

Error analysis for mixed finite element method for the Navier-Stokes equation

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Abstract:

In this paper, we present abstract error estimates of the approximate solution of the incompressible Navier-Stokes problems in two-dimensional domain for mixed finite element method. These estimates are then applied to obtain quasi-optimal error estimate in the energy norm for velocity, pressure and velocity with pressure.

Keywords: Navier-Stokes equation, Mixed finite element, Error estimate.

1. Introduction

Mixed finite method result naturally from finite element approximation of any variational boundary value problem with constraints, these method is useful in the study of approximate methods for treating incompressible fluids and solids, and it is based on formulations having the stresses and/or strain as fundamental variables (in the mechanics of solids), at variance with the classical finite element method where fundamental unknowns are displacements only. So, mixed finite element methods are one of the important approaches for solving problem of partial differential equations, for example, the Navier-Stokes equations.

Reddy, J. N. and Tsay , C. S.[13] derived mixed finite element models which have been based on variants of Reissner's functional for various order polynomial approximations. Johnson, C. and Thomee, V. [8] considered a class of mixed finite element methods for second order elliptic problems introduced by Raviart and Thomas. Douglas, J. and Milner, F. [6] adapted the Nische- Schats approach to the Raviart –Thomas mixed method for scalar second order elliptic problems. Arnold, D. N. [2] treated the basic ideas of mixed finite element methods at an introductory level. Arbogast, T. and Wheeler, M.F. [1] defined a new finite element method, called the characteristic mixed method for approximating the solution to an advection-dominated transport problem.

Lovadina, C. [12] presented a new class of finite elements for the Reissner-Mindlin plate problem based on a modified mixed formulation recently introduced by Arnold and Brezzi (Boundary Value Problems for Partial Differential Equations and Applications, J.L. Baiocchi, eds., Masson, Paris, 1993). Rusten, T. et al. [16] established that an interior penalty method applied to second order elliptic problems gives rise to a local operator which is spectrally equivalent to the corresponding non-local operator arising from the mixed finite element method. Chen, Z. et al. [4] developed domain decomposition algorithms for mixed finite element method for solving the linear second order elliptic problems in \mathbb{R}^2 and \mathbb{R}^3 . Baaijens, F. [3] reviewed the progress made during past decade in the application of mixed finite element methods to solve viscoelastic flow problems using differential constitutive equations and the capabilities of viscoplastic flow to solve predict experimental observations. Liberman, E. [11] presented a posteriori error estimator for mixed finite element method for the Reissner-Mindlin plate model. Russell, T. F. et al. [15] considered control-volume mixed finite element methods for the approximation of second order elliptic problems on rectangular grids. Jun, H. and Shi, Z. [9] presented two stable rectangular

nonconforming mixed finite element methods for the equation of linear elasticity in two dimensional space that produce direct approximations for the stress and displacement.

The aim of this paper is to discuss the error estimate, first to steady-state Navier-Stokes problem and second for unsteady Navier-Stokes of semi-discrete problem.

Notation

Let Ω be an open and bounded domain in \mathbb{R}^2 with Lipschitz continuous boundary Γ . Throughout this paper we will use the standard notation for Sobolev spaces. Specially $H^r(\Omega)$, where r is an integer greater than zero, denotes to the Sobolev space of real-valued functions with square integrable derivatives of order up to r equipped with the usual norm, denoted by $\|\cdot\|_r$. We will denote $H^0(\Omega)$ by $L^2(\Omega)$ and the standard L^2 inner product by (\cdot, \cdot) . Also $H^r(\Omega)$ will denote the space of vector-valued functions each of whose n components and the dual space of $H^r(\Omega)$ will be denoted by $H^{-r}(\Omega)$. As a particular interest we will be consider the constrained space of [17]

$$V = [H_0^1(\Omega)]^2 = \{v = (v_1, v_2) : v_i \in H_0^1(\Omega), \quad i = 1, 2\}$$

