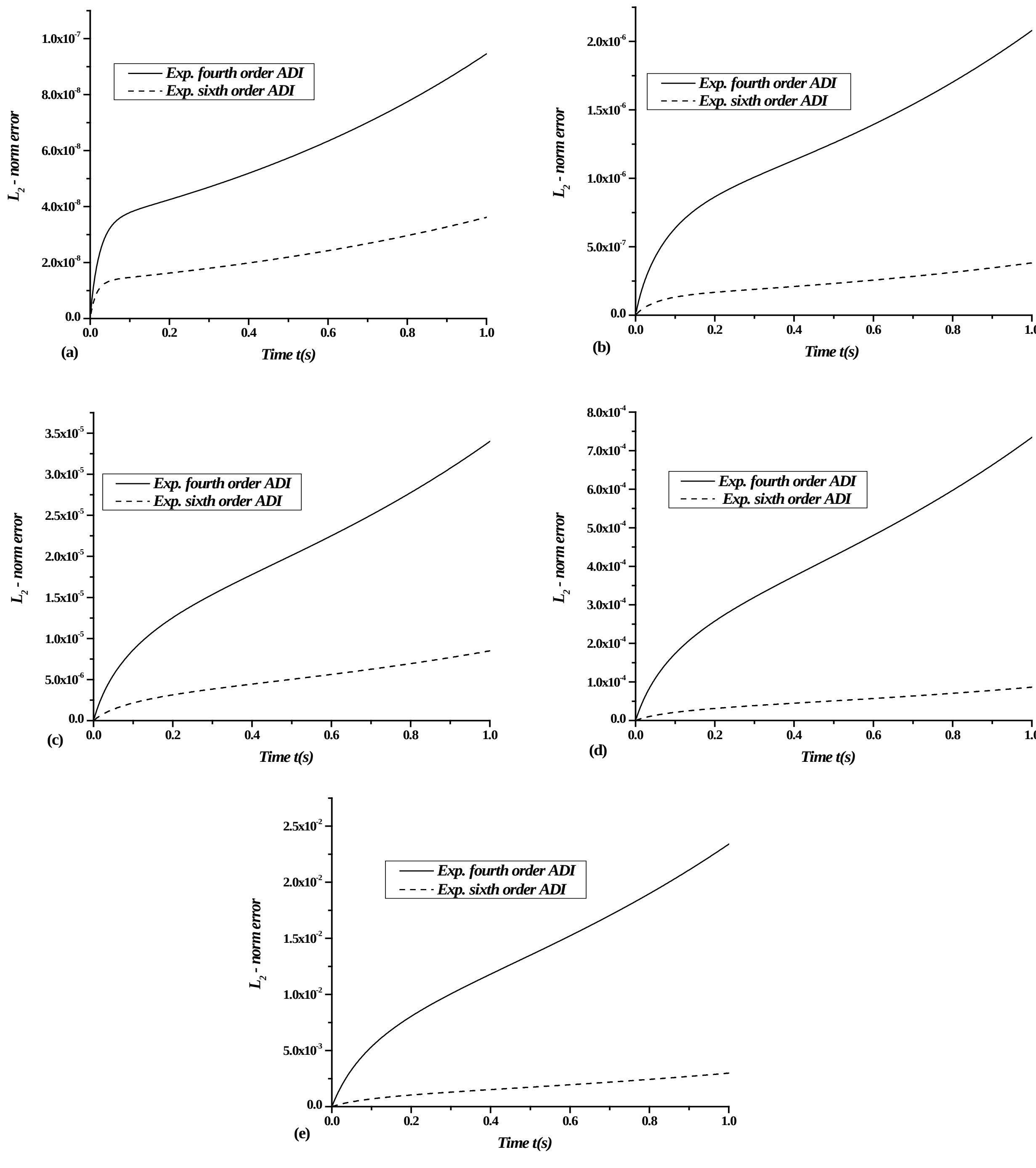

FIG.1. Comparison of errors in L^2 ***_norm*** of the present sixth order scheme with Mahdi fourth order scheme at each time step for, Δt = 0.002. (a) h = 0.1, (b) h = 0.2, (c) h = 0.3, (d) h =0.4, (e) h = 0.5.

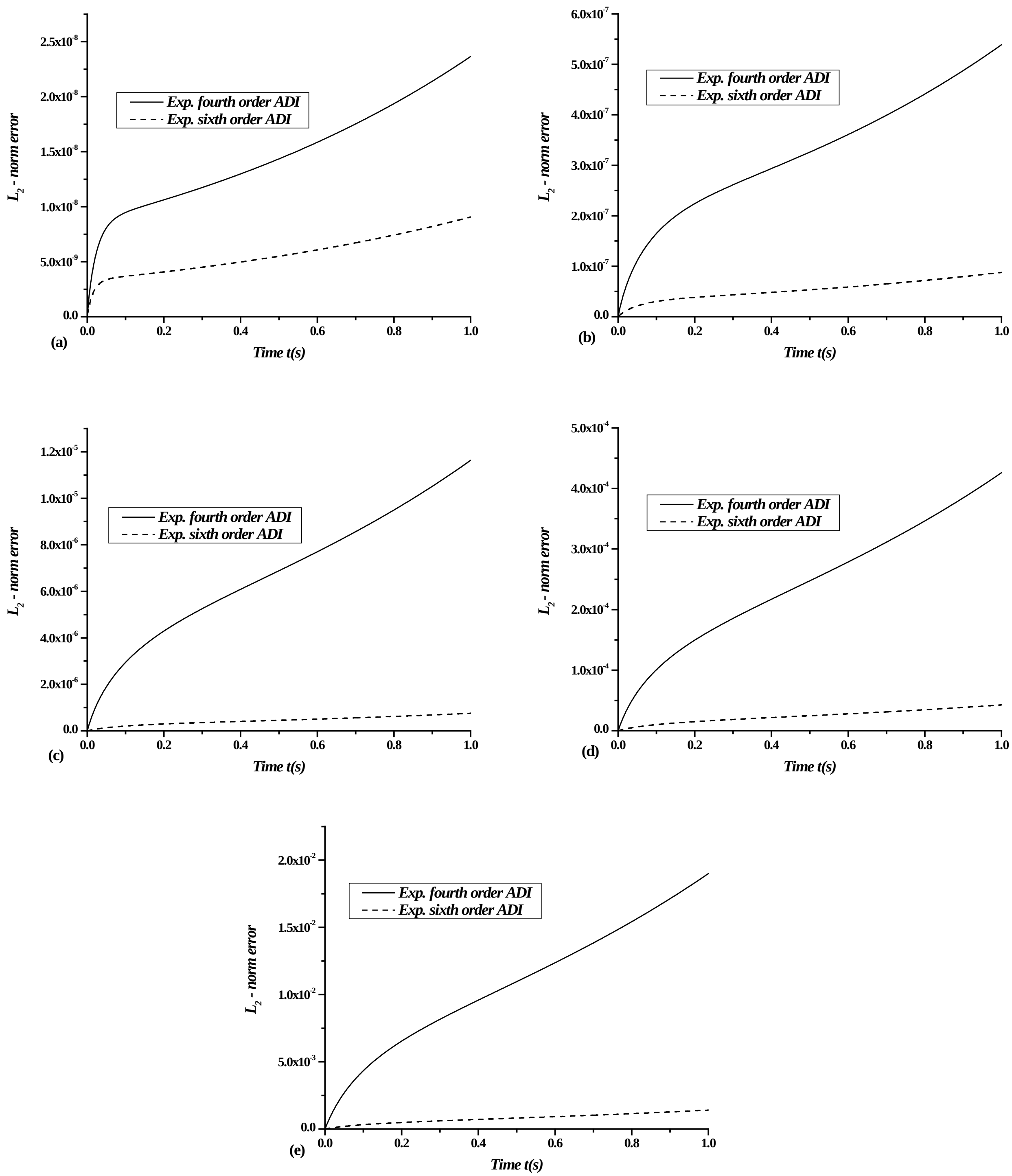

FIG.2. Comparison of errors in ***L^2 _norm*** of the present sixth order scheme with Mahdi. fourth order scheme at each time step for, Δt = 0.001. (a) h = 0.1, (b) h = 0.2, (c) h = 0.3, (d) h =0.4, (e) h = 0.5.

Reference

[1] J. Bear, 1972, Dynamics of fluids of porous media, American Elsevier Publishing Company, New York.

[2] W. Dai and R. Nassar, 2002, Compact ADI methods for solving parabolic differential equations, Numer. Methods Partial Differential Eqs., 18: 129-142.