and

$$Q = \{q \in L_2(\Omega) : \int_{\Omega} q dx = 0\}$$

The weak formulations

We are interested in approximating the solution of the Navier–Stokes equations written in the primitive variable

formulation of the velocity $u = (u_1, u_2)$ and the pressure p . In particular, we consider the unsteady Navier-Stokes problem, see [5]:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u - e \Delta u + \nabla p = f \quad \text{in } \Omega \quad (1.1a)$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega \quad (1.1b)$$

$$u = 0 \quad \text{on } \partial\Omega = \Gamma \quad (1.1c)$$

where $f \in H^1(\Omega)$ is given the body force per unit mass and e is the inverse Reynolds number.

Multiplying (1.1a) and (1.1b) by $v \in V$ and $q \in Q$, respectively, as a test functions and takes integral over Ω

$$\int_{\Omega} \frac{\partial u}{\partial t} v dx - \int_{\Omega} e \Delta u v dx + \int_{\Omega} (\nabla u) u v dx + \int_{\Omega} \nabla p v dx = \int_{\Omega} f v dx \quad ; \quad v \in V,$$

$$\int_{\Omega} (\nabla \cdot u) q dx = 0 \quad ; \quad q \in Q,$$

by using Green's formulation

$$\int_{\Omega} \frac{\partial u}{\partial t} v dx + e \int_{\Omega} \nabla u \nabla v dx - \int_{\Omega} u^2 \nabla v dx - \int_{\Omega} p \operatorname{div} v dx = \int_{\Omega} f v dx ,$$

$$\int_{\Omega} (\nabla \cdot u) q dx = 0 .$$

We consider the following standard weak formulation of non- steady: seek $(u, p) \in V \times Q$ such that

$$(u_t, v) + a(u, v) - n(u, u, v) - b(v, p) = (f, v), \quad v \in V, \quad (1.2a)$$

$$b(u, q) = 0, \quad q \in Q, \quad (1.2b)$$

where,

$$a(u, v) = e \int_{\Omega} \nabla u \nabla v \, dx,$$

$$n(u, u, v) = \frac{1}{2} \int_{\Omega} u^2 \nabla v \, dx,$$

$$b(u, q) = \int_{\Omega} (\nabla \cdot u) q \, dx .$$

Also, the weak formulation of the steady Navier–Stokes equations is as follows:

Seek $(u, p) \in V \times Q$ such that:

$$a(u, v) - n(u, u, v) - b(v, p) = (f, v), \quad v \in V, \quad (1.3a)$$

$$b(u, q) = 0, \quad q \in Q, \quad (1.3b)$$

Continuity of the forms $a(\cdot, \cdot)$, $n(\cdot, \cdot, \cdot)$ and $b(\cdot, \cdot)$ can be demonstrated. These conditions guarantee the existence and uniqueness of the solution (u, p) [5].

The discrete problems

Given finite dimensional spaces $V_h \subset V$ and $Q_h \subset Q$ where $0 < h < 1$ then the approximate solution (u_h, p_h) to (u, p) be the solution of the following problem:

$$(u_{t,h}, v) + a(u_h, v) - n(u_h, u_h, v) - b(v, p_h) = (f, v), \quad v \in V_h, \quad (1.4a)$$

$$b(u_h, q) = 0, \quad q \in Q_h, \quad (1.4b)$$

Also, the discrete problem of the steady Navier–Stokes equations is as follows:

Seek $(u_h, p_h) \in V_h \times Q_h$ such that:

$$a(u_h, v) - n(u_h, u_h, v) - b(v, p_h) = (f, v), \quad v \in V_h, \quad (1.5a)$$

$$b(u_h, q) = 0, \quad q \in Q_h, \quad (1.5b)$$

2. Abstract Results

Let V and Q be two real Banach spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_Q$ respectively. Let

$a(\cdot, \cdot)$ and $b(\cdot, \cdot) \in L^\infty$ be continuous

bilinear forms on $V \times V$ and $V \times Q$ respectively [7], $n(\cdot, \cdot, \cdot) \in L^\infty$ be continuous trilinear form on $V \times V \times V$ [14]:

$$|a(u, v)| \leq \|a\|_{L^\infty} \cdot \|u\|_V \|v\|_V \quad \forall u, v \in V, \quad (2.1)$$

$$|n(u, u, v)| \leq \|n\|_{L^\infty} \cdot \|u\|_V^2 \cdot \|v\|_V \quad \forall u, v \in V, \quad (2.2)$$

$$|b(u, p)| \leq \|b\|_{L^\infty} \cdot \|u\|_V \|p\|_Q \quad \forall u \in V; \forall p \in Q. \quad (2.3)$$

We now state several further assumptions which we will require in the proofs of our main results [7].

(H1) There is a constant $\alpha > 0$ (α independent of h) such that

$$a(v, v) \geq \alpha \|v\|_W^2 \quad \forall v \in Z_h,$$

where

$$Z_h = \{v \in V_h : b(v, q) = 0, \forall q \in Q_h\}.$$

(H2) $S(h)$ is a number satisfying

$$\|v\|_V \leq S(h) \|v\|_W \quad \forall v \in V_h.$$

(H3) There is a linear operator $\Pi_h : Y \rightarrow V_h$ satisfying

$$b(y - \Pi_h y, \varphi) = 0 \quad \forall y \in Y \quad \text{and} \quad \forall \varphi \in Q_h.$$

Definition 2.1 [9] Cauchy-Schwarz inequalities:

$$\left| (v, w)_{L^2(\Omega)} \right| \leq \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}, \quad v, w \in L^2(\Omega) \quad (2.4)$$

and

$$\left| (v, w)_{H^2(\Omega)} \right| \leq \|v\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)}, \quad v, w \in H^1(\Omega) \quad (2.5)$$

Lemma 2.1 There exists a linear operator $\Pi_h : H \rightarrow H_h$ such that, [8]

$$(\text{div } \Pi_h \chi, v_h) = (\text{div } \chi, v_h) \quad \forall v_h \in V_h, \chi \in H,$$

$$\|\Pi_h \chi - \chi\| \leq C h^s \|\chi\|_S \quad \text{for } s = 1, 2.$$

3. Error Estimated

We shall now study the errors $u - u_h$ and $p - p_h$ where u and p are the solution of weak form and u_h and p_h are the solution of the mixed finite element problem (V_h and Q_h).

Theorem 3.1 Let $u \in V$ be the solution of problem (1.3) and $u_h \in V_h$ is the approximated solution of problem (1.5). Then, there exists a constant $C > 0$ independent of h such that:

$$\|u - u_h\| \leq C h. \quad (3.1)$$

Proof: Using (1.3a) we see that

$$\begin{aligned}
 a(\Pi_h u, v) - n(\Pi_h u, \Pi_h u, v) - b(v, p) & \\
 &= a(\Pi_h u - u + u, v) - n(\Pi_h u - u + u, \Pi_h u - u + u, v) - b(v, p) \\
 &= a(\Pi_h u - u, v) - n(\Pi_h u - u, \Pi_h u - u, v) - b(v, p) + a(u, v) - n(u, u, v) \\
 &= a(\Pi_h u - u, v) - n(\Pi_h u - u, \Pi_h u - u, v) + (f, v),
 \end{aligned}$$

let $p_h = q \in Q_h$, by using (1.5a) we get

$$\begin{aligned}
 a(\Pi_h u, v) - n(\Pi_h u, \Pi_h u, v) - b(v, p) & \\
 &= a(\Pi_h u - u, v) - n(\Pi_h u - u, \Pi_h u - u, v) + a(u_h, v) - n(u_h, u_h, v) - b(v, p_h) \\
 a(\Pi_h u - u_h, v) - n(\Pi_h u - u_h, \Pi_h u - u_h, v) - b(v, p - p_h) & \\
 &= a(\Pi_h u - u, v) - n(\Pi_h u - u, \Pi_h u - u, v) \tag{3.2}
 \end{aligned}$$