[3] M. M. Gupta, P. R. Manohar and J. W. Stephenson, 1984, A single cell high-order scheme for the convection-diffusion equation with variable coefficients, Int. J. Numer. Methods Fluids, 4: 641-651.

[4] J. C. Kalita, D. C. Dalal and A. K. Dass, 2002, A class higher order compact schemes for the unsteady two-dimensional convection-diffusion equation, Int. J. Numer. Methods Fluids, 38: 1111-1131.

[5] S. Karaa, A class of two-level compact implicit scheme for solving three-dimensional unsteady convection-diffusion problems, submitted for publication.

[6] S. Karaa and J. Zhang, 2004, High order ADI method for solving unsteady convection-diffusion problems, J. Comput. Phys., 29: 1-9.

[7] S. Karaa, 2006, A high order compact ADI method for solving three-dimensional unsteady convection-diffusion problems, Numer. Methods Partial Differential Eqs., 22: 1-11.

[8] A. R. Mitchell and G. Fairweather, 1964, Improved forms of the alternating direction methods of Douglas, Peaceman, and Rachford for solving parabolic and elliptic equations, Numer Math, 6:285-292.

[9] B. J. Noye and H. H. Tan, 1988, A third-order semi-implicit finite difference method for solving the one-dimensional convection-diffusion equation, Int. J. Numer. Methods Engrg., 26: 1615-1629.

[10] B. J. Noye and H. H. Tan, 1989, Finite difference methods for solving the two-dimensional convection-diffusion equation, Int. J. Numer. Methods Fluids, 9: 75-89.

[11] A. Rigal, 1994, High order difference schemes for solving the one-dimensional convection-diffusion problems, J. Comput. Phys., 114: 59-76.

[12] A. Rigal, 1999, Schemes compacts d'order eleve application aux problemes bidimensionnels de convect diffusion-convection instationnaire I, CR Acad Sci Paris Sr I Math, 328: 535-538.

[13] P. J. Roach, 1976, Computational Fluid Dynamics, Hermosa, Albuquerque, MN.

[14] S. A. Mahdi, Fourth order exponential compact alternating direction implicit (ADI) scheme for solving three-dimensional convection-diffusion equations**,** submitted for publication.

[15] W. F. Spotz and G. F. Carey, 2001, Extantion of high-order schemes to time-

dependent problems, Numer. Methods Partial Differential Eqs., 17: 657-672.

[16] Z. F. Tain, Y. B. Ge, 2003, A fourth-order compact finite difference scheme for the steady stream function-vorticity formulation of the Navier-stokes/ Boussinesq equations, Int. J. Numer. Meth. Fluids, 41: 495-518.

[17] J. W. Thomas, 1995, Numerical Partial Differential Equations: Finite difference methods, Springer, New York.

[18] Z. F. Tain and S. Q. Dai, 2007, High-order compact exponential finite difference methods for convection–diffusion type problems, J. Comput. Phys., 220: 952-974.

[19] Z. F. Tain and Y. B. Ge, 2007, High-order compact exponential finite difference methods for convection–diffusion type problems, J. Comput. Phys, 198: 268-286.

[20] J. Zhang, 1998, An explicit fourth-order compact finite difference scheme for three dimensional convection-diffusion equation, Commun. Numer. Meth. Eng., 14: 209-218.

الأسلوب الآسي الضمني المتناوب الاتجاه من الرتبة السادسة المضغوط لحل معادلة الانتقال والانتشار ثلاثية البعد والمعتمدة على الزمن.

عقيل جاسم حرفش

قسم الرياضيات- كلية العلوم- جامعة البصرة.

الخلاصة

في هذا البحث تم اشتقاق أسلوب الفروقات المحددة الآسية الضمنية المضغوطة من الرتبة السادسة المتناوب الاتجاه لحل معادلة الانتقال والانتشار ثلاثية البعد والمعتمدة على الزمن. وهذا الأسلوب من الرتبة السادسة للحيز والثنائية للزمن. وقد تم برهان إن استقرارية الاسلوب الجديد غير مقيدة بشروط بالاعتماد على القيم الابتدائية فقط. النماذج العددية قدمت وذلك لاختبار دقة الأسلوب الآسي من الرتبة السادسة، والنتائج العددية وضحت بانه ذا دقة عالية مقارنتة مع أسلوب كارا Karaa المضغوط من الرتبة الرابعة وأسلوب مهديMahdi الاسي المضغوط من الرتبة الرابعة.