$$\begin{aligned}
 a(\Pi_h u - u_h, v) - n(\Pi_h u - u_h, \Pi_h u - u_h, v) - b(v, p - p_h) & \\
 &= a(\Pi_h u - u, v) - n(\Pi_h u - u, \Pi_h u - u, v) \tag{3.2}
 \end{aligned}$$

and from (1.3b) and (H3) we see that $b(\Pi_h u, q) = 0$, we have:

$$b(\Pi_h u - u_h, q) = 0, \tag{3.3}$$

choosing $v = \Pi_h u - u_h$ in (3.2) we have

$$\begin{aligned}
 a(\Pi_h u - u_h, \Pi_h u - u_h) - n(\Pi_h u - u_h, \Pi_h u - u_h, \Pi_h u - u_h) - b(\Pi_h u - u_h, p - p_h) & \\
 &= a(\Pi_h u - u, \Pi_h u - u_h) - n(\Pi_h u - u, \Pi_h u - u, \Pi_h u - u_h),
 \end{aligned}$$

by applying (3.3) we get

$$\begin{aligned}
 a(\Pi_h u - u_h, \Pi_h u - u_h) - n(\Pi_h u - u_h, \Pi_h u - u_h, \Pi_h u - u_h) & \\
 &= a(\Pi_h u - u, \Pi_h u - u_h) - n(\Pi_h u - u, \Pi_h u - u, \Pi_h u - u_h) - b(\Pi_h u - u_h, p - q)
 \end{aligned}$$

by using (2.1), (2.2) and (2.3), we get

$$\begin{aligned}
 \alpha \|\Pi_h u - u_h\|^2 + \beta \|\Pi_h u - u_h\|^3 \leq \|a\|_{L^\infty} \|\Pi_h u - u_h\| \|\Pi_h u - u\| + \|n\|_{L^\infty} \|\Pi_h u - u_h\| \|\Pi_h u - u\|^2 & \\
 + \|b\|_{L^\infty} S(h) \|\Pi_h u - u_h\| \|\Pi_h u - u\| \|p - q\|, &
 \end{aligned}$$

$$\alpha \|\Pi_h u - u_h\| + \beta \|\Pi_h u - u_h\|^2 \leq \|a\|_{L^\infty} \|\Pi_h u - u\| + \|n\|_{L^\infty} \|\Pi_h u - u\|^2 + \|b\|_{L^\infty} S(h) \|p - q\|,$$

thus ,

$$\|u - u_h\| + \|u - u_h\|^2 \leq \|\Pi_h u - u\| \|\Pi_h u - u_h\| + \|\Pi_h u - u_h\|^2 \|\Pi_h u - u\|^2,$$

choosing $\beta = \alpha$

$$\|u - u_h\| + \|u - u_h\|^2 \leq \frac{1}{\alpha} [(\|a\|_{L^\infty} + \alpha) \|\Pi_h u - u\| + (\|n\|_{L^\infty} + \alpha) \|\Pi_h u - u\|^2 + \|b\|_{L^\infty} S(h) \|p - q\|],$$

since, $\|u - u_h\|^2 \geq 0$ and $\|\Pi_h u - u\|^2 \geq 0$, then,

$$\|u - u_h\| \leq \frac{1}{\alpha} [(\|a\|_{L^\infty} + \alpha) \|\Pi_h u - u\| + \|b\|_{L^\infty} S(h) \|p - q\|], \quad (3.4)$$

from Lemma 2.1 we have

$$\|u - u_h\| \leq C_1 h_u + C_2 h_p,$$

suppose that $h_u = h_p = h$ and $C = \max\{C_1, C_2\}$, we get

$$\|u - u_h\| \leq Ch.$$

Theorem 3.2 Let $p \in Q$ be the solution of problem (1.3) and $p_h \in Q_h$ is the approximated solution to problem (1.5), then there exists a constant $C > 0$ independent of h such that:

$$\|p - p_h\| \leq C_5 h. \quad (3.5)$$

Proof: Subtracting (1.3a) from (1.5a) we find

$$b(v, p - p_h) = -a(u - u_h, v) - n(u - u_h, u - u_h, v),$$

put $v = u - u_h$, then

$$b(u - u_h, p - p_h) = -a(u - u_h, u - u_h) - n(u - u_h, u - u_h, u - u_h)$$

by using (2.1), (2.2) and (2.3), we get

$$\|b\|_{L^\infty} S(h) \|p - p_h\| \|u - u_h\| \leq \alpha \|u - u_h\|^2 + \beta \|u - u_h\|^3,$$

then,
$$\|p - p_h\| \leq \frac{\alpha}{\|b\|_{L^\infty} S(h)} \|u - u_h\|,$$

where $\|u - u_h\|^2 \geq 0$, by using equation (3.4) we get

$$\|p - p_h\| \leq \frac{1}{\|b\|_{L^\infty} S(h)} [(\|a\|_{L^\infty} + \alpha) \|\Pi_h u - u\| + \|b\|_{L^\infty} S(h) \|p - q\|],$$

from Lemma 2.1, we have

$$\|p - p_h\| \leq C_3 h + C_4 h$$

where, $C_5 = \max\{C_3, C_4\}$ we get: $\|p - p_h\| \leq C_5 h$. □

Theorem 3.3 Let $(u, p) \in V \times Q$ is the solution of problem (1.3) and $(u_h, p_h) \in V_h \times Q_h$ is the approximated solution to problem (1.5), then, there exists a constant $C_6 > 0$ independent of h such that:

$$\|u - u_h\| + \|p - p_h\| \leq C_6 h. \tag{3.6}$$

Proof: We can prove this theorem by using the proof of Theorem 3.1 and Theorem 3.2, where, $C_6 = C + C_5$. \square

approximated solution to problem (1.4). Then, there exists a constant $C_7 > 0$ independent of h such that

Theorem 3.4 Let $u \in V$ is the solution of problem (1.2) and $u_h \in V_h$ is the

$$\|u - u_h\| \leq C_7 h. \tag{3.7}$$

Proof: Using the continuous problems (1.2) and (1.4). Put $p = q \in Q$ and

$u = v \in V$ in equations (1.2a) and (1.2b) respectively, we have

$$(u_t, v) + a(u, v) - n(u, u, v) = (f, v), \tag{3.8}$$

put $p_h = q \in Q_h$ and $u_h = v \in V_h$ in equations (1.4a) and (1.4b) respectively, we have

$$(u_{t,h}, v) + a(u_h, v) - n(u_h, u_h, v) = (f, v), \tag{3.9}$$

subtracting equation (3.9) from equation (3.8) we get

$$(u_t - u_{t,h}, v) + a(u - u_h, v) - n(u - u_h, u - u_h, v) = 0, \tag{3.10}$$

Let $\Pi_h u$ be the interpolate value of u then we can write

$$u - u_h = (u - \Pi_h u) - (u_h - \Pi_h u) = \rho - \theta, \tag{3.11}$$

by using triangle inequality, we have

$$\|u - u_h\| \leq \|\rho\| + \|\theta\|,$$

from Lemma 2.1

$$\|\rho\| \leq Ch \|u\|. \tag{3.12}$$

To estimate θ , from equation (3.10), and use the elliptic projection $(\rho, v) = 0$ and $n(\rho, \rho, v) = 0$, we get

$$(\theta_t, v) + a(\theta, v) - n(\theta, \theta, v) = (\rho_t, v),$$

by choosing $v = \theta$

$$(\theta_t, \theta) + a(\theta, \theta) - n(\theta, \theta, \theta) = (\rho_t, \theta), \tag{3.13}$$

since

$$(\theta_t, \theta) = \int_{\Omega} \frac{d\theta}{dt} \theta dx = \frac{1}{2} \frac{d}{dt} \|\theta\|^2,$$

$$a(\theta, \theta) \geq \alpha \|\theta\|^2.$$

By using Cauchy-Schwartz inequality (2.4) $\varepsilon = C_1$ and $3C$, respectively, we get and applying Young's inequality with

$$n(\theta, \theta, \theta) \leq \beta \|\theta\|^2 \|\theta\| \leq \beta \left[\frac{1}{2C_1} \|\theta\|^4 + \frac{C_1}{2} \|\theta\|^2 \right],$$

$$(\rho_t, \theta) \leq \|\rho_t\| \|\theta\| \leq \frac{1}{6C} \|\rho_t\|^2 + \frac{3C}{2} \|\theta\|^2.$$

Then, equation (3.13) become

$$\frac{1}{2} \frac{d}{dt} \|\theta\|^2 + \alpha \|\theta\|^2 + \beta \left[\frac{1}{2C_1} \|\theta\|^4 + \frac{C_1}{2} \|\theta\|^2 \right] \leq \frac{1}{6C} \|\rho_t\|^2 + \frac{3C}{2} \|\theta\|,$$

by choosing
$$\frac{d}{dt} \|\theta\|^2 + \frac{\beta^2}{C} \|\theta\|^4 \leq \frac{1}{3C} \|\rho_t\|^2,$$

where, $\left(\frac{\beta^2}{C}\right) \|\theta\|^4 \geq 0$, integrate from $t = 0$ to $t = T$

$$\|\theta(T)\|^2 \leq \|\theta(0)\|^2 + \int_0^T \frac{1}{3C} \|\rho_t\|^2 dt,$$

or

$$\|\theta(T)\| \leq \|\theta(0)\| + \left(\int_0^T \frac{1}{3C} \|\rho_t\|^2 dt \right)^{\frac{1}{2}},$$

the first term on the right hand side gives,

$$\begin{aligned} \|\theta(0)\| &= \|u_h(0) - \Pi_h u(0)\| \\ &\leq \|u_h(0) - u(0)\| + \|u(0) - \Pi_h u(0)\| \\ &\leq \|u_h(0) - u(0)\| + Ch \|u(0)\|, \end{aligned} \tag{3.14}$$

whereas, the second term is

$$\begin{aligned} \left(\int_0^T \frac{1}{3C} \|\rho_t\|^2 dt \right)^{\frac{1}{2}} &= \left(\int_0^T \frac{1}{3C} \|u_t - \Pi_h u_t\|^2 dt \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^T Ch^2 \|u_t\|^2 dt \right)^{\frac{1}{2}} \\ &\leq Ch \left(\int_0^T \|u_t\|^2 dt \right)^{\frac{1}{2}} \\ &\leq Ch \|u_t\|_{L^2(H_0^1)}. \end{aligned} \tag{3.15}$$

Then,

$$\|\theta(T)\| \leq \|u_h(0) - u(0)\| + Ch\|u(0)\| + Ch\|u_t\|_{L^2(H_0^1)}, \quad (3.16)$$

$$\|u - u_h\| \leq Ch\|u\| + \|u_h(0) - u(0)\| + Ch\|u(0)\| + Ch\|u_t\|_{L^2(H_0^1)},$$

since, $\|u_h(0) - u(0)\| = 0$, we get

$$\|u - u_h\| \leq Ch\|u\| + Ch\|u(0)\| + Ch\|u_t\|_{L^2(H_0^1)}, \quad (3.17)$$

hence the proof is complete. \square

Theorem 3.5 Let $p \in Q$ is the solution of problem (1.2) and p_h is the approximated solution to problem (1.4).

Then, there exists a constant $C_8 > 0$ independent of h such that

$$\|p - p_h\| \leq C_8 h. \quad (3.18)$$

Proof: By using the continuous equations (1.2) and (1.4), by choosing $v = u_h$ in

equation (1.2a) and $v = u$ in equation (1.4a) we have

$$(u_t, u_h) + a(u, u_h) - n(u, u, u_h) - b(u_h, p) = (f, u_h), \quad (3.19)$$

and

$$(u_{h,t}, u) + a(u_h, u) - n(u_h, u_h, u) - b(u, p_h) = (f, u), \quad (3.20)$$

subtracting equation (3.20) from equation (3.19) yields

$$-n(u - u_h, u - u_h, u_h - u) - b(u_h - u, p - p_h) = (f, u_h - u). \quad (3.21)$$

Let $\Pi_h p$ be the interpolate of p then we can write

$$p - p_h = (p - \Pi_h p) - (p_h - \Pi_h p) = \rho - \phi, \quad (3.22)$$

by using triangle inequality, we have

$$\|p - p_h\| \leq \|\omega\| + \|\phi\|,$$

from Lemma 2.1

$$\|\omega\| \leq Ch\|p\|. \quad (3.23)$$

To estimate ϕ , from equation (3.21) and use the elliptic projection

$$\begin{aligned} n(\rho, \rho, \rho) = 0, \quad n(\rho, \rho, \theta) = 0, \quad n(\rho, \theta, \theta) = 0, \\ b(\rho, \omega) = 0, \quad b(\rho, \phi) = 0, \quad b(\theta, \omega) = 0 \end{aligned}$$

and $(f, \rho) = 0$, we get

$$-n(\theta, \theta, \theta) + b(\theta, \phi) = (f, \theta), \quad (3.24)$$

by using (2.19) and (2.20), we get

$$\beta \|\theta\|^3 + S(h) \|\theta\| \|\phi\| \leq \|f\| \|\theta\|,$$

since, $\|\theta\|^2 \geq 0$ we have

$$\|\phi\| \leq \frac{\|f\|}{S(h)}.$$

$$\|p - p_h\| \leq Ch \|p\| + \left(\frac{\|f\|}{S(h)}\right). \tag{3.25}$$

Hence the proof is complete. \square

Theorem 3.6 Let $u \in V$ and $p \in Q$ are the solutions of problem (1.2) and $u_h \in V_h$ and $p_h \in Q_h$ are the approximated solutions to problem (1.4). Then, there exists a constant $C_9 > 0$ independent of h such that

$$\|u - u_h\| + \|p - p_h\| \leq C_9 h. \tag{3.26}$$

Proof: We can prove this theorem from equations (3.7) and (3.18), where, $C_9 = C_7 + C_8$. \square

4. Conclusions

We consider the discretization of the steady and unsteady states incompressible Navier-Stokes problem in two dimensional domain by a mixed finite element method. We use the standard formulation of this problem in primitive variables and take as

approximation space of the mixed finite element method for the velocity and the pressure.

The error estimate considered for six cases, first to steady-state of our problem for velocity only, we get the following result

$$\|u - u_h\| \leq Ch$$

second for pressure only, we get the following result

$$\|p - p_h\| \leq C_5 h$$

and the third for adding velocity with pressure, we get the following result

$$\|u - u_h\| + \|p - p_h\| \leq C_6 h$$

however, all the other cases are for unsteady state of semi discrete, first for velocity only, we get the following result

$$\|u - u_h\| \leq C_7 h$$

second for pressure only, we get the following result

$$\|p - p_h\| \leq C_8 h$$

and the third for adding velocity with pressure, we get the following result

$$\|u - u_h\| + \|p - p_h\| \leq C_h h.$$

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تحليل الخطأ في طريقة العناصر المحددة المختلطة لمعادلة نافير-ستوكس

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الخلاصة

في هذا البحث، ناقشنا تخمينات خطأ الحل التقريبي لمسألة نافير-ستوكس غير القابلة للأنضغاط في المجال ثنائي البعد لطريقة العناصر المحددة المختلطة. هذه التخمينات تم تطبيقها للحصول على تخمين الخطأ شبه المثالي في معيار كل من السرعة، الضغط والسرعة والضغط.